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ASYMPTOTIC APPROACHES IN THE THEORY OF SHELLS: LONG HISTORY AND NEW TRENDS

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This paper is dedicated to the memory of Professor J.J. Telega

*If no parameters in the world were very large or very small,
science would reduce to an exhaustive list of everything.*

L.N. Trefethen

Abstract. *This paper provides a state-of-the-art review of asymptotic methods in the Theory of Plates and Shells (TPS). Asymptotic methods of solving problems related to TPS have been developed by many authors. The main features of our paper are: (i) it is devoted to the basic principles of asymptotic approaches, and (ii) it deals with both traditional approaches, and less widely used, new approaches. The authors have paid special attention to examples and discussion of results rather than to burying the ideas in formalism, notation, and technical details.*

Key words: *plates, shells, asymptotic methods, homogenization*

1. INTRODUCTION

The theory of Plates and Shells (TPS) is applied usually for technical purposes. However, a role of today's modern TPS is certainly wider. In fact, in many important cases the physical objects cannot be described by equations of 3D theory of elasticity. The examples can be biological membranes, liquid crystals, thin polymeric films, thin-walled objects made from materials with shape memory, as well as various nanostructural devices. TPS does notgive only practically useful results, but it also outlines a general methodology of the transition from 3D to 2D (or 1D) models. It is worth noting that development of mathematical physics in many cases has been motivated by TPS problems, in particular we mean the problems associated with the application of variation and asymptotic methods (AMs). Note that a key (for singular asymptotics) concept of an edge effect appeared in the works of Lamb and Basset in 1890, while the concept of boundary

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layer occurred in Fluid Mechanics only in 1904 [1]. The classical papers by Vishik and Lyusternik are a generalization of some results obtained earlier by Gol'denveizer [2]). On the other hand, TPS problems associated with high technology development of materials and constructions implied development of various homogenization procedures [3-12]. The investigation of rods stability yielded a linearization procedure, whereas Koiter's approach [13] has strongly influenced today's Catastrophe Theory.

Generally, AMs are applied in the field of TPS first for transition from 3D to 2D models, and then to solve 2D problems. Our attention is focused on the latter problem.

2. ON THE PARAMETER OF ASYMPTOTIC INTEGRATION

Almost always while considering any asymptotic behavior, a term "small" or "large" parameter is applied. Since this traditional meaning may lead to confusion, we further apply the term of "asymptotic integration parameters", not restricted to be necessarily small (large). Notice that any asymptotic analysis should begin with normalization of the problem, that is defining it in terms of non-dimensional variables whose typical scale is of the order of one, and the relative magnitude of different physical effects is measured by non-dimensional parameters or dimensionless groups [14]. In particular, in TPS the following parameters are often used: h/R is the ratio of shell thickness to its characteristic size, i.e. radius [2, 15]; a/b is the ratio of characteristic dimensions (i.e. a plate length to its width) [16]; ω^{-1}, where ω is the dimensionless frequency of vibrations [17]; A is the dimensionless amplitude of vibrations [18]; $\varepsilon = w / h$, where w is the normal displacement (the case $\varepsilon << 1$ belongs to Koiter's asymptotics [13], whereas the case $\varepsilon >> 1$ is called Pogorolev's asymptotics [19]); B_1 / B_2 is the ratio of bending stiffnesses of structurally orthotropic shell or the ratio of shear rigidity to membrane rigidity [20]; a small deviation of shell shape from canonical one [21] or a changeable thickness from a constant one; the ratio of shallow shell rise H to curvature radius R, and so on.

For periodically non-homogeneous plates and shells, the small parameter is the ratio of a period of non-homogeneity to a characteristic size of considered structure [3-12].

If it is impossible to define a suitable real physical parameter, it can be introduced to equations in a purely formal manner (artificial parameter of asymptotic integration) [22].

"Let us try to find the asymptotics of some nontrivial solutions. First of all it is necessary to guess (no better word may be chosen) in what form this asymptotics must be sought. This stage – guessing the form of the asymptotics – of course, defines formalization. Analogies, experience, physical considerations, intuition, and 'just lucky' guesses are the toolkit which is used by every investigator". But after the introduction of the parameters of asymptotic integration and after the choice of an AM, it is not necessary to 'reinvent the wheel' – it is better to use some well known and well worked out approach.

3. HOW TO FIND PARAMETERS OF ASYMPTOTIC INTEGRATION

One of the most peculiar aspects of TPS is that associated with the existence of a few parameters of asymptotic integration yielding complexity of the problem being analyzed. In general, this fact is omitted in most studies. Therefore, a domain of application of the results is not clear enough. Gol'denveizer [2] indicated the importance of estimation of the order of coefficients of the PDE and differential operators. In reference [2] the index of variation of a function has been introduced and found to be very convenient. For example [2, 15, 16, 23, 24]

$$w_x \sim \varepsilon^{\alpha} w;\ w_y \sim \varepsilon^{\beta} w;\ w_t \sim \varepsilon^{\gamma} w.$$

To compare the orders of several functions their indices of intensity are introduced in the following way

$$w \sim \varepsilon^{\delta};\ w \sim \varepsilon^{\sigma} u.$$

Parameters of asymptotic integration α,β, etc. are chosen in a way which yields a generalization of the Newton polygon. Notice that one gets finally not only simplified BVP, but also the estimation of application domains for used asymptotic simplifications.

Let us introduce some remarks. Solutions of linear BVP of TPS usually include exponential and trigonometric functions, which cause the efficiency of the described technique. But, for example, the solution of corner boundary layer type can contain powers of coordinates, and in this case the indices of variations should be applied carefully. In addition it should be noted that the described technique gives local estimations.

Although Gol'denveizer's monograph [2] was published long time ago, some of the results reported there have been reconsidered again in the frame of the so called power geometry [25].

Key steps of the method will be illustrated by the example of a membrane lying on an elastic support and governed by the equation

$$\varepsilon\ (w_{1xx} + w_{1yy}) + w = 0\,. \tag{1}$$

The parameters of asymptotic integration α, β are introduced

$$w_{1x} \sim \varepsilon^{\alpha} w,\ \ w_{1y} \sim \varepsilon^{\beta} w,\ \ -\infty < \alpha,\ \beta < \infty\,.$$

Exponents of ε power for all terms of equation (1) follow:

$$1\text{-}2\alpha;\ 1-2\beta;\ 0.$$

Considering plane α β (see Fig. 1), the areas corresponding to the smallest values of exponents associated with all terms of equation (1) are constructed.

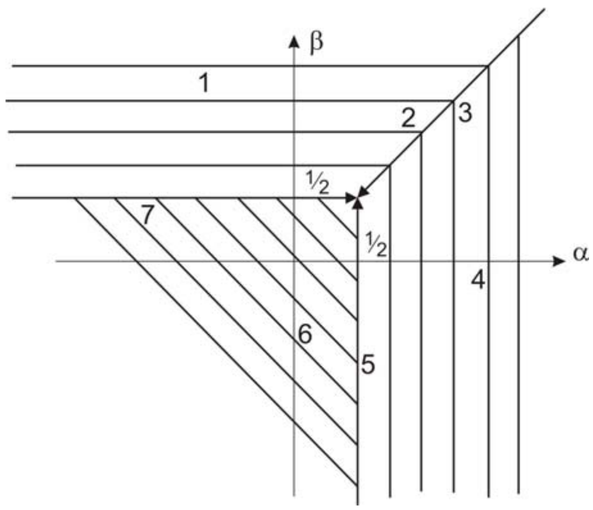

Fig. 1. Newton polygon for equation (1).

Note that exponent $1-2\alpha$ is the smallest one under the choice of α and β values in area 4, exponent $1-2\beta$ – in area 1, and exponent 0 – in area 6 (areas 1, 4, 6 are open sets, i.e. their boundary lines are not included).

In areas 1, 4, 6 the limiting equations follow

$$w_{1yy}=0;\ \ w_{1xx}=0;\ \ w=0\,.$$

The equations include only one term. The values of α and β associated with the equations with two terms are located on boundary lines (without point $\alpha=\beta=1/2$)

$$w_{1xx}+w_{1yy}=0;\ \ \varepsilon\ \ w_{1xx}+w=0;\ \ \varepsilon\ \ w_{1yy}+w=0\,.$$

Finally, for $\alpha=\beta=1/2$ in equation (1) all terms remain. Since there are no blank spaces on the $\alpha\ \beta$ plane, there are no other limiting systems.

Note that the occurrence of more than two parameters of the asymptotic integration results in an increase of the problem complexity. In references [26, 27] the effective algorithms to solve the occurring problems are introduced, whereas in reference [28] a generalization is proposed.

Simultaneous splitting of governing equations should be matched with an appropriate splitting of the associated boundary conditions. This complicated problem is discussed and illustrated in references [2, 15, 16, 23, 24].

4. TIMOSHENKO TYPE PLATE EQUATIONS

Below, we consider an illustrative example showing the efficiency of AM [27]. According to Timoshenko, the effect of a shear deflection occurring for plate vibration is comparable to that of rotary inertia. However, the wave front sets are predicted incorrectly due to the Timoshenko theory. On the other hand, AM shows that a transverse compression effect is comparable with effects of rotary inertia and shear deflection. Correct asymptotic theory gives a proper location of wave fronts as well as averaged characteristics of stress-strain state in the vicinity of the mentioned fronts within two dimensional equations of the form

$$\varphi_{1xx}+a_s^2\varphi_{1yy}+e\varphi_{2xy}+cW_x-8a_s^2(w_x+\varphi_1)-\varphi_{1tt}=0 \tag{2}$$

$$\varphi_{2yy}+a_s^2\varphi_{2xx}+e\varphi_{1xy}+cW_y-8a_s^2(w_y+\varphi_2)-\varphi_{2tt}=0 \tag{3}$$

$$a_s^2(w_{xx}+w_{yy})+e(\varphi_{1x}+\varphi_{2y})+W-w_{tt}=0 \tag{4}$$

$$W+c(\varphi_{1x}+\varphi_{2y})+0.5w_{tt}+\frac{1}{16}W_{tt}=0 \tag{5}$$

$$M_1=\varphi_{1x}+c\varphi_{2y}+cW;\qquad M_2=\varphi_{2y}+c\varphi_{1x}+cW; \tag{6}$$

$$N=W+c\varphi_{1x}+\varphi_{2y} \tag{7}$$

$$\begin{gathered}H=a_s^2(\varphi_{2x}+\varphi_{1y});\qquad Q_1=w_x+\varphi_1=\beta_1;\\ Q_2=w_y+\varphi_2=\beta_2\end{gathered} \tag{8}$$

$$e=\frac{1}{2(1-\nu)},\; c=\frac{\nu}{1-\nu},\quad a_s^2=\frac{1-2\nu}{(1-\nu)^2}.$$

Compare equations (2)-(8) with the equations of Timoshenko plate at the shear coefficient $k^2 = 2/3$:

$$\varphi_{1xx}+\frac{1-\nu}{2}\varphi_{1yy}+\frac{1+\nu}{2}\varphi_{2xy}-4(1-\nu)(w_x+\varphi_1)-\frac{1}{a_1^2}\varphi_{1tt}=0 \tag{9}$$

$$\varphi_{2yy}+\frac{1-\nu}{2}\varphi_{2xx}+\frac{1+\nu}{2}\varphi_{1xy}-4(1-\nu)(w_y+\varphi_2)-\frac{1}{a_1^2}\varphi_{2tt}=0 \tag{10}$$

$$w_{xx}+w_{yy}+\varphi_{1x}+\varphi_{2y}-\frac{3}{2a_s^2}w_{tt}=0 \tag{11}$$

$$\begin{gathered}M_1=a_1^2(\varphi_{1x}+\nu\varphi_{2y});\\ M_2=a_1^2(\varphi_{2y}+\nu\varphi_{1y});\qquad H=a_s^2(\varphi_{2x}+\varphi_{1y})\end{gathered} \tag{12}$$

$$\begin{gathered}Q_1=w_x+\varphi_1=\beta_1;\\ Q_2=w_y+\varphi_2=\beta_2;\qquad a_1^2=\frac{1-2\nu}{(1-\nu)^2}\end{gathered} \tag{13}$$

Note that equations (2)-(8), contrary to (9)-(13), govern the velocities of all wave displacements even in comparison with the 3D case.

Equations (9)-(13) can be obtained from equations (2)-(8), but using the asymptotically inconsistent procedure: the last term of equation (5) as well as the function N in equation (7) should be neglected, and expression $W=-c(\varphi_{1x}+\varphi_{2y})$ should be introduced to equations (3)-(5).

5. Intermediate asymptotics

The idea of an intermediate asymptotics is related to the construction of certain particular self-similar solutions of non-linear problems, being asymptotics of a wide class of other solutions. Dynamic edge effect method (DEEM) proposed by Bolotin [17] gives a good example of the intermediate asymptotics. The main idea of this approach is separation of a continuous elastic system into two parts. In one of them – an interior zone – solutions may be expressed by trigonometric functions with unknown constants. One can use exponential functions in the dynamic edge effect's zone. Then, a matching procedure permits to obtain unknown constants, and a complete solution of dynamic problem may be written in a relatively simple form. This approximate solution is very accurate for high frequency vibrations, but even at low frequency vibrations the error is not excessive. DEEM is naturally generalized for nonlinear case [18, 23].

We should also emphasize that DEEM works properly in connection with variation methods [18, 23]. This is due to the fact that the DEEM gives good approximation of displacements. While finding the eigenvalues the following general rule can be formulated: if you are looking for the eigenforms then asymptotics should be used; if you need an eigenvalue then the found asymptotic function can be used further by one of the variation methods.

6. HOMOGENIZATION APPROACH

The replacement of a non-homogeneous shell by a homogeneous one with some reduced characteristics belongs to one of the most popular approximations in TPS. We can mention structurally orthotropic theories of ribbed, corrugated, perforated PS, PS with many attached masses, etc. For many years a design of similar simplifications depended fully on engineers' intuition, and the obtained quantities differed from each other depending on the theory used. Mathematical difficulties were caused by the occurrence of PDE with rapidly changing coefficients. Beginning from the 70s of the 20th century, the theory of homogenization of PDE has been developed. It should be emphasized that a similar mathematical approach was proposed earlier in the theory of ribbed shells [12].

Using the homogenization approach one must deal with two successively solvable problems: a local problem for periodically repeated element (cell) as well as the global homogeneous problem with some reduced parameters. As a rule, the fundamental difficulty is associated with solution of the cell problem. Although this problem can be solved numerically, an analytical solution is always highly required. The application of AM to solve local problems allowed us to get homogenized solutions for various periodically non-homogeneous TPS with correctly reduced coefficients. The areas of applicability of approximated theories are estimated, and full stress-strain states can be calculated. It is important that one can also predict boundary layers occurring in the vicinity of boundaries. The lack of this knowledge does not allow the shell stress-strain to be fully estimated. Using the homogenization procedure one should take into account the relations between parameters of investigated structures. As an example, a deformation of a reinforced membrane governed by the following equation is analyzed

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = q_1 \ (x, y), \quad \mathrm{k}l \leq \mathrm{y} \leq (\mathrm{k}+1)l . \tag{14}$$

The conditions of conjugations of the neighboring parts of membrane are

$$\lim_{\mathrm{y}\to \mathrm{k}l+0} \mathrm{u} \equiv \mathrm{u}^+ \equiv \lim_{\mathrm{y}\to \mathrm{k}l-0} \mathrm{u} \equiv \mathrm{u}^-, \tag{15}$$

$$\left(\frac{\partial u}{\partial \mathrm{y}}\right)^+ - \left(\frac{\partial u}{\partial y}\right)^- = d_1 \frac{\partial^2 u}{\partial x^2}, \tag{16}$$

$$u = 0 \qquad \text{for } x = 0, \mathrm{H}. \tag{17}$$

Let a characteristic period of external load be $L >> \mathrm{l}$, $\varepsilon = l/L << 1$. We introduce the variables $\eta = y/l$, $y_1 = y/L$ and the following series

$$u = u_0(x,y) + \varepsilon^{\alpha_1}[u_{10}(x,y) + u_1(x,y,\eta)] + \varepsilon^{\alpha_2}[u_{20}(x,y) + u_2(x,y,\eta)] + \ldots, 0 < \alpha_1 < \alpha_2 < \ldots . \tag{18}$$

Substituting (18) into (14)-(17), the following recurrent system is obtained

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \varepsilon^{\alpha-2}\frac{\partial^2 u_1}{\partial \eta^2} + 2\varepsilon^{\alpha-1}\frac{\partial^2 u_0}{\partial y\,\partial\eta} + \varepsilon^{\alpha-2}\frac{\partial^2 u_2}{\partial \eta^2} + 2\varepsilon^{\alpha-1}\frac{\partial^2 u_0}{\partial y\,\partial\eta} + O(\varepsilon^{\alpha}) = q(x,y); \tag{19}$$

$$[u_0 + \varepsilon^{\alpha}(u_{10} + u_1) + \ldots]^+ = [u_0 + \varepsilon^{\alpha}(u_{10} + u_1) + \ldots]^-; \tag{20}$$

$$\varepsilon^{\alpha-1}\left[\left(\frac{\partial u_1}{\partial \eta}\right)^{+}-\left(\frac{\partial u_1}{\partial \eta}\right)^{+}\right] + \mathrm{O}(\varepsilon^{\alpha}) = d\left[\frac{\partial^2 u_0}{\partial x^2} + \mathrm{O}(\varepsilon^{\alpha})\right] \text{where: } q = L^2 q_1;\ d = d_1/L. \quad (21)$$

The character of asymptotics depends essentially on the order of magnitude of d in comparison to ε. Let us introduce the estimation: $d \sim \varepsilon^{\beta}$.

Depending on the value of β one obtains the following limiting systems

$$0 < \alpha < 2, \quad \frac{\partial^2 u_1}{\partial \eta^2} = 0\,; \quad (22)$$

$$\alpha = 2, \quad \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1}{\partial \eta^2} = q(x,y)\,; \quad (23)$$

$$\alpha > 2, \quad \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} = q(x,y)\,, \quad (24)$$

and the following conjugation conditions

$$\beta < \alpha - 1, \quad \frac{\partial^2 u_0}{\partial x^2} = 0\,; \quad (25)$$

$$\beta = \alpha - 1, \quad \left[\left(\frac{\partial u_1}{\partial \eta}\right)^{+}-\left(\frac{\partial u_1}{\partial \eta}\right)^{+}\right] = d\ \varepsilon^{1-\alpha_1}\ \frac{\partial^2 u_0}{\partial x^2}\,; \quad (26)$$

$$\beta > \alpha - 1, \quad \left(\frac{\partial u_1}{\partial \eta}\right)^{+} = \left(\frac{\partial u_1}{\partial \eta}\right)^{+}. \quad (27)$$

The plane of parameters β > 0, α > 0 is divided into nine parts (Fig. 2).

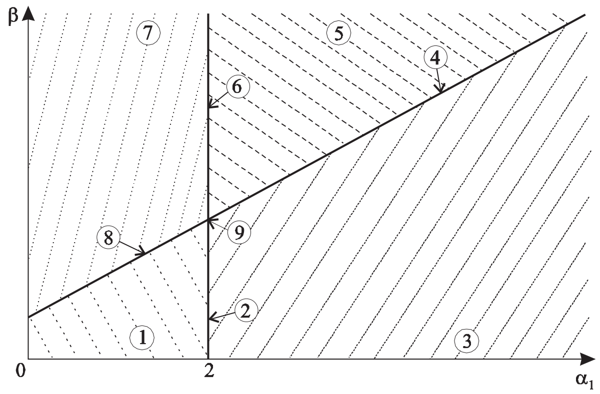

Fig. 2. The plane of parameters β > 0, α > 0.

In zones 1–3 one has

$$\frac{\partial^2 u_1}{\partial \eta^2} = q(x,y)\,.$$

In zones 4–6 the equation has the form of (24). For zones 7 and 9 the limiting systems are incorrect. A particular role plays the case of $\alpha = 2$, $\beta = 1$ (zone 8). The corresponding limiting equation is (23) and

$$u^+ = u^-\,; \qquad \left[\left(\frac{\partial u_1}{\partial \eta}\right)^+ - \left(\frac{\partial u_1}{\partial \eta}\right)^+\right] = d\ \ \varepsilon^{-1}\ \frac{\partial^2 u_0}{\partial x^2}\,.$$

Homogenized BVP is

$$\nabla^2 u_0 + \frac{d_1}{l}\frac{\partial^2 u_0}{\partial\ x^2} = q(x,y)\,.$$

Equation (23) yields

$$u_1 = \frac{d_1(x,y)}{l}\ \frac{\partial^2 u_0}{\partial x^2}\ \ \eta\ \ (\eta\ -\ l)\,.$$

Boundary conditions (17) are not satisfied. In order to construct a boundary layer u_b the new "fast" variable $\xi = x/l$ is introduced and the following series is applied

$u_n = \varepsilon^{\gamma_1} u_{11}(x, y, \xi, \eta) + \varepsilon^{\gamma_2} u_{22}(x, y, \xi, \eta) + \dots, \quad 0 < \gamma_1 < \gamma_2 < \dots .$

BVP for u_{11} are as follows

$$\frac{\partial^2 u_{11}}{\partial \xi^2} + \frac{\partial^2 u_{11}}{\partial \eta^2} = 0\,; \qquad u_{11}|_{\,\eta = kl} = 0,\ k = 0, \pm 1, \pm 2, \dots .$$

Then, the further construction of a boundary layer may be easily carried out.

7. Distributional approach

Terms like $a(x / \varepsilon)$ often occur in the asymptotic problems. In order to introduce parameter ε explicitly, it is useful to apply the distributional approach [29]. As a model problem we consider a transition from 2D ribs to 1D ones. The governing PDE for bending deformation of an infinite plate on the elastic foundation, reinforced by periodic systems of ribs in two main directions, is

$$D\Delta\Delta w + Cw + D_1 F_1(x) w_{xxxx} + D_2 F_2(y) w_{yyyy} = q(x, y),$$

$$F_1(x) = \sum_{n=-\infty}^{\infty} [H(x + nl_1) - H(x + ml_1 + a)];$$

$$F_2(y) = \sum_{n=-\infty}^{\infty} [H(y + nl_2) - H(y + ml_2 + a)].$$

We suppose that the ribs are thin and choose their width a as the parameter of asymptotic integration. To introduce parameters a, b explicitly, we analyze function $f(x) = H(x) - (x + a)$. Applying two-sided Laplace transformation, and using development into a Maclaurin series, one obtains

$$\overline{f}(p) = a + \sum_{n=1}^{\infty}(-1)^n a^{n+1} p^n /(n+1)!,$$

where: $\overline{f}(p)$ is the Laplace transform of f(x) ($x \to p$).

The inverse Laplace transform leads to the following series

$$f(x) = a\delta(x) + \sum_{n=1}^{\infty}(-1)^n a^{n+1}\delta^{(n)}(x)/(n+1)!.$$

Functions $F_1(x)$ and $F_2(y)$ can be expanded in a similar way. As a result, we obtain the following equation:

$$D\Delta\Delta w + Cw + D_1\Phi_1(x)w_{xxxx} + D_2\Phi_2(y)w_{yyyy} = q(x,y),$$

$$\Phi_1(x) = \Phi_{10}(x) + \Phi_{11}(x) + \Phi_{12}(x) = \sum_{n=-\infty}^{\infty} a\delta(x+nl_1) - 0.5\sum_{n=-\infty}^{\infty} a^2\delta'(x+nl_1) + \sum_{n=-\infty}^{\infty}\sum_{k=2}^{\infty}(-1)^k a^{k+1}\delta^{(n)}(x+nl_1),$$

$$\Phi_2(y) = \Phi_{20}(y) + \Phi_{21}(y) + \Phi_{22}(y) = \sum_{n=-\infty}^{\infty} a\delta(y+nl_2) - 0.5\sum_{n=-\infty}^{\infty} a^2\delta'(y+nl_2) + \sum_{n=-\infty}^{\infty}\sum_{k=2}^{\infty}(-1)^k a^{k+1}\delta^{(n)}(y+nl_2).$$

A solution to the equation can be sought in the form of the following series:

$$w = w_0 + \sum_{i=0}^{\infty} a^i w_i \ .$$

In the zero order approximation one gets a plate with 1D ribs

$$D\Delta\Delta w_0 + Cw + D_1\Phi_{10}(x)w_{0xxxx} + D_2\Phi_{20}(y)w_{0yyyy} = q(x,y).$$

An influence of the ribs width appears in the next approximations.

8. Real and asymptotic errors

Accuracy of AM is usually estimated by an asymptotic error, i.e. owing to the order of estimation of the last omitted term. However, a TPS analyst is more interested in a real rather than asymptotic error. It may happen that in order to increase real accuracy of the obtained solution one has to omit the asymptotic character of constructed solutions. Some methods for decreasing the real error of constructed approximate solutions follow.

1. Asymptotically accurate semi-membrane theory of cylindrical shells can be developed using the condition of absence of shear and ring deformations in the shell middle surface. However, the condition of absence of shear deformations is realized with less accuracy than for ring deformation. Although, a theory, constructed on the basis of only ring deformation absence is asymptotically inaccurate, practically it gives more accurate results.
2. Owing to the asymptotic splitting of BVP, a fundamental error is introduced by simplification of the boundary conditions. In many cases one may analytically obtain a general solution of edge effect equations. Using this solution, it is possible to exclude exactly the terms of edge effect solution from boundary conditions and avoid splitting of the boundary conditions.

3. The method of composite equations is devoted to constructing uniformly suitable solutions on the basis of various limiting cases. A fundamental idea of the method can be formulated in the following way. First, the components of the governing equations are detected, which, when neglected, lead to non-homogeneity in a zero order approximation. Second, the mentioned components are defined in a relatively simple way (they must include essential properties in the non-homogeneous states). Matching of the limiting relations leads to uniformly suitable equations. In the TPS a composite equation of the stress-strain fundamental state has been obtained, unifying the semi-membrane and membrane theories and a plane plate deformation. A simple edge effect and bending of the plate are included in a composite equation of the edge effect type. The obtained composite equations are of the fourth order because of a longitudinal variable and are applicable in the whole range of different loadings [18, 23].
4. In order to improve the accuracy with a help of AM of the zero order approximation, one may apply either variation methods or Newton-Kantorovitch approaches.

9. BEYOND THE SERIES LOCALITY

The principal shortcoming of AMs is the local nature of solutions based on them. Problems of elimination of the expansion locality, evaluation of the convergence domain and construction of uniformly suitable solutions are very urgent.

There are many approaches to these problems [18, 22, 23, 30-32]: the method of analytic continuation, Borel summation procedure, Euler transformation, Domb-Sykes diagram. As a rule, they need a significant number of the expansion components.

Not diminishing the merits of the mentioned techniques, let us, however, note that in practice only a few of the first components of the expansion of perturbations are usually known. Lately, the situation has indeed changed a little due to computer application. It may happen that a number of terms of asymptotic series can be increased without any serious problems. For instance, computing improvement terms with respect to an eigenvalue are usually successfully defined by eigenvalues and eigenfunctions. The knowledge of the n-th eigenfunction allows us to define $2n+1$ eigenvalues [33]. However, until now there are usually 3-5 components available in a perturbation series, and exactly from this segment of the series we have to extract all available information. To this end the method of Padé approximants (PA) may be very useful.

Let

$$F(\varepsilon) = \sum_{i=0}^{\infty} C_i \varepsilon^i ,$$

$$F_{mn}(\varepsilon) = \sum_{i=0}^{m} a_i \varepsilon^i / \sum_{i=0}^{m} b_i \varepsilon^i ,$$

where the coefficients a_i, b_i are determined from the following condition: the first $(m+n)$ components of the expansion of the rational function $F_{mn}(\varepsilon)$ in a Maclaurin series coincide with the first $(m+n+1)$ components of the series $F(\varepsilon)$. PA performs meromorphic continuation of the function given in the form of the power series. If the PA sequence converges to a given function, then the roots of its denominators tend to singular points.

A wide application of the PA is observed due to its suitable properties. Among others, we should mention the effect of error autocorrection: even very significant errors in the coefficients of PA do not affect the accuracy of the approximation. This is because the errors in the numerator and the denominator of PA compensate each other, because the errors in the coefficients of the PA are not distributed in an arbitrary way, but from the coefficients of a new approximant to the approximated function.

PA can be used for a heuristic evaluation of the domain of applicability of a perturbation series. The ε values, up to which the difference between calculations according to the truncated perturbation series and its diagonal PA do not exceed a given value (e.g. 5%), can be considered as a limiting value for applicability of the perturbation series.

10. HOMOTOPY PERTURBATION TECHNIQUE

Dorodnitzyn (1969) proposed a method of introducing the parameter ε into the input BVP in such a way that for $\varepsilon = 0$ the simplified problem is obtained, whereas for $\varepsilon = 1$ the input problem is governed. Then, the perturbation method can be used. Now this approach is known as a homotopy perturbation technique [31, 34]. The main problem for it is the divergence of perturbation series for $\varepsilon = 1$. In order to overcome the occurring difficulties, the PA can be used effectively [18, 22, 23].

Let us focus on the application of the homotopy perturbation method [18, 22, 23] when solving mixed BVP – the vibration of a rectangular plate ($-0.5k \le x \le 0.5k$, $-0.5 \le y \le 0.5$) , simply supported at $x = \pm 0.5k$, and having mixed boundary conditions of the "clamped - simple supported" type, symmetrical to the y axis or the sides $y = \pm 0.5k$ (Fig. 3). The governing equation is

$$\nabla^4 w - \lambda w = 0.$$

The boundary conditions after introducing a homotopy parameter have the form

$$w = 0, \qquad w_{xx} = 0 \quad for \quad x = \pm 0.5k;$$
$$w = 0, \quad w_{yy} = \bar{H}(x)\varepsilon(w_{yy} \pm w_y) \quad for \quad y = \pm 0.5,$$

where: $\bar{H}(x) = -H(x-\mu) + H(-x-\mu)$.

Substituting w and λ in the form of ε-series:

$w = w_0 + \varepsilon w_1 + \ldots, \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots,$

and after applying the usual perturbation procedure, one has

$$\lambda_0 = \pi^4\psi^2, \lambda_1 = 4\pi^2 n^2 \gamma_{mm},$$

$$\lambda_2 = 4\pi^2 n^2 \gamma_{mm}\left\{1 - \frac{\gamma_{mm}}{\pi^2\psi}\left[\frac{\pi\alpha}{2} cth^{(-1)^m}\left(\frac{\pi\alpha}{2}\right) + \frac{n^2}{\psi} - \frac{3}{2}\right]\right\} - \frac{2n^2}{\psi}\sum_{\substack{\{i=1,3,5,\ldots\} \\ \{i=2,4,6,\ldots\}}}^{\infty} \gamma_{im}\left[\alpha_i cth^{(-1)^i}\left(\frac{\alpha_i}{2}\right)\right.$$
$$\left. + \begin{Bmatrix} -\phi_i cth^{(-1)^i}(\phi_i/2) \\ \beta_i cth^{(-1)^i}(\beta_i/2) \end{Bmatrix}\right], \qquad \begin{Bmatrix} i^2 > m^2 + n^2 k \\ i^2 < m^2 + n^2 k \end{Bmatrix}.$$

where:

$$\psi = n^2 + \frac{m^2}{k^2}, \quad \alpha = \sqrt{2\frac{m^2}{k^2} + n^2}, \quad \alpha_i = \sqrt{\frac{i^2 + m^2}{k^2} + n^2}, \quad \beta_i = \pi\sqrt{\frac{m^2 - i^2}{k^2} + n^2},$$

$$\gamma_{im} = \begin{cases} 2(0.5 - \mu) + \dfrac{(-1)^m}{\pi m}\sin(2\pi\mu m), & for \quad i = m \\ \dfrac{4}{\pi}\dfrac{1}{(m^2 - i^2)}\left[\begin{Bmatrix} i \\ m \end{Bmatrix}\sin(\pi\mu i)\cos(\pi\mu m) - \right. & \\ \left. -\begin{Bmatrix} m \\ i \end{Bmatrix}\sin(\pi\mu m)\cos(\pi\mu i)\right], & for \quad i \neq m, \end{cases}$$

$\sum'$ is the sum without the component $i = m$.

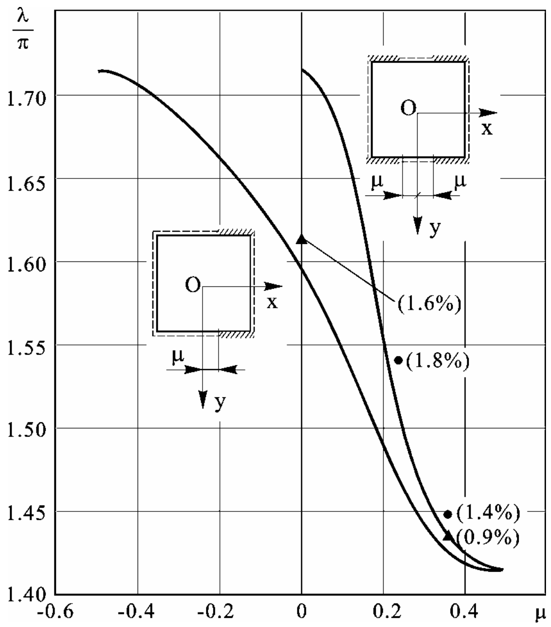

Fig. 3. Relationship between the vibration frequency λ and the clamped segment length.

Truncated perturbation series for $\mu = 0$ (both sides $y = \pm\, 0.5$ are completely clamped) for the square plate gives $(1.4783\pi)^4$. PA is

$$\lambda_p(\varepsilon) = \frac{a_0 + a_1\varepsilon}{1 + b_1\varepsilon}, \quad a_0 = \lambda_0, \quad a_1 = \lambda_1 + b_1\lambda_0, \quad b_1 = -\lambda_2/\lambda_1,$$

and for $\varepsilon = 1$ one obtains $\lambda_p = (1.7081\pi)^4$, while numerical value $\lambda = (1.7050\pi)^4$ Figure 3 presents the relation of λ versus μ and some experimental data (dots and triangles).

11. THEORIES OF HIGHER ORDER APPROXIMATIONS

In order to increase approximation accuracy the terms of higher order may remain in the input equations, but such an approach can increase the order of the approximate PDE. This problem can be overcome by PA. Let us consider vibrations of a stretched beam

$$w_{\tau\tau} - w_{\xi\xi} + \varepsilon w_{\xi\xi\xi\xi} = 0\,; \tag{28}$$

$$w = w_{\xi\xi} = 0 \quad \textit{for} \quad \xi = 0,1. \tag{29}$$

A string type model is obtained from (28) for $\varepsilon = 0$

$$w_{\tau\tau} - w_{\xi\xi} = 0\,; \tag{30}$$

$$w = 0 \quad \textit{for} \quad \xi = 0,1\ . \tag{31}$$

Using PA for differential operator, one obtains

$$(1 + \varepsilon \frac{\partial^2}{\partial \xi^2}) w_{\tau\tau} - w_{\xi\xi} = 0\,. \tag{32}$$

The associated boundary conditions have the form (31). Observe that if the model (30), (31) approximates eigenvalues of the initial problem up to the order of ε, then model (32),(31) includes the second order approximation of ε^2 preserving the equation order with respect to the spatial coordinates.

12. MATCHING OF LIMITING ASYMPTOTICS

It happens often that solutions related to two limiting values of a certain parameter can be easily constructed. In this case one can define a solution valid for all parameter values with a help of two-point PA [18, 22, 23]. Let

$$F(\varepsilon) = \begin{cases} \sum\limits_{i=0}^{\infty} a_i \varepsilon^i & \text{when} \quad \varepsilon \to 0, \\ \sum\limits_{i=0}^{\infty} b_i \varepsilon^{-i} & \text{when} \quad \varepsilon \to A. \end{cases} \tag{33}$$

The TPPA is represented by the following rational function

$$F(\varepsilon) = \frac{\sum\limits_{k=0}^{m} a_k \varepsilon^k}{\sum\limits_{k=0}^{n} b_k \varepsilon^k}\,,$$

where: k+1 (k = 0,1, ...,n+m+1) are the coefficients of a Taylor expansion if $\varepsilon \to 0$, and m+n+1-k are the coefficients of a Laurent series if $\varepsilon \to A$ coincide with the corresponding coefficients of the series (33).

As an example we consider the problem of nonlinear deformation of a sphere. The solution

$$Q = 0.42\varepsilon + 0.3\varepsilon^3 + 0(\varepsilon^5)\,, \tag{34}$$

$$\varepsilon = 2(w/h)\sqrt{3\sqrt{1-\nu^2}}\,,\quad Q = \frac{0.5qR^2 3\sqrt{1-\nu^2}}{Eh^2}\,,$$

has been obtained by means of the AM for a closed sphere subjected to the uniform external pressure q [19]. Here w is the amplitude of postbuckling axisymmetric equilibrium form.

In the region of small displacements the Koiter approach is valid

$$Q = 1 + 0(\varepsilon^{-4})\,. \tag{35}$$

By matching expansions (34) and (35) with the TPPA, one obtains the following solution [19]

$$Q = \frac{A}{A+2.19}\,,\; A = \varepsilon^4 + 0.082\varepsilon^3 + 0.386\varepsilon^2 + 0.92\varepsilon\,. \tag{36}$$

Curves 1 and 2 in Fig. 4 correspond to solutions (34), (36), respectively. Accuracy of solution (36) is confirmed by comparison with the precise numerical solution.

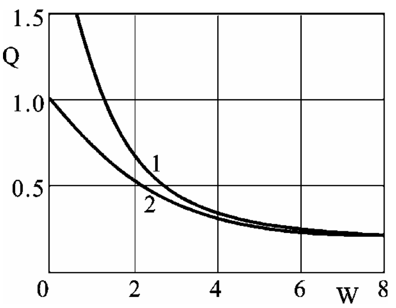

Fig. 4. Matching of quasi-linear and essentially nonlinear asymptotics.

In Figure 5 the results of comparison of experimental data for postbuckling equilibrium states of shallow elliptic parabolic-shaped shells under external pressure [35] with the solution based on TPPA [19] are shown, where: $\bar{w} = w/h$;

$$\bar{\mathrm{P}} = \frac{0.5qR_1R_2 3\sqrt{1-\nu^2}}{Eh^2}.$$

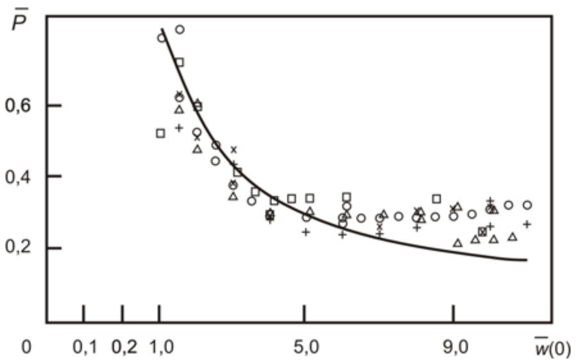

Fig. 5. Comparison of TPPA solution (solid line) with experimental results.

The second example is associated with homogenization of a rectangular plate with circular perforations. Analytical solutions for small and large holes were obtained [12] by using the AM perturbation of the domain and boundary form. For the coefficients A and B of the homogenized equation

$$A(W_{xxxx} + W_{yyyy}) + 2BW_{xxyy} = q(x, y)$$

one has the following expressions (for ν = 0.3):

$$A = \frac{1-\lambda}{1-0.5785\lambda}, \qquad B = \frac{1-\lambda}{1-0.6701\lambda},$$

where $\lambda = b/a$, b is the diameter of the hole, a is the length of the square cell side.

Figure 6 shows the numerical results for A and B.

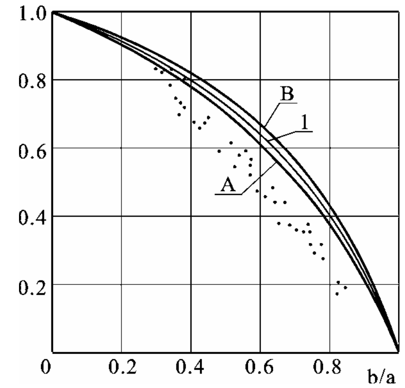

Fig. 6. Homogenized coefficients of perforated plates

The values of coefficients are compared with theoretical results, obtained by means of the two-period elliptic functions (curve 1 in Fig. 5) and experimental results (points in Fig. 5).

Evidently, the TPPA is not a panacea. As a rule, one of the limit expansions ($\varepsilon \to 0$ or $\varepsilon \to A$) contains logarithmic or exponential terms. In this case one can use the method of asymptotically equivalent functions (AEF). Suppose that we have a perturbation approach in powers of ε for $\varepsilon \to 0$ and asymptotic expansions $F(\varepsilon)$ containing, logarithm for $\varepsilon \to A$. By definition AEF is the Ra ratio with unknown coefficients a_i, b_i, containing both powers of ε and function F(ε). The coefficients a, b are chosen in such a way, that the expansion of Ra in powers of ε matches the corresponding perturbation expansion and the asymptotic behavior of Ra for $\varepsilon \to \infty$ coincides with $F(\varepsilon)$.

13. Merits and demerits of AM

Advantages of AMs follow:

1. Essentially simplified solutions, which in many cases can be obtained in an analytical way.
2. AMs are easily matched with other approaches, i.e. numerical, variation ones, etc. Owing to the introduced simplification of the input BVP and separation of the associated peculiarities of the considered problem one may effectively apply numerical approaches. AMs allow us to exhibit the structure of solution and the type of approximating functions in Bubnov-Galerkin, Ritz, Trefftz and Kantorovich approaches. Owing to the construction of zero order solution, it can be applied as a starting solution for other iteration processes like the Newton-Kantorovich method.
3. AMs are strictly associated with a physical aspect of the analyzed problem allowing for it easier understanding.
4. AMs allow us to explain mathematical and physical bases of approximated engineering methods, increasing their accuracy and reliability of obtained results.
5. AMs give a possibility of a unified approach to various different problems exhibiting their common aspects and internal unity.

However, the main drawback of AM is generated by insufficiently accurate results of low approximations, while the construction of successive approximations is not always easy. Also the accuracy of the estimation of AMs and intervals of their applicability in many cases causes serious difficulties.

Conclusions

A choice of discussed and illustrated AM is mainly motivated by authors' subjective choice. Many important methods like WKB (Bauer et al., 1994) or matched asymptotic expansion [36] are omitted. Other interesting problems about the junction of plates and shells with 1D and 3D bodies or the junction of two shells [37, 38], the solutions of shell problems in singular domains [36, 39] have not been considered either. Finally, let us mention the AM application in the localization problems [16, 40].

It is expected that further development of AM is associated with combined numerical-analytical approaches and include them in standard codes. This is important, because an accurate numerical computation of shells with arbitrarily small thickness is impossible in practice. Standard finite element codes usually fail to give accurate results for h/R~0.01 or 0.001.

Nowadays in order to compute thin-walled structures, the standard finite element codes are used. It seems that an asymptotic information is rather rarely applied. On the other hand, AM belongs to fundamental ones during the construction of mathematical models of physical processes [14, 34]. "Design of computational or experimental schemes without the guidance of asymptotic information is wasteful at best, dangerous at worst, because of possible failure to identify crucial (stiff) features of the process and their localization in coordinate and parameter space. Moreover, all experience suggests that asymptotic solutions are useful numerically far beyond their nominal range of validity, and can often be used directly, at least at a preliminary product design stage, for example, saving the need for accurate computation until the final design stage where many variables have been restricted to narrow ranges" [34].

Finally, there is an ocean of books and papers devoted to the considered problems, and only some of them are cited. On the other hand, a reader may find additional references in [16, 22, 24, 32, 36, 40-44].

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ASIMPTOTISKI PRISTUPI TEORIJI LJUSKI: DUGA ISTORIJA I NOVI TRENDOVI

I.V. Andrianov, J. Awrejcewicz

Rad pruža kompleksan prikaz asimptotskih metoda u primeni na Teoriju ljuski i ploča (TPS). Asimptotiske metode rešavanja problema koji se odnose na teoriju ploča i ljuski razvili su mnogi autori. Glavne osobine ovog rada su: (i) posvećen je osnovnim principima asimptotskih metoda, i (ii) i bavi se i tradicionalnim i, manje rasprostranjenim, novim pristupima. Autori posebnu pažnju posvećuju primerima i diskusiji rezultata i ne opterećuju ideje formalizmom, komentarima i tehničkim detaljima

Ključne reči: *ploče, ljuske, asimptotske metode, homogenizacija*