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ON INVESTIGATION OF DYNAMICAL SYSTEMS WITH CONSTRAINTS

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Viktor G. Veretennikov, Pavel S. Krasilnikov, Valery A. Sinictsyn

Department of Theoretical Mechanics, Moscow Aviation Institute
(State Technical University), Moscow, 125871, Russia,
E-mail: kras@k804.mainet.msk.su

Abstract. *The dynamical systems with constraints (differential-algebraic systems) are investigated by methods of analytical mechanics. So, the well-known mechanical principle of release from constraints is extended to DAE systems. The definition of ideal constraint is formulated for these systems. It is shown that a necessary and sufficient condition for constraint forces to have a representation by Lagrange's multipliers is that the constraint be ideal. It is obtained the condition of ideality for the constraint dependents on the method of physical realization of restriction. Therefore one and the same constraint may be as ideal so nonideal. The examples are considered. The principal equation for dynamical systems with ideal constraints is obtained. For Chetaev's systems, the principal equation is also derived. As an example, the problem of the construction of periodic solutions for average Lorenz's dynamical system is considered.*

1. INTRODUCTION

It is well known that a lot of dynamical systems don't have any restrictions to phase coordinates. However some of the systems have the form of differential-algebraic equations (DAE) containing restrictions to variables. For example, it is often complex to specify the generalized variables for multibody mechanical systems. To investigate these systems, the redundant variables are applied. Therefore there exist the holonomic constraints. In other case, to investigate some problems of the simulation, the system of high dimension is partitioned to subsystems of smaller dimensions provided that there exist an algebraic connections between subsystems. By methods of mechanics, we calculate the constraint forces, and obtain the equations which don't contain these forces.

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2. The principle of release from constraints. The ideal constraints.

Suppose the equations of motion of some object can be written in the form

$$\dot{x} = f(x)\ , \tag{1}$$

where $x \in R^n$ is the vector of phase variables, f: $R^n \to R^n$ is a smooth map.

If there are not any restrictions to x, then we say that (1) is *a free system***.** Suppose there exists a restriction to x

$$\varphi(x,t) = 0, \quad \varphi = (\varphi_1, \ldots, \varphi_s), \;\; s < n\ , \tag{2}$$

where φ is a smooth map; then dynamical system is not free, but it is DAE.

It is obvious that velocity $\dot{x}$ of free system (1) is different from velocity of dynamical system under restriction (2). Therefore, this restriction leads to the appearance of additional term $r(x) = (r_1 \ldots r_n)^{\mathrm{T}}$ in the right - hand side of system (1). The term $r(x)$ is called a *constraint force* of restriction (2).

Hence it follows that dynamical system under restriction (2) is of type:

$$\dot{x} = f(x) + r(x)\ . \tag{3}$$

Equality (2) is the first (or special) integral of system (3). These reasoning is the abstract description for the well-known mechanical *principle of release from constraints.* We say that (3) is not *a free system.*

Using the methods of classical mechanics, let us define more exactly the concept for constraint forces. It is known that, in mechanics, constraints are considered as objective actions. They are contained in some equations of motion if and only if these equations have the highest order for derivatives of variables. For example, Lagrange's equations of second kind and Hamilton's system in generalized impulses include constraint forces.

Without the loss of generality it can be assumed that, according to some properties of system, the constraint forces change the derivatives of some selected variables $x_1, x_2, \ldots, x_m$ $(m < n)$ coinciding with m first variables. The corresponding vector of constraint forces is of the following form: $r = (r^*(x), 0, \ldots 0)$, $r^*(x) = (r_1, r_2, \ldots, r_m)$. So, if some variables $\boldsymbol{x}_s$ are to be derivatives of another variables x_k, then constraint forces should be input into the equations in x_s. For example, if dynamical system contains the mechanical subsystem with generalized variables x_k, k=1,...,m, then the constraint forces belong to equations in generalized velocities $\dot{x}_k$.

In other case, by physical principal, the constraint forces belong to some subsystem only. For example, the equations of motion for rigid body with fixed point can be written in the form

$$\frac{d\,K_0}{d\,t} = M_0, \qquad \frac{\tilde{d}\,k_1}{d\,t} + [\omega, k_1] = 0\ .$$

Here $K_0 = \omega(L)$ is the moment of momentum for body with respect to fixed point O, ω is the angular velocity, (L) is the tensor of inertia, M_0 is the principal moment of external forces, k_1 is the unit vector of fixed vertical axis Oz, $\tilde{d}(\,)/dt$ is the operator of local differentiation. It is clear the constraint forces belong to the first set of equations.

To find equations including constraint forces, in the abstract case when variables x_i don't have clear mechanical sense, we can consider the problem of the transformation of system (1) to Lagrange's or Hamilton's types. The necessary and sufficient conditions for existence of such transformation is that equations (1) should be self-conjugated [1]. We can also consider the transformation of system (1) to Chetaev's system [2]:

$$\frac{d}{dt}\left(\frac{\partial T(q,\dot{q},p)}{\partial \dot{q}_j}\right)-\frac{\partial T(q,\dot{q},p)}{\partial q_j}=Q(q,\dot{q},p)\ , \tag{4}$$

$$\dot{p}=F(q,\dot{q},p)\ . \tag{5}$$

These equations have Lagrange's (or Hamilton's) subsystem (4) with varying parameters p. By [2], it follows that the constraint forces pertain to equations in generalized velocities (or generalized impulses), moreover constraint forces are contained in equations in parameters (5). Note that the problem of existence for this transformation has not been solved yet.

The definition of constraint forces needs correction because of the problem of practical realisation of constraints. Indeed, the different realisations of the same constraint can produce different kinds of constraint forces. Using methods of analytical mechanics, in order to avoid the ambiguity under the calculation of $r(x)$, we introduce the concept of ideal constraints.

Suppose $s < m$. Let $x^* = (x_1,\ldots,x_m)$ be the vector of selected variables, $\delta x^* = (\delta x_1,\ldots,\delta x_m)$ vector of virtual movements satisfying the condition

$$\sum_{i=1}^{m}\frac{\partial\varphi_j}{\partial x_i}\delta x_i = 0 \quad (j=1,\ldots,s) \tag{6}$$

Restrictions (2) are called *ideal* if, for arbitrary virtual movements δx^* from current position, virtual work δA of constraint force $r^*(x)$ is equal to zero:

$$\delta A(r^*) \equiv r^*(x)\circ\delta x^* = 0\ . \tag{7}$$

Note that the variation of vector x is the same as Jourdain's variation well-known in classical mechanics. Indeed, under the condition of Jourdain's variation, if the equations in some variables don't have any constraint forces, we consider these variables as constants. Therefore equations (6) in virtual movements do not contain the variations of variables x_j, $j = m+1,\ldots,n$.

Let $\lambda = (\lambda_1,\ldots,\lambda_s)$ be the vector of Lagrange's multipliers, $\varphi_{x^*} = \left\|\partial\varphi_i/\partial x_j\right\|_{i,j=1}^{s,m}$ a nondegenerate (s x m) Jacobian matrix, $(\varphi_k)_{x^*} = (\partial\varphi_k/\partial x_1,\cdots,\partial\varphi_k/\partial x_m)$.

Lemma. For restrictions (2) to be ideal it is necessary and sufficient to have

$$r^* = \lambda\,\varphi_{x^*} = \sum_{k=1}^{s}\lambda_k(\varphi_k)_{x^*} \tag{8}$$

Proof. Suppose $r^*(x)$ satisfies (8). By (6), (7), it follows that

$$\delta A(r^*) = \sum_{j=1}^{s} \lambda_j \sum_{k=1}^{m} (\partial \varphi_j / \partial x_k) \delta x_k = 0 ,$$

that is restrictions (2) are ideal.

Suppose restrictions (2) satisfy (7). Multiplying the equation with number j from (6) by λ_j, summing all results and subtracting this sum from (7), we get

$$\sum_{i=1}^{m} (r_i^* - \sum_{j=1}^{s} \frac{\partial \varphi_j}{\partial x_i} \lambda_j) \delta x_i = 0$$

If we use the Lagrange's indeterminate multipliers and the condition of nondegeneracy for matrix φ_{x^*}, we obtain

$$r_i^* = \sum_{j=1}^{s} \frac{\partial \mathrm{j}_j}{\partial x_i} \lambda_j .$$

Hence it follows (8). This completes the proof.

Thus the constraint force $r^*(x)$ equals the sum of s vectors. Each vector $\lambda_k (\varphi_k)_{x^*}$ is directed along the normal to manifold $\varphi_k = 0$ if we suppose that variables $x_{m+1},\dots,x_n$ are equal to constant values corresponding to flowing time t provided that $x_1,\dots,x_m$ are varying coordinates.

Suppose $s = m$; then using (6), we obtain $\delta x^* = 0$, therefore the definition of ideal restriction (2) by (7) is not valid. By continuity, restriction (2) is called *ideal* if condition (8) holds.

If $s > m$, virtual movement δx^* is undefined. It means the condition $m \geq s$ should take place only.

To find the Lagrange's multipliers, let us differentiate (2) with respect to t, and replace $\dot{x}^*$ by $(f^*(x) + r^*)$. We obtain the following equation in λ:

$$\lambda \varphi_{x^*} (\varphi_{x^*})^{\mathrm{T}} = -\varphi_t - f \varphi_x^{\mathrm{T}} \tag{9}$$

Let us remark that there isn't enough analytical formula (2) to represent the constraint force $r^*(x)$ in form (8). For example, restriction

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l^2 \tag{10}$$

can be both ideal and nonideal.

Indeed, suppose two massive points $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ are connected by rigid rod. Let us consider the rod by itself. We have the following classical theorems describing the motion of the rod

$$ma_c = R_1 + R_2 + mg, \quad J_c \dot{\omega} = M_c(R_1) + M_c(R_2)$$

where point C is a barycenter of the rod, m is the mass of the rod, J_c is its moment of inertia with respect to C, R_k are the external constraint forces, a_c is the acceleration of C, ω is the angular velocity of the rod, $M_c(R_k)$ are the moments of R_k.

It is well known, virtual work δA of R_k, mg is of the type

$$\delta A = F \circ \delta r_c + M_c \circ \delta \varphi \,,$$
$$F = R_1 + R_2 + mg \,,\; M_c = M_c(R_1) + M_c(R_2)\,.$$

Here δr_c is the vector of virtual movement for barycenter C, $\delta\varphi$ is the vector of virtual rotation about the instantaneous axis passing through point C. By condition $m = 0$, $J_c = 0$, it follows $F = 0$, $M_c = 0$ therefore $\delta A = 0$, i.e., restriction (10) is ideal.

Suppose two points $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ are moving along line L under force F applying to point M_1. Another point M_2 is under servoforce P conserving the distance between points dynamically. We have $\delta A = (F + P)\delta x \neq 0$ because of $(F + P) \neq 0$ in generally. Thus constraint (10) is not ideal in this case.

3. The principal equation of dynamical systems

Let us consider the ideal restriction (2) only. With the help of (7), if we find constraint force $r^*(x)$ from equations (3) and substitute it for $r^*(x)$ in (7), we have

$$\sum_{k=1}^{m} (\dot{x}_k - f_k(x))\,\delta x_k = 0\,. \tag{11}$$

We shall say that equation (11) is the *principal equation of dynamical system* (2), (3).

The equation (11) must be considered together with equations (2), (6), and the remaining "kinematic equations"

$$\dot{x}_i - f_i(x) = 0, \quad i = m+1, \ldots, n\,. \tag{12}$$

The principal equation does not contain the ideal constraint forces. It produces so many independent equations of motion, how many independent virtual movements are supposed by equations (6). Thus equation (11) is necessary for the elimination of ideal constraint forces from equations (3).

Theorem. Let restrictions (2) be ideal, $\varphi_{x^*} = \left\| \partial \varphi_i / \partial x_j \right\|_{i,j=1}^{s,m}$ a nondegenerate matrix; then equations (11), (12) are equivalent to equations (3).

Proof. Arguing as above, we see that (11), (12) follows from (3).

Suppose equations (11), (12) hold. Multiplying equation with number j from (6) by λ_j, summing all results, and subtracting this sum from (11), we then have

$$\sum_{k=1}^{m} \Big(\dot{x}_k - f_k(x) - \sum_{j=1}^{s} \frac{\partial \varphi_j}{\partial x_k} \lambda_j\Big)\delta x_k = 0\,.$$

By the condition of nondegeneracy for matrix φ_{x^*}, we find the Lagrange's multipliers λ_j such that

$$\dot{x}_k - f_k(x) - \sum_{j=1}^{s} \frac{\partial \varphi_j}{\partial x_k} \lambda_j = 0$$

By (8), it follows that equations (3) are the corollary of (11), (12). This completes the proof.

The mechanical system with varying parameters is a special case of dynamical systems in question. Suppose some subsystem is described by Newton's differential equations of second order. We shall say that dynamical system is of *Chetaev's type* if it has the form [2]

$$m_k a_k = F_k(t, r_1, v_1, \ldots, r_n, v_n, p_1, \ldots, p_l) \quad k = 1, \ldots, n,$$

$$\mu_i \dot{p}_i = S_i(t, r_1, v_1, \ldots, r_n, v_n, p_1, \ldots, p_l) \quad i = 1, \ldots, l,$$

where m_k, a_k are the mass and acceleration of mass point with subscript k, F_k are the active forces, μ_i are the positive coefficients, p_i are varying parameters, S_i are the additional forces. Note that, by Chetaev, S_i are called *compulsions*.

Suppose there exists the following constraint:

$$\varphi(t, r_1, v_1, \ldots, r_n, v_n, p_1, \ldots, p_l) = 0, \tag{13}$$

where v_k are the velocities of mass points; then the equations of motion are as follows:

$$\begin{aligned} m_k a_k &= F_k + R_k, \quad k = 1, \ldots, n \\ \mu_i \dot{p}_i &= S_i + P_i, \quad i = 1, \ldots, l \end{aligned} \tag{14}$$

Here R_k, P_i are the constraint forces. Varying (13) by Jourdain, we get

$$\sum_k \frac{\partial \varphi}{\partial v_k} \delta v_k + \sum_i \frac{\partial \varphi}{\partial p_i} \delta p_i = 0 \tag{15}$$

For constraint (13) to be ideal it is necessary and sufficient to have

$$\sum_k R_k \, \delta v_k + \sum_i P_i \, \delta p_i = 0 \tag{16}$$

For ideal constraint force, we get

$$R_k = \lambda \frac{\partial \varphi}{\partial v_k}, \quad P_i = \lambda \frac{\partial \varphi}{\partial p_i}.$$

If we exclude constraint forces R_k, P_i from system (14) and substitute the result into (16), we obtain the following principal equation

$$\sum_k (m_k a_k - F_k) \delta v_k + \sum_i (\mu_i \dot{p}_i - S_i) \delta p_i = 0. \tag{17}$$

From equation (17), there follow so many equations, how many independent virtual variations δv_k, δp_i are supposed by (15). This equation does not contain the constraint forces. By theorem, the equation (17) with kinematic relations $\dot{r}_i = v_i$ is equivalent to (14).

When parameters p_j vanish, we have classical mechanical systems. Using $\delta r_k = \delta v_k \delta t$, hence it follows d'Alembert - Lagrange equation

$$\sum_k (m_k a_k - F_k) \delta r_k = 0.$$

Note that, for dynamical systems of Chetaev's type, the substitution δr_k for δv_k in (15) is not valid [3], because of the nonhomogeneous of equation (15) with respect to δv_k.

4. ON SOME PERIODIC MOTIONS IN LORENZ'S SYSTEMS.

The dynamical system of the third order such as that

$$\dot{x}_1 = -\sigma(x_1 - x_2)\,, \quad \dot{x}_2 = a x_1 - x_2 - x_1 x_3\,,$$

$$\dot{x}_3 = -b x_3 + x_1 x_2\,, \tag{18}$$

where σ, a, b are positive coefficients is called Lorenz's system. It is well known, equations (18) were obtained as a mathematical model of convective motion in heated up layer of liquid [4]. There are too many researchers investigating these system [5].

Lorenz's system (18) can be transformed to Chetaev's system. Indeed, let us take the following change of variables [5]:

$$\xi = \mu x_1 / \sqrt{2}(\sigma+1)\,, \quad \eta = \frac{\mu^2}{(\sigma+1)^2}(\sigma x_3 - \frac{1}{2}x_1^2)\,,$$

$$\tau = \sqrt{\sigma(a-1)}\,t\,, \quad \mu = (\sigma+1)/\sqrt{\sigma(a-1)}\,;$$

then we obtain

$$\xi'' + \mu\xi' + (\eta - 1)\xi + \xi^3 = 0\,,$$

$$\eta' = -\mu f(\xi,\eta) \tag{19}$$

$$f(\xi,\eta) = [b\eta - \left(2\sigma - \beta\right)\xi^2]/(\sigma+1)$$

Here the prime denotes the derivative with respect to τ.

Let us consider the problem of the construction of some periodic solutions for (19). As the generating system, let us consider the dynamical system (19) with the ideal restrictions

$$\varphi_1 \equiv \xi'^2 + (\eta-1)\xi^2 + \frac{\xi^4}{2} = 2E = const,$$
$$\varphi_2 \equiv \eta = const, \tag{20}$$

where E takes the form of full mechanical energy. Thus, we have the case $s = m = 2$ and $x^* = (\xi', \eta)$.

Equations (6) in virtual movements are of the following form:

$$2\xi'\delta\xi' + \xi^2\delta\eta = 0, \quad \delta\eta = 0.$$

Calculating the constraint forces by means of (8), we get

$$r_1 = \lambda_1 \frac{\partial\varphi_1}{\partial\xi'} + \lambda_2 \frac{\partial\varphi_2}{\partial\xi'} = 2\lambda_1\xi',$$

$$r_2 = \lambda_1 \frac{\partial\varphi_1}{\partial\eta} + \lambda_2 \frac{\partial\varphi_2}{\partial\eta} = \lambda_1\xi^2 + \lambda_2\,. \tag{21}$$

If we substitute (21) into right-hand sides of equations (19), we get

$$\xi'' + \mu\xi' + (\eta - 1)\xi + \xi^3 = 2\lambda_1\xi'$$
$$\varsigma' = -\mu f(\xi,\eta) + \lambda_1\xi^2 + \lambda_2 \tag{22}$$

It follows from (9) that λ_1, λ_2 take the form

$$\lambda_1 = \frac{\mu}{2}, \quad \lambda_2 = \mu[f(\eta,\xi) - \frac{\xi^2}{2}] \tag{23}$$

It is obvious, if $\mu = 0$, equations (19) are the same as equations (22), (23) including parameter $\mu \neq 0$. Let (22), (23) be a generating system, E, η slow variables, and ξ fast variable.

Let change (ξ,ξ',η) to (ξ,E,η). Free equations (19) in new variables E, η, ξ can be written in the form

$$E' = -\mu F(\xi,E,\eta) - \frac{1}{2}\mu f(\xi,\eta)\xi^2$$
$$\eta' = -\mu f(\xi,\eta), \quad \xi' = \pm\sqrt{F(\xi,E,\eta)} \tag{24}$$

where

$$F(\xi,E,\eta) = 2E - (\eta - 1)\xi^2 - \frac{1}{2}\xi^4 .$$

We shall see that equations (22), (23) in new variables are as follows

$$E' = -\mu F(\xi,E,\eta) - \frac{1}{2}\mu f(\xi,\eta)\xi^2 + r_E$$
$$\eta' = -\mu f(\xi,\eta) + r_{\eta}, \quad \xi' = \pm\sqrt{F(\xi,E,\eta)} \tag{25}$$

Here r_E, r_η are the constraint forces taking the form

$$r_E = 2\lambda_1\xi'^2 + \xi^2(\lambda_1\xi^2 + \lambda_2)/2, \quad r_{\eta} = \lambda_1\xi^2 + \lambda_2$$

By (20), it follows that

$$\tau - \tau_0 = \pm\int_0^{\xi} \frac{d\xi}{\sqrt{F(\xi)}}, \tag{26}$$

For periodic motion, fast variable ξ takes maximum D, if

$$2E = (\eta - 1)D^2 + D^4/2 .$$

If we write integral (26) on terms of variable φ, where $\xi = D^2(1 - \varphi^2)$, we get the normal Legendre's form of integral (26)

$$\omega\tau = \int_0^{\varphi} \frac{d\varphi}{\sqrt{(1-\varphi^2)(1-k^2\varphi^2)}},$$

$$\omega = \sqrt{\eta - 1 + D^2}, \; k^2 = \frac{d^2}{2\omega^2}, \; \eta + D^2 > 1 \tag{27}$$

Converting this integral, we obtain the solution

$$E = const,\ \eta = const,\ \xi = D\ cn(\omega\tau, k) \tag{28}$$

of generating system (25). Here, $cn(\omega\tau,k)$ is a Jacobi's function with k modulus, $T = 4K(k)/\omega$ is a period of this function.

Let us average the constraint forces r_E, r_η over a period T, and vanish the results:

$$\frac{1}{T}\int_0^T r_\delta(\xi)\big|_{\xi=D\ cn(\omega\tau,\ k)} d\tau = 0,\ \ \delta = E, \eta\ . \tag{29}$$

It follows from (24), (25), (29) that average equations (24) are the same as average equations (25). If we consider conditions (29), (20), (27) as the equations in η, D, they define the set of initial data such as that (28) to be a periodic solution of average equations (24).

Equalities (29) can be represented in the form

$$\begin{gathered}\int_0^D \sqrt{F(\xi)}\, d\xi - \frac{1}{2}\int_0^D \xi^2 f(\eta,\xi)\frac{d\xi}{\sqrt{F(\xi)}} = 0,\\ \int_0^D f(\eta,\xi)\frac{d\xi}{\sqrt{F(\xi)}} = 0\end{gathered} \tag{30}$$

Finally, we note that integrals (30) were obtained in [5] also by another way.

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O ISTRAŽIVANJU DIMAMIČKIH SISTEMA POD DEJSTVOM VEZA

Viktor G. Veretennikov, Pavel S. Krasilnikov, Valery A. Sinictsyn

Dinamički sistemi pod dejstvom veza (diferencijalni algebarski sistemi DAES) istražuju se metodama analitičke mehanike. Tako, dobro poznati princip mehanike - oslobadjanja od dejstva veza proširen je na DAE sistema. Definicija idealnih veza je formulisana za ove sisteme. Pokazuje se da su potrebni i dovoljni uslovi da da bi sile otpora veza bile predstavljene Lagranžeovim množiocima veza da su veze idealne. Došlo se do uslova idealnosti veza koji zavisi od metode fizičke realizacije veza. Stoga jedna i ista veza može da bude idealna i neidelna. Razmatreni su primeri. Princip jednakosti za dinamičke sisteme sa idealnim vezama se dobija. Za Četajeve sisteme, princip jednakosti je takodje izveden. Kao primer je proučen problem konstrukcije periodičnih rešenja za usrednjen Lorencov dinamički sistem.