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Invited Paper

THE LAGRANGIAN GEOMETRICAL MODEL AND THE ASSOCIATED DYNAMICAL SYSTEM OF A NONHOLONOMIC MECHANICAL SYSTEM

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Abstract. *One considers a Lagrangian nonholonomic mechanical system* $\Sigma = (M, L(x,y), Q_\sigma(x,dx), F_i(x,\dot{x}))$ *, with* $y = \dot{x}$*, whose evolution equations are (1.3). One associates to system* Σ *a canonical semispray* S^* *on the phase space TM, which has the integral curves given by the evolution equations of* Σ*. The Lagrange geometry of system* Σ *is the geometry of semispray* S^* *which is a dynamical system, on TM, intrinsically associated to* Σ*. The obtained results are new and original.*

Key words: *Lagrange spaces, Semispray, Dynamical System, Lagrangian mechanical systems*

1. INTRODUCTION

In this paper we propose to study a new Lagrangian model for nonholonomic mechanical systems $\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y))$ in the most general case when $L(x,y)$ is a regular Lagrangian, $F_i(x,y)$ are the external forces defined on the phases space TM, and $Q_\sigma = 0$, ($\sigma = m + 1,\ldots,n = \dim M$) are the kinematic nonholonomic constrains defined on the configuration space M. Σ will be named *Lagrangian nonholonomic mechanical system*.

The classical nonholonomic mechanical systems are the particular cases of Σ, obtained for $L(x,y) = 2T(x,y) = g_{ij}(x)y^i y^j$, $y^i = \dot{x}^i$ the kinetic energy of a Riemannian metric $ds^2 = g_{ij}(x)dx^i dx^j$ and the external forces F_i depend on the material points $(x^i) \in M$.

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In the case when $L(x,y) = F^2(x,y)$, where $F(x,y)$ is the fundamental function of a Finsler space, Σ is a new class of nonholonomic mechanical systems – which have not been studied yet. It is called Finslerian nonholonomic mechanical system.

Some particular properties of Σ were investigated by us in the paper [10].

So, we study tangent bundle of configurations space M, Lagrangian scleronomic nonholonomic mechanical systems Σ, canonical semispray and canonical nonlinear connection of system Σ, N^*- canonical metrical connection, h- and v- electromagnetic tensors, gravitational field, examples: classical nonholonomic mechanical systems and Finslerian mechanical systems. Consequently, the obtained results are new and original.

Recalling that the geometrization of holonomic mechanical systems was done by Levi-Civita, [1], [3], [14], [17], while, in 1926, Gh. Vrănceanu, by introducing the notion of Riemannian nonholonomic space, realized a first geometric model for the nonholonomic, scleronomic mechanical system. He considers as evolution the equations of system, the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = \sum_{\sigma=m+1}^{n} \lambda_\sigma a_{\sigma i}(x) + F_i(x) \tag{1.1}$$

where $Q_\sigma(x,dx) = a_{\sigma i}(x)dx^i = 0$ $(\sigma = m+1,...,n)$ give the kinematic constraints.

At the International Congress of Mathematicians from Bologna, 1928, Elie Cartan showed that the equations (1.1) are not sufficient. He gives the geometrization of these systems by fixing the normal distribution to the distribution $Q_\sigma = 0$.

But, one proves that these new elements are not enough.

Mendel Haimovici [3] completed E. Cartan, supposing that the system of Pfaff equations $Q_\sigma = 0$ has the first derivate system identically null.

In his Ph.D. Thesis (1956), [5], R. Miron solved the general case in which for the system $Q_\sigma = 0$, a number $r < m$ of derivate subsystems exist. E. Cartan considered this case unrecheable, because of the calculating difficulties [2].

The holonomic mechanical Finsler systems was studied recently by R. Miron and C. Frigioiu [9]. They are given by $\Sigma = (M,F(x,y),F_i(x,y))$, where $F^n = (M,F(x,y))$ is a Finsler space and $F_i(x,y)$, with $y = \dot{x}$, are the external forces depending on material point (x^i) and his velocity $(\dot{x}^i)$.

The general case was investigated by the second author in [13], [14].

We notice that the previous geometrical study can be extended to nonholonomic case.

So that, in this paper, we study Lagrangian nonholonomic scleronomic mechanical systems

$$\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y)),\ (y = \dot{x}), \tag{1.2}$$

where $L^n = (M,L(x,y))$ is a Lagrange space, [7], $F_i(x,\dot{x})$ are external forces and the Pfaff equations $Q_\sigma(x,dx) = a_{\sigma i}(x)dx^i = 0$, $(\sigma = m+1,...,n)$ are the kinematic constrains of the system.

The equations of evolution of system Σ are Lagrange equations (1.1)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^i}\right) - \dfrac{\partial L}{\partial x^i} = \displaystyle\sum_{\sigma=m+1}^{n} \lambda^\sigma(x) a_{\sigma i}(x) + F_i(x,y), & y^i = \dfrac{dx^i}{dt}, \\ Q_\sigma(x,dx) = a_{\sigma i}(x)dx^i = 0, & \end{cases} \tag{1.3}$$

where $\lambda^\sigma(x)$ are Lagrange multipliers.

Finslerian nonholonomic mechanical systems are obtained for $L(x,y) = F^2(x,y)$, where $F(x,y)$ is the fundamental function of a Finsler space. Finslerian or Lagrangian nonholonomic mechanical systems will be named *Lagrangian nonholonomic mechanical systems*.

Evidently, we study only scleronom systems associating a canonical semispray S^* to them, whose integral curves are given by the evolution equations (1.3) of Σ.

The vector field S^* is a dynamical system on the phase space TM. Then, the problems concerning the equilibrium of Σ and the stability of its solutions can be approached on the phase space TM, in a classical manner, [7], [14].

The geometry of the pair $(S^*, Q_\sigma = 0)$ represents the Lagrange geometry of Σ on the phase space TM. We highlighted the fundamental geometric objects of this geometry as N^*-metrical canonical connection, its structure equations, etc.

2. THE TANGENT BUNDLE OF THE CONFIGURATIONS SPACE

Let M be a real differentiable manifold of dimension n. A point $x \in M$ has local coordinates $(x^i),(i = 1,...,n)$. The tangent bundle (TM,π,M) is the differentiable manifold TM of dimension $2n$, real and orientable. The points $u = (x,y) \in TM$ have the local coordinates (x^i,y^i) and $\pi(u) = x$. M is called the configuration space and TM the phase space.

A change of coordinates on TM is given by

$$\begin{aligned} \tilde{x}^i &= \tilde{x}(x^1,...,x^n), \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j . \end{aligned} \tag{2.1}$$

The natural base of the tangent space $T_u(TM)$ is $\left.\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)\right|_u, (i = 1,...,n)$.

The vertical distribution $V : u \in TM \to V(u) \subset T_u(TM)$ is locally generated by the vector fields $\left.\left(\frac{\partial}{\partial y^1},...,\frac{\partial}{\partial y^n}\right)\right|_u$.

There exists a vector field $C = y^i \frac{\partial}{\partial y^i}$ on TM, called the Liouville vector field, belonging to vertical distribution V. C does not have singular points on the differentiable manifold $\widetilde{TM} = TM \setminus \{0\}$. Also, on TM there exists an integrable tangent structure J, given by:

$$J = \frac{\partial}{\partial y^i} \otimes dx^i , \tag{2.2}$$

J has the property

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right) = 0; J^2 = 0 . \tag{2.2'}$$

A vector field $S \in \chi(TM)$ with the property

$$JS = C \tag{2.3}$$

is called a *semispray*. If M is a paracompact manifold then on TM there exists semisprays.

Locally, a semispray S is expressed in the form:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}. \tag{2.4}$$

The function $G^i(x,y)$ are called the coefficients of S. A change of coordinates (2.1) change G^i as follows:

$$2\tilde{G}^i = \frac{\partial \tilde{x}^i}{\partial x^j} 2G^j - \frac{\partial \tilde{y}^i}{\partial x^j} y^j. \tag{2.5}$$

The integral curves of the vector field S are given by

$$\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} + 2G^i(x,y) = 0. \tag{2.6}$$

A nonlinear connection on TM is a regular distribution $N : u \in TM \to N(u) \in T_u(TM)$ supplementary to the vertical distribution V, that is:

$$T_u(TM) = N(u) \oplus V(u), \quad \forall u \in TM. \tag{2.7}$$

A local base adapted to (2.7) is $\left.\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)\right|_u$ $(i = 1,...,n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}. \tag{2.8}$$

Its dual base is $\left.(dx^i, \delta y^i)\right|_u$ where

$$\delta y^i = dy^i + N_j^i dx^j \tag{2.8'}$$

The functions (N_j^i) are called the coefficients of the nonlinear connection N. It is known that the integrability of N distribution is characterized by the vanishing of the d-tensor field

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j} \tag{2.9}$$

Autoparallel curves of the nonlinear connection N are given by the equations

$$\frac{dx^i}{dt} = y^i, \frac{\delta y^i}{dt} = 0 \tag{2.10}$$

If S is a semispray with the coefficients G^i, then the functions $N_j^i = \frac{\partial G^i}{\partial y^j}$ determine the coefficients of a nonlinear connection.

3. LAGRANGIAN NONHOLONOMIC, SCLERONOMIC MECHANICAL SYSTEMS

We will apply the theory from the preceding paragraph and the variational problem in case of the scleronomic Lagrangian systems Σ.

The evolution equations of these systems will be given in the classical form (1.3), but more generally, because the exterior forces $F_i\,(x,y)$ are considered as the components of a d-covectors fields on the phases space TM, [14].

Let Σ (1.2) be a Lagrangian nonholonomic, scleronomic mechanical systems with the evolution equations (1.3), (1.3'). Σ determines, in a canonic way, a semispray S^* on the phase space TM, which we will study in this section.

We denote with $g_{ij}\,(x,y)$ the fundamental tensor of the Lagrange space $L^n = (M, L(x,y))$ and with $g^{ij}(x,y)$ its contravariant. As it is known [7], $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$, rank$\|g_{ij}\| = n$ on $TM\setminus\{0\}$ and g_{ij} has constant signature.

External forces $F_i\,(x,y)$ determine a d-covariant vector field and

$$F_{ij} = \frac{\partial F_j}{\partial y^i} - \frac{\partial F_i}{\partial y^j} \tag{3.1}$$

is an antisymmetric d-tensor field, named *elicoidal tensor of system* Σ.

The functions that determine the constrains of the system

$$Q_\sigma(x,y) = a_{\sigma i}(x) y^i, \ \ (\sigma = m+1,...,n)$$

are scalars with respect to the changes of the coordinates on TM.

So, $a_{\sigma i}(x)$ are $n-m$ covector fields on M and

$$\sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x,y) \tag{3.2}$$

is also a scalar function on TM. The functions $\lambda^\sigma(x)$ are the Lagrange multipliers.

Let L^* be the Lagrangian

$$L^*(x,y) = L(x,y) + \sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x,y)\,. \tag{3.3}$$

We have:

1° $g^*_{ij}(x,y) = g_{ij}(x,y)$

2° $L^*\left(x, \dfrac{dx}{dt}\right) = L\left(x, \dfrac{dx}{dt}\right)$ on the distribution $Q_\sigma(x,dx) = 0$.

3° The integral of action of the Lagrangian L^* is

$$\int_0^1 L^*(x,\dot{x})dt = \int_0^1 [L(x,\dot{x}) + \lambda^\sigma(x) Q_\sigma(x,\dot{x})]dt\,. \tag{3.4}$$

4° The Euler – Lagrange equations of L^*:

$$\frac{\partial L^*}{\partial x^i} - \frac{d}{dt}\frac{\partial L^*}{\partial y^i} = 0, \ \ y^i = \frac{dx^i}{dt}$$

are given by

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial y^i}+[\frac{\partial(\lambda^\sigma Q_\sigma)}{\partial x^i}-\frac{d}{dt}\frac{\partial}{\partial y^i}(\lambda^\sigma Q_\sigma)]=0, y^i=\frac{dx^i}{dt}$$

or by

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial y^i}+\left(\frac{\partial \lambda^\sigma}{\partial x^i}Q_\sigma+\lambda^\sigma\frac{\partial Q_\sigma}{\partial x^i}-\frac{d}{dt}\left(\lambda^\sigma\frac{\partial Q_\sigma}{\partial y^i}\right)\right)=0 .$$

But $\frac{\partial Q_\sigma}{\partial y^i}=a_{\sigma i}(x)$.

We obtain

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)+\left[\frac{\partial \lambda^\sigma}{\partial x^i}Q_\sigma+\lambda^\sigma\frac{\partial Q_\sigma}{\partial x^i}-\frac{d}{dt}\left(\lambda^\sigma a_{\sigma i}\right)\right]=0$$

or, equivalently:

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)+\left[\frac{\partial \lambda^\sigma}{\partial x^i}a_{\sigma j}-\frac{\partial \lambda^\sigma}{\partial x^j}a_{\sigma i}+\lambda^\sigma\left(\frac{\partial a_{\sigma j}}{\partial x^i}-\frac{\partial a_{\sigma i}}{\partial x^j}\right)\right]y^j=0,\ \ y^i=\frac{dx^i}{dt} . \tag{3.5}$$

The Lagrangians L^*, (3.3) and L are equivalent if corresponding solutions of Lagrange equations:

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial y^i}=0, \frac{\partial L^*}{\partial x^i}-\frac{d}{dt}\frac{\partial L^*}{\partial y^i}=0$$

are equal and $g_{ij}=g^*_{ij}$.

Theorem 3.1. *The Lagrangians $L(x,y)$ and $L^*(x,y)=L(x,y)+\lambda^\sigma a_{\sigma i}y^i$ are equivalent if and only if one of the following equations are satisfied.*

$$\frac{\partial \lambda^\sigma}{\partial x^i}a_{\sigma j}-\frac{\partial \lambda^\sigma}{\partial x^j}a_{\sigma i}+\lambda^\sigma\left(\frac{\partial a_{\sigma j}}{\partial x^i}-\frac{\partial a_{\sigma i}}{\partial x^j}\right)=0 \tag{3.6}$$

$$d(\lambda^\sigma Q_\sigma(x,dx))=0 \tag{3.7}$$

Proof. **The first method**: The Lagrangians $L(x,y)$ and $L^*(x,y)$ have the property

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}=\frac{1}{2}\frac{\partial^2 L^*}{\partial y^i \partial y^j}=g^*_{ij} \tag{3.8}$$

From (3.5) it results that the Euler - -Lagrange equations for L and L^* hold if and only if

$$\left\{\frac{\partial \lambda^\sigma}{\partial x^i}a_{\sigma j}-\frac{\partial \lambda^\sigma}{\partial x^j}a_{\sigma i}+\lambda^\sigma\left(\frac{\partial a_{\sigma j}}{\partial x^i}-\frac{\partial a_{\sigma i}}{\partial x^j}\right)\right\}y^j=0 .$$

Deriving with respect to y^i, we obtain equations (3.6).

The second method: The Lagrangians $L(x,y)$ and $L^*(x,y)=L(x,y)+\lambda^\sigma Q_\sigma(x,y)$ are equivalent if and only if 1-form $\lambda^\sigma Q_\sigma(x,dx)$ is closed (theorem of Carathéodory) [14].

So

$$d(\lambda^\sigma Q_\sigma(x,dx)) = d(\lambda^\sigma a_{\sigma j} dx^j) = 0 \tag{3.9}$$

We have, exterior differentiating:

$$d(\lambda^\sigma Q_\sigma(x,dx)) = \frac{1}{2}\left[\frac{\partial\lambda^\sigma}{\partial x^i}a_{\sigma j} - \frac{\partial\lambda^\sigma}{\partial x^j}a_{\sigma i} + \lambda^\sigma\left(\frac{\partial a_{\sigma i}}{\partial x^i} - \frac{\partial a_{\sigma i}}{\partial x^j}\right)\right]dx^i \wedge dx^j$$

which are the equations (3.6).

Using Theorem 3.1, we can introduce:

Postulate. *The equations of evolution of the Lagrangian nonholonomic,scleronomic mechanical system* $\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y))$ *are:*

$$\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i(x,y) + \lambda^\sigma(x)a_{\sigma i}(x),\ \ y^i = \frac{dx^i}{dt}, \tag{3.10}$$

where the multipliers $\lambda^\sigma(x)$ satisfy the equation

$$d[\lambda^\sigma(x)Q_\sigma(x,dx)] = 0 \tag{3.11}$$

4. THE CANONICAL SEMISPRAY AND NONLINEAR CONNECTION OF SYSTEM Σ

The canonical semispray S^* of the system

$$\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y)) \tag{4.1}$$

is a vector field S^* on the phase space

$$S^* = y^i\frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i}, \tag{4.2}$$

whose integral curves are given by the evolution equations of the system Σ, (3.10), (3.11).

But the equations (3.10) can be written in the following form

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = \frac{1}{2}\left[F^i(x,y) + \lambda^\sigma(x)a^i_\sigma(x,y)\right], y^i = \frac{dx^i}{dt}, \tag{4.3}$$

where

$$2G^i(x,y) = \frac{1}{2}g^{ij}(x,y)[\frac{\partial^2 L}{\partial y^j\partial x^k}y^k - \frac{\partial L}{\partial x^j}] \tag{4.4}$$

$$F^i(x,y) = g^{ij}(x,y)F_j(x,y) \tag{4.5}$$

$$a^i_\sigma(x,y) = g^{ij}(x,y)a_{\sigma j}(x)\ . \tag{4.6}$$

Let the system of functions be

$$2G^{*i}(x,y) = 2G^i(x,y) - \frac{1}{2}[F^i(x,y) + \lambda^\sigma(x)a^i_\sigma(x,y)]\ . \tag{4.7}$$

Then

Theorem 4.1. *The system of functions $G^{*i}(x,y)$ from (4.7) are the coefficients of a semispray determined only by the Lagrangian nonholonomic mechanical system* Σ.

Proof. Since

$$(*)\ 2\widetilde{G}^i = \frac{\partial\tilde{x}^i}{\partial x^j}2G^i - \frac{\partial\tilde{y}^i}{\partial x^j}y^j$$

and $F^i, \lambda^\sigma a^i_\sigma$ are d-contravariants vectorial fields, it results that $G^{*i}(x,y)$, from (4.7) is transformed by (2.1) in the same manner as $G^i(x,y)$.

The semispray S^* (4.2) with the coefficients G^{*i} from (4.7) and the multipliers λ^σ verifying (3.11) is called the *canonical semispray* of the system Σ.

The vector fields S^* defines a dynamic system on TM. It has the following important property:

Theorem 4.2. *The integral curves of the canonic semispray S^* are given by the evolution equations of the system* Σ (4.3), (3.11).

Proof. The integral curves of S^* are given by

$$\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} = -2G^{*i},$$

λ^σ verifying (3.11) and G^{*i} from (4.7).

These equations are equivalent to (4.3), (3.11), q.e.d..

We define the Lagrange geometry of system Σ as being the Lagrange geometry on the phase space TM of the canonic semispray S^*.

The nonlinear connection N^* of S^* is called *canonical nonlinear connection* of system Σ.

Theorem 4.3. *The canonical nonlinear connection N^* of system Σ, has the coefficients:*

$$N^{*i}_j = \frac{\partial G^{*i}}{\partial y^j} = N^i_j - \frac{1}{4}\left(\frac{\partial F^i}{\partial y^j} + \lambda^\sigma \frac{\partial a^i_\sigma}{\partial y^j}\right) \tag{4.8}$$

Theorem 4.4. *The Berwald connection of system Σ has the coefficients*

$$B^{*i}{}_{jk} = B^i{}_{jk} - \frac{1}{4}\left(\frac{\partial^2 F^i}{\partial y^j \partial y^k} - \lambda^\sigma \frac{\partial^2 a^i{}_\sigma}{\partial y^j \partial y^k}\right) \tag{4.9}$$

Corollary 4.1. *The weak torsion $t^{*i}_{jk} = \dfrac{\partial N^i_j}{\partial y^k} - \dfrac{\partial N^i_k}{\partial y^j}$ of N^* vanishes.*

The nonlinear connection N^* on TM gives a direct decomposition of tangent space $T_u(TM)$:

$$T_u(TM) = N^*(u) \oplus V(u), \quad \forall u \in TM \tag{4.10}$$

Therefore, it admits a local adapted basis $\left.\left(\dfrac{\delta^*}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)\right|_u$, where

$$\frac{\delta^*}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{*j}_i \frac{\partial}{\partial y^j} = \frac{\delta}{\delta x^i} + \frac{1}{4}\left(\frac{\partial F^i}{\partial y^i} + \lambda^\sigma \frac{\partial a^j_\sigma}{\partial y^i}\right)\frac{\partial}{\partial y^i}, \tag{4.11}$$

with $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$.

Dual basis $(dx^i, \delta^* y^i)$ has 1-forms $\delta^* y^i$:

$$\delta^* y^i = dy^i + N_j^{*i} dx^j = \delta y^i - \frac{1}{4}\left(\frac{\partial F^i}{\partial y^j} + \lambda^\sigma \frac{\partial a_\sigma^i}{\partial y^j} \right) dx^j \tag{4.13}$$

and where $\delta y^i = dy^i + N_j^i dx^j$.

Corollary 4.2. *The integrability conditions of nonlinear connection N^* are given by*

$$R^{*i}{}_{jk} = \frac{\delta^* N_j^{*i}}{\delta x^k} - \frac{\delta N^{*i}}{\delta x^j} = 0 .$$

5. N^* - CANONICAL METRICAL CONNECTION

Let " $_|$ " and " $|$ " h and v covariant derivates defined by a N^*- linear connection, [7].

The first author, in [6], [7], proves:

Theorem 5.1. *There exists only one N^*- linear connection $C\Gamma(N^*) = (L_{jk}^{*i}, C_{jk}^{*i})$, having the properties:*

$$\begin{aligned} &g_{ij\overset{*}{|}k} = 0, \quad g_{ij} \overset{*}{|}_k = 0 \\ &T_{jk}^{*i} = L_{jk}^{*i} - L_{kj}^{*i} = 0, S_{jk}^{*i} = C_{jk}^{*i} - C_{kj}^{*i} = 0, \end{aligned} \tag{5.1}$$

where $g_{ij\overset{*}{|}k} = \frac{\delta^* g_{ij}}{\delta x^k} - g_{sj} L_{ik}^{*s} - g_{is} L_{jk}^{*s} ; g_{ij} \overset{*}{|}_k = \frac{\partial g_{ij}}{\partial y^k} - g_{sj} C_{ik}^{*s} - g_{is} C_{jk}^{*s}$.

Theorem 5.2. *The connection $C\Gamma(N^*)$ has the coefficients given by the generalized Christoffel symbols:*

$$\begin{aligned} L^{*i}{}_{jk} &= \frac{1}{2} g^{ih} \left(\frac{\delta^* g_{hj}}{\delta x^k} + \frac{\delta^* g_{hk}}{\delta x^j} - \frac{\delta^* g_{jk}}{\delta x^h} \right) \\ C^{*i}{}_{jk} &= \frac{1}{2} g^{ih} \left(\frac{\partial g_{hj}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h} \right) \end{aligned} \tag{5.2}$$

This N^*- linear connection $C\Gamma(N^*)$ will be called the N^* - canonical metrical connection of Lagrangian nonholonomic mechanical system Σ.

$C\Gamma(N^*)$ is related to N-metrical connection of Lagrange space $L^n = (M, L(x,y))$, $C\Gamma(N) = (L_{jk}^i, C_{jk}^i)$ where

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2} g^{ih} \left(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right) \\ C^i{}_{jk} &= \frac{1}{2} g^{ih} \left(\frac{\partial g_{hj}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h} \right) \end{aligned} \tag{5.2'}$$

by the relations from the following theorem:

Theorem 5.3. We have:

$$L^{*i}{}_{jk} = L^i_{jk} + \mathcal{U}^i_{jk}, C^{*i}{}_{jk} = C^i_{jk} \tag{5.3}$$

where

$$\mathcal{U}^i_{jk} = g^{ih}[K^l_k C_{hjl} + K^l_j C_{hkl} - K^l_h C_{jkl}] \tag{5.3'}$$

$$K^j_i = \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i} + \lambda^\sigma \frac{\partial a^j_\sigma}{\partial y^i}\right) \tag{5.4}$$

and C_{ijk} is Cartan tensor of L, $C_{ijk} = \frac{1}{4}\frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k}$.

Proof. With notation (5.4) we obtain

$$\frac{\delta^*}{\delta x^i} = \frac{\delta}{\delta x^i} + K^l_i \frac{\partial}{\partial y^l} \tag{5.5}$$

Then,

$$\frac{\delta^* g_{hj}}{\delta x^k} = \frac{\delta g_{hj}}{\delta x^k} + K^l_k \frac{\partial g_{hj}}{\partial y^l}.$$

Substituing in (5.2) we have:

$$L^{*i}{}_{jk} = L^i_{jk} + \frac{1}{2} g^{ih}[K^l_k \frac{\partial g_{hj}}{\partial y^l} + K^l_j \frac{\partial g_{hk}}{\partial y^l} - K^l_h \frac{\partial g_{jk}}{\partial y^l}]$$

or

$$L^{*i}{}_{jk} = L^i_{jk} + g^{ih}[K^l_k C_{hjl} + K^l_j C_{hkl} - K^l_h C_{jkl}].$$

The fact that $C^{*i}{}_{jk} = C^i_{jk}$ results from (5.2) and (5.2').

Now, we can use the h- and v-covariant derivatives [7] with respect to connection $C\Gamma(N^*)$. Also, we observe that in the particular case of nonholonomic mechanical systems, which have the properties

$$g_{ij}(x,y) = g_{ij}(x), F_i(x,y) = F_i(x).$$

We have $\lambda^\sigma a_{\sigma i}(x)$ with $\frac{\partial}{\partial y^i}(\lambda^\sigma a_{\sigma i}) = 0$. Consequently, we obtain the tensors $K^j_i = 0$ and $\mathcal{U}^i_{jk} = 0$. The connections $C\Gamma(N^*)$ and $C\Gamma(N)$ have the coefficients $L^{*i}{}_{jk} = L^i_{jk}$, and $C^{*i}{}_{jk} = C^i_{jk} = 0$.

In the general case, the Ricci identities, with respect to N^*- canonical metrical connection $C\Gamma(N^*) = (L^{*i}{}_{jk}, C^i_{jk})$, for a d-vector field $X^i(x,y)$ on TM, are given by

$$\begin{aligned} &X^i{}_{\underset{|h}{*}\underset{|k}{*}} - X^i{}_{\underset{|k}{*}\underset{|h}{*}} = X^l R^{*i}_l{}_{hk} - X^i{}_{\underset{|l}{*}} T^{*l}{}_{hk} - X^i|_l R^{*l}{}_{hk} \\ &X^i{}_{\underset{|h}{*}}|_k - X^i|_{\underset{k|h}{*}} = X^l P^{*i}_l{}_{hk} - X^i{}_{\underset{|l}{*}} C^{*l}{}_{hk} - X^i|_l P^{*l}{}_{hk} \\ &X^i|_h|_k - X^i|_k|_h = X^l S_l{}^i{}_{hk} - X^i|_l S^l{}_{hk} \end{aligned} \tag{5.6}$$

where d-tensors of curvature and torsion of N^*- canonical metrical connection $C\Gamma(N^*)$ are:

$$\begin{aligned}
R^{*i}_{j\ kh} &= \frac{\delta^* L^{*i}{}_{jk}}{\delta x^h} - \frac{\delta^* L^{*i}{}_{jh}}{\delta x^k} + L^{*i}{}_{jk} L^{*i}{}_{jh} - L^{*l}{}_{jh} L^{*i}{}_{lk} + C^{*i}{}_{jl} R^{*l}{}_{kh} \\
P^{*i}_{j\ kh} &= \frac{\partial L^{*i}{}_{jk}}{\partial y^h} - C^i_{jh|k} + C^i_{jl} P^{*l}_{k\ l} \\
S_j{}^i{}_{kh} &= \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^l_{jk} C^i_{lh} - C^l_{jh} C^i_{lk}
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
&T^{*i}{}_{jk} = 0, S^i_{jk} = 0, C^{*i}_{jk} = C^i_{jk}, \\
&R^{*i}{}_{jk} = \frac{\delta^* N^{*i}_j}{\delta x^k} - \frac{\delta^* N^{*i}_k}{\delta x^j}, P^{*i}{}_{jk} = \frac{\partial N^{*i}}{\partial y^k} - L^{*i}_{jk}
\end{aligned} \tag{5.8}$$

In these formulas we use the formulas (5.3), (5.4), (5.5). The formulas (5.7) and (5.8) and (5.3), (5.4) and (5.5) can also be obtained directly using

$$\begin{cases} N^{*i}_j = N^{*i}_j, \\ L^{*i}_{jk} = L^i_{jk} + \mathcal{U}^i_{jk}, \\ C^{*i}_{jk} = C^i_{jk}. \end{cases} \tag{5.9}$$

We have the following result:

Theorem 5.4. *The structure equations of the N^*- canonical metrical connection of nonholonomic mechanical system Σ are given by*

$$\begin{aligned}
&d(dx^i) - dx^l \wedge \omega^{*i}_l = -\overset{(0)}{\Omega}{}^{*i}, \\
&d(\delta^* y^i) - \delta^* y^l \wedge \omega^{*i}_l = -\overset{(1)}{\Omega}{}^{*i}, \\
&d(\omega^{*i}_j) - \omega^{*l}_j \wedge \omega^{*i}_l = -\Omega^{*i}_l,
\end{aligned} \tag{5.10}$$

where $\overset{(0)}{\Omega}{}^{*i}$ and $\overset{(1)}{\Omega}{}^{*i}$ are 2-torsion forms

$$\begin{aligned}
\overset{(0)}{\Omega}{}^{*i} &= C^{*i}_{jk} dx^j \wedge \delta^* y^k \\
\overset{(1)}{\Omega}{}^{*i} &= \frac{1}{2} R^{*i}{}_{jk} dx^j \wedge dx^k + P^{*i}{}_{jk} dx^j \wedge \delta^* y^k
\end{aligned} \tag{5.10'}$$

and Ω^{*i}_j are 2-curvature forms

$$\Omega^{*i}_j = \frac{1}{2} R^{*i}_{j\ kh} dx^k \wedge dx^h + P^{*i}_{j\ kh} dx^k \wedge \delta^* y^h + \frac{1}{2} S_j{}^i{}_{kh} \delta y^{*k} \wedge \delta y^{*h}, \tag{5.10''}$$

$R^{*i}_{j\ kh}, P^{*i}_{j\ kh}, S_j{}^i{}_{kh}$ being d-curvature tensors of connection $C\Gamma(N^*)$, [7].

Bianchi identities satisfied by the N^*- canonical metrical connection $C\Gamma(N^*)$ are obtained from the structure equations (5.10) by exterior differentiation.

6. *H*- AND *V*-ELECTROMAGNETICS TENSORS

The canonical metrical connection $C\Gamma(N^{*})$ of the neolonomic mechanical system Σ allows to determine the *h*- and *v*-deflection tensor fields [7]:

$$\begin{aligned} D^i_j &= y^i_{\overset{*}{|}j} = y^h L^{*i}_{hj} - N^{*i}_j \\ d^i_j &= y^i|_j = \delta^i_j + y^h C^i{}_{hj}. \end{aligned} \tag{6.1}$$

Using the formulas (5.3), (5.3') for the coefficients L^{*i}_{hj} and C^i_{hj} we obtain

$$D^i_j = y^s L^i_{sj} + \frac{1}{2} y^s g^{ih}[K^l_j \frac{\partial g_{hs}}{\partial y^l} + K^l_s \frac{\partial g_{hj}}{\partial y^l} - K^l_h \frac{\partial g_{sj}}{\partial y^l}] =$$
$$= y^s L^i_{sj} + y^s g^{ih}[K^l_j C_{hsl} + K^l_s C_{hjl} - K^l_h C_{sjl}]$$

and

$$d^i_j = \delta^i_j + y^s C^i_{sj} = \delta^i_j + y^s g^{ih} C_{shj}.$$

Then, the covariant deflection tensors are given by:

$$\begin{cases} D_{ij} = g_{ir} D^r_j = g_{ir} y^s L^i_{sj} + y^s (K^l_j C_{isl} + K^l_s C_{ijl} - K^l_i C_{sjl}) \\ d_{ij} = g_{ir} d^r_j = g_{ij} + y^s C_{sij}. \end{cases} \tag{6.2}$$

But, these tensors satisfy fundamental identities from (5.6) for Liouville vector field y^i. Then, we have:

Theorem 6.1. *h- and v-covariant deflection tensors* D_{ij} *and* d_{ij} *satisfy the identities:*

$$\begin{cases} D_{ij\overset{*}{|}k} - D_{ik\overset{*}{|}j} = y^s R^{*}_{sijk} - d_{ir} R^{*r}_{jk} \\ D_{ij}|_k - d_{ik}\overset{*}{|}_j = y^s P^{*}_{sijk} - D_{is} C^s_{jk} - d_{is} P^{*s}_{jk} \\ d_{ij}|_k - d_{ik}|_j = y^s S_{sijk}. \end{cases} \tag{6.3}$$

These identities give the Lorentz equations for electromagnetic tensor fields for Lagrangian nonholonomic mechanical system Σ.

Definition 6.1. *The following d-tensors*

$$\begin{aligned} \mathcal{F}_{ij} &= \frac{1}{2}(D_{ij} - D_{ji}) \\ f_{ij} &= \frac{1}{2}(d_{ij} - d_{ji}) \end{aligned} \tag{6.4}$$

are *h*- and *v*-electromagnetic tensors of Σ.

From (6.2) we see that v-deflection tensor d_{ij} is symmetric. Therefore we have:

Proposition 6.1. *The v-electromagnetic tensor* f_{ij} *of nonholonomic mechanical system* Σ *vanishes.*

We study only *h*-electromagnetic tensor $\mathcal{F}_{ij}$ for determining the Lorentz equations that are satisfied.

We observe that the d-tensor $\mathcal{F}_{ij}$ do not coincide with the elicoidal tensor F_{ij} from (3.1). So, from (6.2), $\mathcal{F}_{ij}$ is given by

$$\mathcal{F}_{ij} = \frac{1}{2}\{(g_{ir}L^r_{sj} - g_{ir}L^r_{si})y^s + 2(K^l_j C_{isl} - K^l_j C_{jsl})y^s\} \tag{6.5}$$

But, from (5.4), we deduce:

$$K^j_i = \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i} + \lambda^\sigma \frac{\partial a^j_\sigma}{\partial y^i}\right)$$

and we obtain

$$\mathcal{F}_{ij} = \frac{1}{2}y^s\left(g_{ir}L^r_{sj} - g_{jr}L^r_{si}\right) + \frac{1}{4}y^s\left[\left(\frac{\partial F^l}{\partial y^j} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^j}\right)C_{isl} - \left(\frac{\partial F^l}{\partial y^i} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^i}\right)C_{jsl}\right],$$

where $F^l = g^{il}F_i$ and $C_{isl} = g_{il}C^h_{sl}$. Thus

$$\mathcal{F}_{ij} = \frac{1}{2}y^s\left(g_{ir}L^r_{sj} - g_{jr}L^r_{si}\right) + \frac{1}{4}\left[\left(\frac{\partial F^l}{\partial y^j} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^j}\right)g_{ir} - \left(\frac{\partial F^l}{\partial y^i} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^i}\right)g_{jr}\right] \tag{6.6}$$

From Theorem 6.1, (6.4) and Bianchi identities for $C\Gamma(N^*)$ we have

Theorem 6.2. *The h-electromagnetic tensor* $\mathcal{F}_{ij}$ *of nonholonomic mechanical system* Σ *with respect to* $C\Gamma(N^*)$ *satisfies the following generalized Maxwell equations*:

$$\begin{aligned} &\mathcal{F}_{\overset{*}{ij|k}} + \mathcal{F}_{\overset{*}{jk|i}} + \mathcal{F}_{\overset{*}{ki|j}} = -\sum_{isr}^{C} y^s R^{*r}{}_{jk} \\ &\mathcal{F}_{ij}\,|_k + \mathcal{F}_{jk}\,|_i + \mathcal{F}_{ki}\,|_j = 0. \end{aligned} \tag{6.7}$$

We remark that, if the electromagnetic tensor $\mathcal{F}_{ij}$ does not depend of F_i and $a_{\sigma i}$, then it is given by

$$\mathcal{F}_{ij} = \frac{1}{2}y^s(g_{ir}L^r_{sj} - g_{jr}L^r_{si}) \tag{6.8}$$

Thus, we have $N^{*i}_j = N^i_j$ and the Maxwell equations are those that appear, in general, in Lagrange spaces theory. An easier method of determination of Maxwell equations is given by the almost hermetian model of nonholonomic mechanical system Σ.

So, we consider the 2-form

$$\theta = g_{ij}\delta^* y^i \wedge dx^j \tag{6.9}$$

Using 1-forms $\delta^* y^i$ from (4.13) and (5.4) or from the equivalent formula

$$\delta^* y^i = \delta y^i - K^i_j dx^j \tag{6.10}$$

we have

Theorem 6.3. *The 2-form* θ *has the following properties:*

1° θ *depend only on nonholonomic mechanical system* Σ.

2° θ *is an almost symplectic structure on phase space TM.*

3° θ *depends on symplectic structure of associated Lagrange space* $L^n = (M, L(x,y))$ $\overset{0}{\theta} = g_{ij}\delta y^i \wedge dx^j$.

The relation between θ and $\overset{0}{\theta}$ is given by

$$\theta = \overset{0}{\theta} - g_{ij}K^i_s dx^s \wedge dx^j . \tag{6.11}$$

Proof. 1° In (6.9) the fundamental tensor g_{ij} and the nonlinear connection $\overset{*}{N}$ of Σ appear.

2° θ is 2-form of rank $2n$ with $\det\begin{Vmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{Vmatrix} \neq 0$. So, θ is a nonsingular 2-form on *TM*. It is an almost symplectic structure on the manifold given by the phases space.

3° Using $\overset{*}{\delta} y^i$ from (6.10) in the expression of 2-form θ we obtain (6.11).

In order to study the case when θ is a symplectic structure, we will examine the exterior differential of θ.

Proposition 6.2. *The exterior differential of 2-form* θ *is given by*

$$d\theta = -\frac{1}{2}[d(g_{sj}K^s_i - g_{si}K^s_j)]dx^i \wedge dx^j \tag{6.12}$$

Proof. We exterior differentiation (6.11) and remark that $\overset{0}{\theta}$ is a closed 2-form, so $d\overset{0}{\theta} = 0$ and we obtain (6.12). □

Since we have

$$\begin{aligned} d\theta = -\frac{1}{3!}\Big[&\frac{\overset{*}{\delta}}{\delta x^h}(g_{sj}K^s_i - g_{si}K^s_j) + \frac{\overset{*}{\delta}}{\delta x^i}(g_{sh}K^s_j - g_{sj}K^s_h) + \\ &+ \frac{\overset{*}{\delta}}{\delta x^i}(g_{si}K^s_h - g_{sh}K^s_i)\delta x^h \wedge dx^i \wedge dx^j\Big] + \frac{1}{2}\frac{\partial}{\partial y^h}(g_{sj}K^s_i - g_{si}K^s_j)\overset{*}{\delta} y^h \wedge dx^i \wedge dx^j \end{aligned} \tag{6.13}$$

Thus we state:

Theorem 6.4. *The almost symplectic structure* θ *is integrable if and only if the nonholonomic mechanical system* Σ *has the properties:*

$$\sum_{ijh} \frac{\overset{*}{\delta}}{\delta x^h}(g_{sj}K^s_i - g_{si}K^s_j) = 0,$$

$$\frac{\partial}{\partial y^h}(g_{sj}K^s_i - g_{si}K^s_j) = 0, \quad \sum_{ijh} \text{ being cyclic sum.}$$

Finally, we observe that the exterior equations $d\,(d\theta) = 0$ give us the Maxwell equations for nonholonomic mechanical system Σ. So, it is sufficient that to evidence the electromagnetic tensor $\mathcal{F}_{ij}$ in equations (6.12).

Then, $d\,(d\theta) = 0$ give the generalized Maxwell equations (6.7).

7. THE GRAVITATIONAL FIELD

The nonholonomic mechanical system $\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y))$ has the gravitational potentials given by the system of functions

$$g_{ij}^* = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{7.1}$$

We remark that this field does not depend on the external forces $F_i(x,y)$ and on the nonholonomic constrains $Q_\sigma(x,y) = a_{\sigma i}(x)y^i$. So that the gravitational potentials $g_{ij}^*(x,y)$ do not depend on the Lagrange multipliers λ_i^σ $(\sigma = p+1,...,n)$.

This fact results from

$$g_{ij}^* = g_{ij}(x,y), \tag{7.1'}$$

where $g_{ij}(x,y)$ is the fundamental tensor of the Lagrange space associated to system Σ, $L^n = (M, L(x,y))$.

The Theorem 5.1 shows that the canonical metrical connection $C\Gamma(N^*) = (L_{jk}^{*i}, C_{jk}^i)$ has the properties

$$g_{ij\overset{*}{|}k} = 0, \quad g_{ij}|_k = 0 \tag{7.2}$$

and it is unique in the following conditions:

$$T^{*i}{}_{jk} = 0, S_{jk}^i = 0.$$

We rewrite the coefficients of the connection $C\Gamma(N^*)$:

$$\begin{cases} L_{jk}^{*i} = \dfrac{1}{2}g^{ih}\left(\dfrac{\delta^* g_{jh}}{\delta x^k} + \dfrac{\delta^* g_{kh}}{\delta x^j} - \dfrac{\delta^* g_{jk}}{\delta x^h}\right) \\ C_{jk}^i = \dfrac{1}{2}g^{ih}\left(\dfrac{\partial g_{jh}}{\partial y^k} + \dfrac{\partial g_{kh}}{\partial y^j} - \dfrac{\partial g_{jk}}{\partial y^h}\right) \end{cases} \tag{7.3}$$

where L_{jk}^{*i} is given by (5.3)

$$\begin{aligned} L_{jk}^{*i} &= L_{jk}^i + \mathcal{U}_{jk}^i \\ \mathcal{U}_{jk}^i &= g^{ih}(K_k^l C_{hjk} + K_j^l C_{hjl} - K_h^l C_{jkl}) \end{aligned} \tag{7.4}$$

We apply the Ricci identities to the fundamental tensor g_{ij} and we use (7.2):

$$\begin{aligned} 0 = R_{ijkh}^* + R_{jikh}^* = 0 \\ P_{ijkh}^* + P_{jikh}^* = 0; S_{ijkh} + S_{jikh} = 0 \end{aligned} \tag{7.5}$$

where

$$\begin{aligned} R_{ijkh}^* &= g_{jl}R_i^{*l}{}_{kh}; P_{ijkh}^* = g_{jl}P_i^{*l}{}_{kh}; \\ S_{ijkh} &= g_{jl}S_i^*{}_{kh} \end{aligned} \tag{7.5'}$$

We must calculate the curvature tensors $R^{*i}_{j\ kh}, P^{*i}_{j\ kh}$ by means of the curvature tensors of the connection $C\Gamma(N)$.

So, we suppose that $C\Gamma(N)$ is the metrical connection of the associated Lagrange space L^n. Therefore the formulas (5.2') hold:

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{jh}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h}\right) \\ C^i_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{jh}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^h}\right) \end{aligned} \tag{7.6}$$

and

$$g_{ij|k} = 0, g_{ij}|_k = 0 \tag{7.7}$$

$$T^h_{\ ij} = 0, S^h_{\ ij} = 0 \tag{7.7'}$$

$$\begin{aligned} &R_{ijkh} + R_{jikh} = 0, P_{ijkh} + P_{jikh} = 0 \\ &S_{ijkh} + S_{jikh} = 0. \end{aligned} \tag{7.7''}$$

The Ricci tensors of $C\Gamma(N)$ are:

$$R_{ij} = R^h_{i\ jh};\ {}'P_{ij} = P^h_{i\ jh},\ {}''P_{ij} = P^h_{i\ hj} \tag{7.7'''}$$

and the curvature scalars are:

$$R = g^{ij} R_{ij}, S = g^{ij} S_{ij} \tag{7.7''''}$$

The following theorem is known [7]

Theorem 7.1. *The Einstein equations of the Lagrange space $L^n = (M,L(x,y))$, $n > 2$, corresponding to the canonical metrical connection $C\Gamma(N)$ are given by*

$$\begin{cases} R_{ij} - \dfrac{1}{2} R g_{ij} = \overset{0}{k} \overset{H}{T}_{ij},\ {}'P_{ij} = \overset{0}{k} \overset{1}{T}_{ij} \\ S_{ij} - \dfrac{1}{2} S g_{ij} = \overset{0}{k} \overset{V}{T}_{ij},\ {}''P_{ij} = -\overset{0}{k} \overset{2}{T}_{ij} \end{cases} \tag{7.8}$$

The energy momentum tensors $\overset{H}{T}_{ij}, \overset{V}{T}_{ij}$ satisfy the conditions (5.5) from §5 of the book [7].

Using the previous theory we determine the Einstein equations of the Lagrange space $L^{*n} = (M,L^*(x,y))$ named the Einstein equations of the nonholonomic mechanical system Σ.

Theorem 7.2. *The Einstein equations of the nonholonomic mechanical system Σ corresponding to the canonical metrical connection $C\Gamma(N^*)$ for $n \geq 2$ are the following:*

$$\begin{aligned} R^*_{ij} - \frac{1}{2} R^* g_{ij} &= k \overset{H*}{T}_{ij},\ {}'P^*_{ij} = k \overset{1^*}{T}_{ij} \\ S_{ij} - \frac{1}{2} S g_{ij} &= k \overset{V}{T}_{ij},\ {}''P_{ij} = -k \overset{2^*}{T}_{ij} \end{aligned} \tag{7.9}$$

where R^*_{ij}, $'P^*_{ij}$, $''P_{ij}$, S_{ij} are the Ricci tensors of the system Σ, R^* and S are the scalar curvature, $\overset{H*}{T}_{ij}, \overset{V}{T}_{ij}, \overset{1\,*}{T}_{ij}, \overset{2\,*}{T}_{ij}$ are the energy momentum tensors of the system Σ and k is a constant.

In order to determine the tensors R^*_{ij}, $'P^*_{ij}$, $''P_{ij}$ and the curvature scalars R^*, we consider the transformation of nonlinear connection $N \to \overline{N}$ given by (4.8)

$$\overline{N}^i_j = N^i_j - K^i_j \tag{7.10}$$

Then the connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ is transformed into the connection $\overline{L}\Gamma(N^*) = (\overline{L}^i_{jk}, \overline{C}^i_{jk})$ given by (7.10) and its coefficients are:

$$\overline{L}^i_{jk} = L^i_{jk} + K^s_k C^i_{js}, \overline{C}^i_{jk} = C^i_{jk} \tag{7.11}$$

We observe that $\overline{L}\Gamma(N^*)$ is also a metrical connection with respect to g_{ij}. So, we have

$$g_{ij\,\overline{|}\,k} = g_{ij|k} + g_{ij}\,|_r\, K^r_k, g_{ij}\ \overline{|} = g_{ij}\,|_k \tag{7.11'}$$

A new transformation $\overline{L}\Gamma(N^*) \to L^*\Gamma(N^*)$ given by

$$\begin{aligned} N^{*i}_j &= N^i \\ L^*_{jk} &= \overline{L}^i_{jk} + g^{ih}(K^s_j C_{hks} - K^s_h C_{jks}) \\ C^{*i}_{jk} &= C^i_{jk} \end{aligned} \tag{7.12}$$

determines, also, a metrical connection with respect to g_{ij}.

Consequently, we have the following succesive transformations

$$C\Gamma(N) \overset{(7.11)}{\to} \overline{L}\Gamma(\overline{N}) \overset{(7.12)}{\to} C^*\Gamma(N^*) \tag{7.13}$$

and, by their composition, we obtain:

$$C\Gamma(N) = (L^i_{jk}, C^i_{jk}) \to C^*\Gamma(N^*) = (L^{*i}_{jk}, C^i_{jk}) .$$

The curvature tensors have the same transformation of connections (7.13).

Theorem 7.3. *The following formulas hold:*

$$\begin{aligned} &\overline{N}^i_j = N^i_j - K^i_j, \overline{L}^i_{jk} = L^i_{jk} + K^s_k C^i_{js}, \\ &\overline{C}^i_{jk} = C^i_{jk} \\ &\overline{T}^i_{jk} = C^i_{js} K^s_k - C^i_{ks} K^s_j, \overline{S}^i_{jk} = 0 \\ &P^i_{jk} = P^i_{jk} - \left(\frac{\partial K^i_j}{\partial y^k} + C^i_{ks} K^s_j \right) \end{aligned} \tag{7.14}$$

$$\left\{ \overline{R}^i_{jk} = R^i_{jk} + \left(K^s_k \frac{\partial N^i_j}{\partial y^s} - K^s_j \frac{\partial N^i_j}{\partial y^s} \right) - \frac{\delta K^i_j}{\delta x^k} - \frac{\delta K^i_k}{\delta x^j} + K^r_j \frac{\partial K^i_k}{\partial y^r} - K^r_k \frac{\partial K^i_j}{\partial y^r} \right\} \tag{7.15}$$

and

$$\begin{aligned}\bar{R}^i_{jkh} &= R^i_{jkh} + P^i_{jks} - P^i_{jhs}K^s_k + S^h_{irs}K^r_hK^s_k,\\ \bar{P}^i_{jkh} &= P^i_{jkh} + S^i_{jks}K^s_h\\ \bar{S}^i_{jkh} &= S^i_{jkh}\end{aligned} \tag{7.16}$$

Last formulas can be established directly from complicated calculus.

Theorem 7.4. *Also, we have the formulas*:

$$\begin{cases} N^{*i}_j = \bar{N}^i_j, L^{*i}_{jk} = \bar{L}^i_{jk} + \bar{B}^i_{jk}, C^{*i}_{jk} = \bar{C}^i_{jk}\\ \bar{B}^i_{jk} = g^{ih}(K^s_jC_{hks} - K^s_hC_{jks})\end{cases} \tag{7.17}$$

and

$$\begin{aligned}R^{*i}_{jkh} &= \bar{R}^i_{jkh} + \bar{S}^i_{jkh},\\ P^{*i}_{jkh} &= \bar{P}^i_{jkh} + \bar{\Pi}^i_{jkh}\\ S^{*i}_{jkh} &= \bar{S}^i_{jkh}\end{aligned} \tag{7.18}$$

where

$$\begin{aligned}\bar{S}^i_{jkh} &= B^i_{jk\,\bar{|}\,h} - B^i_{jh\,\bar{|}\,k} + B^r_{jk}B^i_{rh} - B^r_{jh}B^i_{rh} + B^i_{jr}\bar{T}^r_{kh} + C^i_{jr}\bar{R}^r_{kh}\\ \Pi^i_{jkh} &= B^i_{jk}\,|_h - C^i_{jk\,\bar{|}\,k} + B^r_{jk}C^i_{rh} - C^r_{jh}B^i_{rk} + B^i_{jr}C^r_{kh} + C^i_{jr}\bar{P}^r_{kh}\end{aligned} \tag{7.19}$$

We consider also that these formulas can be established directly.

In order to determine the relations between the curvature tensors of N^*- canonical metrical $C\Gamma(N^*)$ and N-canonical metrical connection $C\Gamma(N)$, we must eliminate $\bar{N}, \bar{R}, \bar{P}, \bar{S}$ from (7.14) – (7.19).

Theorem 7.5. *The curvatures of the connections* $C\Gamma(N^*)$ *and* $C\Gamma(N)$ *are related by the following formulas:*

$$\begin{aligned}R^{*i}_{jkh} &= R^i_{jkh} + S^i_{jkh},\\ P^{*i}_{jkh} &= P^i_{jkh} + \Pi^i_{jkh}\\ S^{*i}_{jkh} &= S^i_{jkh}\end{aligned} \tag{7.20}$$

where

$$\begin{aligned}S^{*i}_{jkh} &= P^i_{jks}K^s_h - P^i_{jhs}K^s_k + S^h_{irs}K^r_hK^s_k + \bar{S}^i_{jkh}\\ \Pi^{*i}_{jkh} &= S^i_{jks}K^s_h + \bar{\Pi}^i_{jkh}\end{aligned} \tag{7.21}$$

Now, we can determine the Ricci tensors and the curvature scalars.

Theorem 7.6. *The Ricci tensors and the curvature scalars of the connections* $C\Gamma(N^*)$ *and* $C\Gamma(N)$ *satisfy the following relations:*

$$\begin{aligned}&R^*_{ij} = R_{ij} + S^{*h}_{ijh};\ {}'P^*_{ij} = {}'P_{ij} + \Pi^{*h}_{ihj}\\ &{}'P^*_{ij} = {}''P_{ij} + \Pi^{*h}_{ihj}; S^*_{ij} = S_{ij}\\ &R^* = R + g^{ij}S^{*k}_{ijh}, S^* = g^{ij}S_{ij}\end{aligned} \tag{7.22}$$

We also have

Theorem 7.7. *The Einstein equations of the nonholonomic mechanical system* $\Sigma = (M, L(x,y), F_i(x,y), Q_\sigma(x,y))$ *with respect to the* N^**- canonical metrical connection* $C\Gamma(N^*)$ (7.9), *for* $n \geq 2$*, are given by:*

$$\begin{aligned} & R_{ij} - \frac{1}{2} R g_{ij} + \{S^{*h}_{ijh} - \frac{1}{2} g^{rs} S^{*h}_{rsh} g_{ij}\} = k \overset{H^*}{T}_{ij} \\ & {}' P_{ij} + \Pi^{*h}_{ijh} = k \overset{1*}{T}_{ij}; {}'' P_{ij} + \Pi^{*h}_{ihj} = -k \overset{2*}{T}_{ij} \\ & S_{ij} - \frac{1}{2} S g_{ij} = k \overset{V}{T}_{ij} \end{aligned} \tag{7.23}$$

where S^* and Π^* are expressed in (7.21) and (7.19).

Remarks:

1. In (7.23) complementary terms to the Einstein equations of the Lagrange space L^n appear. The external forces $F_i(x,y)$ and $Q_\sigma(x,y)$ are those that determine these terms.
2. Evidently, it is not simple to obtain the Einstein equations (7.22) of the system Σ, but this procedure is an algorithm imposed by the transformations of connection (7.13).

In Lagrange geometries, there are many situations where the calculus of the Ricci tensors from the Einstein equations is very difficult. The most eloquent example is given by the Randers or the Ingarden spaces [6].

8. EXAMPLES

8.1. Classical nonholonomic mechanical systems

We consider nonholonomic scleronomic mechanical systems

$$\Sigma = (M, L(x,y), F_i(x), Q_\sigma(x,y)) \tag{8.1}$$

where $L(x,y)$ is given by the kinetic energy

$$L(x,y) = g_{ij}(x) y^i y^j, \quad y^i = \frac{dx^i}{dt} \tag{8.2}$$

$g_{ij}(x)$ being the fundamental tensor of a Riemann space $R^n = (M, g_{ij}(x))$, the external forces $F_i(x)$ give a d-covectors field on the base manifold M and Q_σ give the cinematic constrains $Q_\sigma = a_{\sigma i}\dot{x}^i \quad (\sigma = m+1, \ldots, n)$.

The space R^n will be named the associated Riemann space of the system Σ. It coincides with the space $L^n = (M, L(x,y))$ whose fundamental tensor is $g_{ij}(x)$ (depending only by the material points x of Σ).

We observe that the elycoidal tensor of the system Σ, F_{ij} given by (3.1) vanishes

$$F_{ij}(x,y) = 0 \tag{8.3}$$

The Lagrangian $L^*(x,y)$ from (3.3) has the classical form:

$$L^*(x,y) = g_{ij}(x)y^i y^j + \lambda^\sigma(x)a_{\sigma i}(x)y^i \tag{8.5}$$

where $\lambda^\sigma a_{\sigma i}$ means $\sum_{\sigma=m+1}^{n} \lambda^\sigma a_{\sigma i}$.

So, we have

Proposition 8.1. *The Lagrangian $L^*(x,y)$ of a classical nonholonomic mechanical system Σ has the form of a Lagrangian from electrodynamics where the electromagnetics potentials $A_i(x)$ are given by*

$$A_i(x) = \lambda^\sigma(x)a_{\sigma i}(x) \tag{8.6}$$

Proposition 8.2. *The Euler – Lagrange equations of the Lagrangian $L^*(x,y)$ are given by*

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} + [\frac{\partial \lambda^\sigma}{\partial x^i}Q_\sigma + \lambda^\sigma \frac{\partial Q_\sigma}{\partial x^i} - \frac{d}{dt}(\lambda^\sigma a_{\sigma i})] \tag{8.7}$$

or

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} + [\frac{\partial \lambda^\sigma}{\partial x^i}a_{\sigma j} - \frac{\partial \lambda_\sigma}{\partial x^j}a_{\sigma i} + \lambda^\sigma(\frac{\partial a_{\sigma i}}{\partial x^i} - \frac{\partial a_{\sigma j}}{\partial x^j})]y^j = 0 \tag{8.7'}$$

Proposition 8.3. *The Lagrangians $L^*(x,y)$ and $L(x,y)$ are equivalent if and only if the Lagrange multipliers $\lambda^\sigma(x)$ satisfy the exterior equations:*

$$d[\lambda^\sigma a_{\sigma i}] \wedge dx^i = 0 \tag{8.8}$$

Evidently (8.8) give restrictions for the multipliers λ^σ.

We have

Theorem 8.1. *The equations of evolution of the nonholonomic mechanical system Σ are:*

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}g^{is}[F_s(x) + \lambda^\sigma a_{\sigma s}(x)] \tag{8.9}$$

$\Gamma^i_{jk}(x)$ being the Christoffel symbols of $g_{ij}(x)$.

Theorem 8.2. *The canonical semispray of the system Σ is given by*

$$S^* = y^i \frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i} \tag{8.10}$$

where

$$2G^{*i}(x,y) = \Gamma^i_{jk}(x)y^j y^k - \frac{1}{2}(F^i(x) + \lambda^\sigma a^i_\sigma(x)) \tag{8.11}$$

$\Gamma^i_{jk}(x)$ are the Christoffel symbols of the associated Riemann space R^n and

$$F^i(x) = g^{ih}(x)F_h(x), a^i_\sigma(x) = g^{ih}(x)a_{\sigma h}(x) .$$

The Theorem 4.2 leads to:

Theorem 8.3. *The integral curves of the semispray S^* are the evolution curves of the nonholonomic mechanical system* Σ.

Because the tensor $\dfrac{\partial F^i}{\partial y^j}$ vanishes, the Theorem 4.3 gives us

Theorem 8.4. *The canonical nonlinear connection N^* of the system Σ coincides with the canonical nonlinear connection N with the coefficients $N^i_j = \Gamma^i_{jk}(x)y^k$ of the associated Riemann space.*

So, the nonlinear connection N^* does not depend on the external forces $F_i(x)$ or on the nonholonomy forces $Q_\sigma(x,y)$. This property simplifies the whole theory because:

N^* - canonical metrical connection $C\Gamma(N^*)$ has the coefficients

$$L^{*i}_{jk} = \Gamma^i_{jk}(x), C^i_{jk} = 0\,.$$

So, $C\Gamma(N^*)$ is a Cartan connection [7].

8.2. Finslerian nonholonomic mechanical systems

The Finslerian holonomic mechanical systems were studied in the paper [9]. But, the Finslerian nonholonomic mechanical systems have not been studied until now.

Their geometrical theory appears here for the first time. It is a particular case of the theory from precedent paragraphs.

Let a Finsler space $F^n=(M,F(x,y))$ be. Its canonical Cartan nonlinear connection N has the coefficients $N^i{}_j \cdot F^n$ is endowed with N-canonical metrical connection $C\Gamma(N) = (F^i_{jk}, C^i_{jk})$, [7].

The function $F^2(x,y)$ is a regular Lagrangian and the fundamental metric tensor $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ has the property

$$F^2(x,y) = g_{ij}(x,y)y^i y^j \tag{8.12}$$

Then, the absolute energy of the space F^n is exactly (8.12) and its kinetic energy is

$$F^2(x,\frac{dx}{dt}) = g_{ij}(x,\frac{dx}{dt})\frac{dx^i}{dt}\frac{dx^j}{dt} \tag{8.12'}$$

Definition 8.1. *A Finslerian nonholonomic mechanical system is a quadruple:*

$$\Sigma = (M, F^2(x,y), F_i(x,y), Q_\sigma(x,y)) \tag{8.13}$$

where $F^2(x,y)$ is the kinetic energy (8.12') *$F_i(x,y)$ are the external forces and $Q_\sigma(x,dx) = a_{\sigma i}(x)dx^i$, ($\sigma = m + 1,...,n$).*

The Pfaff equations

$$Q_\sigma(x,dx) := a_{\sigma_i}(x)dx^i = 0, \quad (\sigma = p+1,...,n) \tag{8.14}$$

determine the kinematic nonholonomic constrains of the system Σ.

The elycoidal tensor of the system is

$$F_{ij} = \frac{\partial F_j}{\partial y^i} - \frac{\partial F_i}{\partial y^j} \tag{8.15}$$

Let the Lagrangian be

$$L^*(x,y) = F^2(x,y) + \lambda^\sigma(x) Q_\sigma(x,y) = F^2(x,y) + \lambda^\sigma(x) a_{\sigma i}(x) y^i \tag{8.16}$$

We observe that:

1. The Lagrangian $L^*(x,y)$ of the Finslerian nonholonomic mechanical system Σ is not a homogeneous function with respect to y^i.
2. The Lagrangian $L^*(x,y)$ has the fundamental tensor g^*_{ij} equal with the fundamental tensor g_{ij} of Finsler space F^n:

$$g^*_{ij}(x,y) = g_{ij}(x,y) \tag{8.17}$$

3. The Euler – Lagrange equations of $L^*(x,y)$ are:

$$\frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\frac{\partial F^2}{\partial y^i} + [\frac{\partial \lambda^\sigma}{\partial x^i} a_{\sigma j} - \frac{\partial \lambda^\sigma}{\partial x^j} a_{\sigma i} + \lambda^\sigma (\frac{\partial a_{\sigma j}}{\partial x^i} - \frac{\partial a_{\sigma i}}{\partial x^j})] = 0 \tag{8.18}$$

Under the condition that the Lagrangians $L^*(x,y)$ and $F^2(x,y)$ give the same Euler – Lagrange equations, we obtain

$$d(\lambda^\sigma Q_\sigma(x,dx)) = 0 \tag{8.19}$$

The exterior equations (8.19) give the restrictions of he Lagrange multipliers $\lambda^\sigma(x)$.

The canonical semispray S^* of the Finslerian nonholonomic mechanical system Σ is given by (4.2):

$$S^* = y^i \frac{\partial}{\partial y^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i} \tag{8.20}$$

where

$$2G^*(x,y) = 2G^i(x,y) - \frac{1}{2}(F^i(x,y) + \lambda^\sigma(x) a^i_\sigma(x)) \tag{8.21}$$

$$2G^i = \gamma^i_{jk}(x,y) y^j y^k$$
$$F^i(x,y) = g^{ij}(x,y) F_j(x,y)$$
$$a^i_\sigma(x,y) = g^{ij}(x,y) a_{\sigma j}(x)$$

where $\gamma^i_{jk}(x,y)$ is the Christoffel symbols of the fundamental tensor $g_{ij}(x,y)$ of the space F^n.

Regarding S^* we have

Theorem 8.5. *The integral curves of the canonical semispray S^* of the Finslerian nonholonomic mechanical system Σ are the solutions curves of the system of differential equations*

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,\frac{dx}{dt})\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}(F^i(x,\frac{dx}{dt}) + \lambda^\sigma(x) a^i_\sigma(x,\frac{dx}{dt})) \tag{8.23}$$

But (8.23) gives the equations of evolution of the system Σ.

In particular, if $F_i \equiv 0, \lambda^\sigma(x) \equiv 0$, then the equations (8.23) are reduced to the equations of the geodesics of the Finsler space F^n.

The conclusion is the following:

Theorem 8.6. *The Finslerian nonholonomic mechanical system* Σ *may be considered as a dynamical system given by the semispray vector field* S^* *on the phases space TM, where the Lagrange multipliers* $\lambda^\sigma(x)$ *satisfy the exterior equation (8.13).*

Consequently, the geometry of system Σ *is the geometry of the canonical semispray* S^* *on the phase space TM.*

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GEOMETRIJSKI MODEL LAGRANŽIJANA I PRIDRUŽENI DINAMIČKI SISTEM NEHOLONOMNOG MEHANIČKOG SISTEMA

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Razmatra se geometrijski model Lagranžiana i pridruženi dinamički sistem mehaničkog sistema $\Sigma = (M, L(x,y), Q_\sigma(x,dx), F_i(x,\dot{x}))$ *, sa* $y = \dot{x}$ *, čije su evolucione jednačine (1.3). Kanonski semisprej* S^* *udružuje se u system* Σ *na prostoru faze TM, koja ima integralne krive date evolucionim jednačinama Σ. Lagranžeova geometrija sistema Σ je geometrija* S^* *koja je dinamički system, na TM, suštinski pridružen u Σ. Dobijeni rezultati su novi i originalni.*

Ključne reči: *Lagranžeov prostor, semisprej, dinamički sistem, Lagranžijan mehaničkog sistema*