FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 4, No 17, 2005, pp. 373 - 376

Letter to Editor

A NOTE ON TRACELESS METRIC TENSOR

UDC 514.763.5(045)=20

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Abstract. *We present the coordinate transformations which transform the diagonal Minkowski metric tensor in a metric tensor with all zero diagonal components. Some of the obtained coordinate transformations provide the singular metric tensors and we point out those ones which provide the nonsingular metrics.*

Key words: *the diagonal Minkowski metric tensor, zero diagonal components, coordinate transformation.*

We shall try to make a traceless metric tensor starting from the diagonal metric tensor of Minkowski. In fact, we shall try to find out the coordinate transformations which will provide the metric tensor with zero diagonal components, and we use the term "traceless" in that sense.

We start with the metric of Minkowski. We try to find out the coordinate transformations which will provide the metric tensor with all zero diagonal components.

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$$

$$\{\bar{g}_{\alpha\beta}\}_M = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix} \qquad \{g_{\rho\nu}\} = \begin{Bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{Bmatrix}$$

In order to find out such a coordinate transformations, we shall use the usual tensor transformation:

$$g_{\rho\nu} = \bar{g}_{\alpha\beta} \frac{\partial \bar{x}^\alpha}{\partial x^\rho} \frac{\partial \bar{x}^\beta}{\partial x^\nu}$$

Received October 23, 2003

The conditions that make the diagonal components of the metric $g_{\rho\nu}$ zero, yield the following equations:

$$
\begin{aligned}
g_{11} &= \left(\frac{\partial \bar{x}^1}{\partial x^1}\right)^2 + \left(\frac{\partial \bar{x}^2}{\partial x^1}\right)^2 + \left(\frac{\partial \bar{x}^3}{\partial x^1}\right)^2 - \left(\frac{\partial \bar{x}^4}{\partial x^1}\right)^2 = 0 \\
g_{22} &= \left(\frac{\partial \bar{x}^1}{\partial x^2}\right)^2 + \left(\frac{\partial \bar{x}^2}{\partial x^2}\right)^2 + \left(\frac{\partial \bar{x}^3}{\partial x^2}\right)^2 - \left(\frac{\partial \bar{x}^4}{\partial x^2}\right)^2 = 0 \\
g_{33} &= \left(\frac{\partial \bar{x}^1}{\partial x^3}\right)^2 + \left(\frac{\partial \bar{x}^2}{\partial x^3}\right)^2 + \left(\frac{\partial \bar{x}^3}{\partial x^3}\right)^2 - \left(\frac{\partial \bar{x}^4}{\partial x^3}\right)^2 = 0 \\
g_{44} &= \left(\frac{\partial \bar{x}^1}{\partial x^4}\right)^2 + \left(\frac{\partial \bar{x}^2}{\partial x^4}\right)^2 + \left(\frac{\partial \bar{x}^3}{\partial x^4}\right)^2 - \left(\frac{\partial \bar{x}^4}{\partial x^4}\right)^2 = 0
\end{aligned}
\tag{1}
$$

We deal with flat, Minkowski metric and we do not need a general solution of this system of equations. The metric we will obtain will be the flat metric as well. Therefore a feasible choice is that the coordinate transformation solution could be linear ones. Furthermore, we need nonsingular coordinate transformations so we have to test whether the Jacobian of the obtained transformations is different from zero. We present a few solutions and the appropriate Jacobians.

$$
R_{\rho\nu\xi\psi} = \bar{R}_{\alpha\beta\gamma\delta} \frac{\partial \bar{x}^\alpha}{\partial x^\rho} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \frac{\partial \bar{x}^\gamma}{\partial x^\xi} \frac{\partial \bar{x}^\delta}{\partial x^\psi}
$$

$$
\begin{aligned}
\bar{x}^1 &= Ax^1 + Mx^4 \\
\bar{x}^2 &= Bx^2 \\
\bar{x}^3 &= Cx^3 \\
\bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + Mx^4
\end{aligned}
\qquad
\left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right| = \begin{vmatrix} A & 0 & 0 & M \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ A & B & C & M \end{vmatrix} = 0
\tag{2}
$$

Where A, B, C, and M are the constants (different from zero).

Similar inconvenience, $J = \left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right|$, we have in the following, similar examples of the solutions:

$$
\begin{aligned}
\bar{x}^1 &= Ax^1 & \bar{x}^1 &= Ax^1 \\
\bar{x}^2 &= Bx^2 + Mx^4 & \bar{x}^2 &= Bx^2 \\
\bar{x}^3 &= Cx^3 & \bar{x}^3 &= Cx^3 + Mx^4 \\
\bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + Mx^4 & \bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + Mx^4
\end{aligned}
\tag{3}
$$

There are some more solutions, also linear, which provide the nonsingular coordinate transformations and we like to emphasize them:

$$\bar{x}^1 = Ax^1 + Mx^4$$
$$\bar{x}^2 = Bx^2 + Nx^4$$
$$\bar{x}^3 = Cx^3$$
$$\bar{x}^4 = Ax^1 + Bx^2 + Cx^3 + \sqrt{M^2 + N^2}\,x^4$$

$$\left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right| = \begin{vmatrix} A & 0 & 0 & M \\ 0 & B & 0 & N \\ 0 & 0 & C & 0 \\ A & B & C & \sqrt{M^2+N^2} \end{vmatrix} \neq 0$$

We introduced a constant $N \neq 0$. For brevity, we shall use $\gamma \equiv \sqrt{M^2 + N^2}$.
This solution provides the following metric tensor which we preset in the form of matrix:

$$\{g_{\rho\nu}\} = \begin{Bmatrix} 0 & -AB & -AC & -A(M-\gamma) \\ -AB & 0 & -BC & -B(N-\gamma) \\ -AC & -BC & 0 & -C\gamma \\ -A(M-\gamma) & -B(N-\gamma) & -C\gamma & 0 \end{Bmatrix} \qquad (4)$$

We preset three more coordinate transformations as the solutions which provide the nonsingular metric tensor.

Two of them are similar to the previous one. We use again $\gamma \equiv \sqrt{M^2 + N^2}$.

$$\begin{aligned} \bar{x}^1 &= Ax^1 + Mx^4 & \bar{x}^1 &= Ax^1 \\ \bar{x}^2 &= Bx^2 & \bar{x}^2 &= Bx^2 + Mx^4 \\ \bar{x}^3 &= Cx^3 + Nx^4 & \bar{x}^3 &= Cx^3 + Nx^4 \\ \bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + \gamma x^4 & \bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + \gamma x^4 \end{aligned} \qquad (5)$$

The nonsingularity of these metrics is easier to check through their Jacobians. We present the two Jacobians:

$$\left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right| = \begin{vmatrix} A & 0 & 0 & M \\ 0 & B & 0 & 0 \\ 0 & 0 & C & N \\ A & B & C & \gamma \end{vmatrix} = 0 \qquad \left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right| = \begin{vmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & M \\ 0 & 0 & C & N \\ A & B & C & \gamma \end{vmatrix} = 0 \qquad (6)$$

These coordinate transformations provide the following metric tensors. The coordinate transformation presented on the left side of (5) provides "left" Jacobian of (6) and the following metric tensor:

$$\{g_{\rho\nu}\} = \begin{Bmatrix} 0 & -AB & -AC & -A(M-\gamma) \\ -AB & 0 & -BC & -B\gamma \\ -AC & -BC & 0 & -C(N-\gamma) \\ -A(M-\gamma) & -B\gamma & -C(N-\gamma) & 0 \end{Bmatrix} \qquad (7)$$

The coordinate transformation and Jacobian presented on the right side of (5) and (6), provide the following metric tensor:

$$\{g_{\rho\nu}\} = \begin{Bmatrix} 0 & -AB & -AC & -A\gamma \\ -AB & 0 & -BC & B(M-\gamma) \\ -AC & -BC & 0 & -C(N-\gamma) \\ -A\gamma & B(M-\gamma) & -C(N-\gamma) & 0 \end{Bmatrix} \tag{8}$$

There is one more example of the nonsingular coordinate transformation we would like to present here:

$$\begin{aligned} \bar{x}^1 &= Ax^1 + Mx^4 \\ \bar{x}^2 &= Bx^2 + Nx^4 \\ \bar{x}^3 &= Cx^3 + Px^4 \\ \bar{x}^4 &= Ax^1 + Bx^2 + Cx^3 + \sqrt{M^2+N^2+P^2}\,x^4 \end{aligned} \qquad \left|\frac{\partial \bar{x}^\alpha}{\partial x^\rho}\right| = \begin{vmatrix} A & 0 & 0 & M \\ 0 & B & 0 & N \\ 0 & 0 & C & P \\ A & B & C & \sqrt{M^2+N^2+P^2} \end{vmatrix}$$

We introduced one more constant, $N \neq 0$. For brevity, we use $\sqrt{M^2+N^2+P^2} \equiv \psi$. This solution provides the following metric tensor:

$$\{g_{\rho\nu}\} = \begin{Bmatrix} 0 & -AB & -AC & -A(M-\psi) \\ -AB & 0 & -BC & B(N-\psi) \\ -AC & -BC & 0 & -C(P-\psi) \\ -A(M-\psi) & B(N-\psi) & -C(P-\psi) & 0 \end{Bmatrix} \tag{9}$$

We found the coordinate transformations which transform the diagonal Minkowski metric tensor in a metric tensor with all zero diagonal components. Some of the obtained coordinate transformations provide the singular metric tensors and we point out those ones, (4), (7), (8) and (9), which provide the nonsingular metrics.

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METRIČKI TENZOR SA NULTIM ELEMENTIMA NA DIJAGONALI

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Prikazujemo koordinatne transformacije koje metriku Minkovskog prevode u metriku čiji su svi dijagonalni elementi jednaki nuli. Našli smo nekoliko rešenja i istakli smo ona rešenja koja daju nesingularan metrički tenzor.

Ključne reči: *dijagonalni metrički tenzor Minkowskog, nulti dijagonalni elementi, transformacije koordinata.*