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TWO-FREQUENCY NONLINEAR VIBRATIONS OF ANTISYMMETRIC LAMINATED ANGLE-PLY PLATE

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Abstract. *In this paper two-frequency vibrations of laminated angle-ply rectangular plate which is freely supported on its own edges are analyzed. The classical Kirhhoff theory is used and the vibration equations of Karman type are analyzed using the Airy function. Asymptotic solution in the first approximation is given. Numerical example includes analysis of the two-frequency plate vibrations under non-stationary conditions and under the activity of time-dependent external impulse. Amplitude-frequency and phase-frequency characteristics of plate under non-stationary conditions for different laminate characteristics are presented graphically.*

Key words: *two-frequency nonlinear vibrations, laminated plate, amplitude, phase, frequency, Method Крылов-Боголюбов-Митропольский.*

1. INTRODUCTION

The problem of laminated composite vibrations has been the object of consideration during the past five decades. The equations of laminated plate vibrations are essentially identical to those for a single-layer orthotropic plate. Jones [6] gave the fundamental basis for tension-deformation state of laminated plates and differential equations of linear plates vibrations. Khdeir and Reddy [7] consider the free vibrations of laminated composite plates, for different boundary conditions, comparing the Kirhhoff theory with the applied one. Tylikovski [10] considers stability of nonlinear symmetrical laminated cross-ply plates. The equation of vibration of a cross-ply laminated plate is derived by introduction of Airy function. Ghazarian and Locke [1] with the invoking of Galerkin method determine equations of laminated plate vibrations, which are simple for analysis.

Very applicable asymptotic method of Крылов-Боголюбов-Митропольского [8] for solving of nonlinear vibrations continuum problems is applied in papers of K. Hedrih [2], [3], [4], [5], Pavlović [2], [9] and Kozić and Sl. Mitić [2]. Janevski [11], [12] analyzes a single frequency vibration laminated plate and considers influence of mechanical and other characteristics on the amplitude and phase of the asymptotic solution.

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In the present paper two-frequency vibrations of a laminated plate under the time dependent external force effect are considered. Also, the influence of mechanical and other characteristics on the amplitude and phase of the asymptotic solution is given in the first approximation.

2. PROBLEM FORMULATION

Components of the deformation tensor and components of the curvature of the plate middle surface are defined as follows:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \ \{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \tag{1}$$

where u(x,y,t), v(x,y,t) are the in-plane displacements and w(x,y,t) is the displacement normal to the middle surface of plate.

Membrane forces, moments of bending and torsion moment in the cross section along the axes can be presented as:

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}, \ \{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}. \tag{2}$$

The relationship between the forces and the moments in the middle surface of the plate is expressed by the equation

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = [C]\begin{Bmatrix} \{\varepsilon\} \\ \{\kappa\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix}\begin{Bmatrix} \{\varepsilon\} \\ \{\kappa\} \end{Bmatrix}, \tag{3}$$

Matrix of stiffness [C] for antisymmetric angle-ply laminates has the form:

$$[C] = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & \mathbf{D}_{11} & \mathbf{D}_{12} & 0 \\ 0 & 0 & B_{26} & \mathbf{D}_{12} & \mathbf{D}_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & \mathbf{D}_{66} \end{bmatrix}, \tag{4}$$

and the matrices of extensional stiffness [A], coupling stiffness [B] and bending stiffness [D] are defined as:

$$[A]=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix},\ [B]=\begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix},\ [D]=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}. \qquad (5)$$

Elements the matrix of stiffness are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij} dz ,$$

where Q_{ij} are the reduced in-plane stiffness of an individual lamina, and h is the thickness of the plate.

Differential equations describing the plate vibrations are obtained from the condition that forces and moments in the coordinate direction are balanced dynamically

$$\begin{gathered} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 , \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 , \qquad (6) \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = \rho h \frac{\partial^2 w}{\partial t^2} + 2\beta\rho h \frac{\partial w}{\partial t} , \end{gathered}$$

where ρ is the density of the plate material and d is the damping coefficient.

From equation (3) the moment of bending as well as the moment of torsion components can be expressed in terms of the transverse displacement of the middle surface plate:

$$\begin{aligned} M_x &= B_{16}\gamma_{xy} - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}, \\ M_y &= B_{26}\gamma_{xy} - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}, \qquad (7) \\ M_{xy} &= B_{16}\varepsilon_x + B_{26}\varepsilon_y - 2D_{66}\frac{\partial^2 w}{\partial x \partial y}. \end{aligned}$$

Introducing the function of tension $\psi = \psi(x, y, t)$ so that

$$N_x = \frac{\partial^2 \psi}{\partial y^2},\ N_y = \frac{\partial^2 \psi}{\partial x^2},\ N_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y} , \qquad (8)$$

the first and the second equation of the system Eq. (6) are satisfied. The condition of the deformation compatibility can be expressed as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} . \qquad (9)$$

and according to Eq. (1) it can be rewritten in the form:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}=\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{10}$$

From equation (3) it follows that

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = [C]^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}, \tag{11}$$

where $[C]^{-1}$ is the inverse of the matrix of stiffness [C]:

$$[C]^{-1}=\begin{bmatrix} A_{11}^* & A_{12}^* & 0 & 0 & 0 & B_{16}^* \\ A_{12}^* & A_{22}^* & 0 & 0 & 0 & B_{26}^* \\ 0 & 0 & A_{66}^* & B_{16}^* & B_{26}^* & 0 \\ 0 & 0 & B_{16}^* & D_{11}^* & D_{12}^* & 0 \\ 0 & 0 & B_{26}^* & D_{12}^* & D_{22}^* & 0 \\ B_{16}^* & B_{26}^* & 0 & 0 & 0 & D_{66}^* \end{bmatrix} \tag{12}$$

From Eqs. (8), (11) and (12) the components of the tensor of deformation can be expressed in terms of the function of tension

$$\begin{aligned} \varepsilon_x &= A_{11}^*\frac{\partial^2 \psi}{\partial y^2}+A_{12}^*\frac{\partial^2 \psi}{\partial x^2}+B_{16}^* M_{xy}, \\ \varepsilon_y &= A_{12}^*\frac{\partial^2 \psi}{\partial y^2}+A_{22}^*\frac{\partial^2 \psi}{\partial x^2}+B_{26}^* M_{xy}, \\ \gamma_{xy} &= -A_{66}^*\frac{\partial^2 \psi}{\partial x \partial y}+B_{16}^* M_x+B_{26}^* M_y, \end{aligned} \tag{13}$$

Substituting Eqs. (7) and (8) into the third equation of the system Eq. (6) and including Eq. (13), after its differentiation, into the left-hand side of Eq. (10) results in:

$$\rho h\frac{\partial^2 w}{\partial t^2}+2\beta\rho h\frac{\partial w}{\partial t}+L_{AU}(w)+e_1\frac{\partial^4 \psi}{\partial x^3 \partial y}+e_2\frac{\partial^4 \psi}{\partial x \partial y^3}-L(w,\psi)=q(x,y,t), \tag{14}$$

$$\Theta_{AU}\psi=-\frac{1}{2}L(w,w)-k_1\frac{\partial^4 w}{\partial x^3 \partial x}-k_2\frac{\partial^4 w}{\partial x \partial y^3}, \tag{15}$$

where q(x,y,t) is external disturbing force. The following denotations is used:

$$L(w,\psi)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \psi}{\partial x\partial y} \tag{16}$$

$$L_{AU}(w)=g_{11}\frac{\partial^4 w}{\partial x^4}+g_{12}\frac{\partial^4 w}{\partial x^2\partial y^2}+g_{22}\frac{\partial^4 w}{\partial y^4}$$

$$g_{11}=b_6B_{16}(B_{16}^*D_{11}+B_{26}^*D_{12})+D_{11},$$

$$g_{12}=b_6B_{16}(B_{16}^*D_{12}+B_{26}^*D_{22})+b_6B_{26}(B_{16}^*D_{11}+B_{26}^*D_{12})+2[D_{12}+2b_6D_{66}],$$

$$g_{22}=b_6B_{26}(B_{16}^*D_{12}+B_{26}^*D_{22})+D_{22},$$

$$e_1=b_6(B_{16}A_{66}^*-2(B_{16}A_{12}^*+B_{26}A_{22}^*)),$$

$$e_2=b_6(B_{26}A_{66}^*-2(B_{16}A_{11}^*+B_{26}A_{12}^*)),$$

$$\Theta_{AU}\psi=h_{11}\frac{\partial^4 \psi}{\partial x^4}+h_{12}\frac{\partial^4 \psi}{\partial x^2\partial y^2}+h_{22}\frac{\partial^4 \psi}{\partial y^4}, \tag{17}$$

$$h_{11}=b_6B_{26}^*(B_{16}A_{12}^*+B_{26}A_{22}^*)+A_{22}^*,$$

$$h_{12}=b_6B_{16}^*(B_{16}A_{12}^*+B_{26}A_{22}^*)+b_6B_{26}^*(B_{16}A_{11}^*+B_{26}A_{12}^*)+(b_6A_{66}^*+2A_{12}^*),$$

$$h_{22}=b_6B_{16}^*(B_{16}A_{11}^*+B_{26}A_{12}^*)+A_{11}^*,$$

$$k_1=b_6((B_{16}^*D_{11}+B_{26}^*D_{12})-2B_{26}^*D_{66}),$$

$$k_2=b_6((B_{16}^*D_{12}+B_{26}^*D_{22})-2B_{16}^*D_{66}),$$

Eqs. (14) and (15) are differential equations of the plate vibrations. Solving the Eqs. (14) and (15), under known boundary and initial conditions, one can determine transverse displacements of the middle surface $w(x,y,t)$ of a laminated plate, as well as the function of tension $\psi(x,y,t)$. Also according to the equations (7), (8) and (13) all necessary components of the tensor of deformation, forces and moments are determined.

3. TWO-FREQUENCY VIBRATIONS OF ANTISYMMETRIC LAMINATED ANGLE-PLY PLATES

Let us consider plate vibration described by the system of differential equations (14)-(15). Suppose that disturbing force q(x,y,t) is acting on the system. The force is 2π -periodical of $\theta_1(t)$ and $\theta_2(t)$ with the constant amplitude P_1^* and P_2^* in the form

$$q(x,y,t) = \varepsilon(P_1^* \sin\theta_1 \cdot w_{11}(x,y) + P_2^* \sin\theta_2 \cdot w_{12}(x,y)), \tag{18}$$

where $\frac{d\theta_i}{dt} = \nu_i(t)$ (i = 1,2) is momentary frequency and ε is a small positive parameter. For the laminated plate, freely supported along edges, boundary conditions are

$$\begin{aligned} &\left.\begin{matrix} x=0 \\ x=a \end{matrix}\right| \to w=0;\ M_x=0,\ N_x=0,\ N_{xy}=0; \\ &\left.\begin{matrix} y=0 \\ y=b \end{matrix}\right| \to w=0;\ M_y=0,\ N_y=0,\ N_{xy}=0; \end{aligned} \tag{19}$$

Let the initial conditions be

$$\begin{aligned} w(x,y,t)\Big|_{t=0} &= \sum_{j=1}^{2} p_{1j} w_{1j}(x,y), \\ \left.\frac{\partial w(x,y,t)}{\partial t}\right|_{t=0} &= \sum_{j=1}^{2} q_{1j} w_{1j}(x,y), \end{aligned} \tag{20}$$

where $w_{ij}(x,y) = \sin(\frac{i\pi}{a}x)\sin(\frac{j\pi}{b}y)$ are arbitrary normal functions and p_{1j} and q_{1j} are real numbers. According to the boundary and initial conditions (Eqs. (19), (20)), in the two-frequency regime of the plate vibrations, the transverse displacement w(x,y,t) as the solution of the system Eqs. (14)-(15) is supposed in the form

$$w(x,y,t) = f_1(t)\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) + f_2(t)\sin(\frac{\pi x}{a})\sin(\frac{2\pi y}{b}), \tag{21}$$

where $f_1(t)$ is the unknown function of time, which will be determined from the equation of vibration.

Taking Eq. (16) into consideration, function L(w,w) is evaluated in the form:

$$L(w,w) = 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2,$$

and included in the Eq. (15). Solving of partial differential equation one determines the function of tension in the form

$$\Psi(x,y,t)=-\frac{1}{4\lambda^2\cdot a_{22}}f_1(t)f_2(t)\cos(\frac{\pi y}{b})+\frac{\lambda^2}{32\cdot a_{11}}(f_1^2(t)+4f_2^2(t))\cos(\frac{2\pi x}{a})+$$
$$\frac{1}{32\cdot\lambda^2\cdot a_{22}}f_1^2(t)\cos(\frac{2\pi y}{b})+\frac{1}{36\cdot\lambda^2\cdot a_{22}}f_1(t)f_2(t)\cos(\frac{3\pi y}{b})+\frac{1}{128\cdot\lambda^2\cdot a_{22}}f_2^2(t)\cos(\frac{4\pi y}{b})+$$
$$+\frac{9}{4}\frac{\lambda^2}{16a_{11}+4\lambda^2\cdot a_{12}+\lambda^4\cdot a_{22}}f_1(t)f_2(t)\cos(\frac{2\pi x}{a})\cos(\frac{\pi y}{b})- \tag{22}$$
$$-\frac{1}{4}\frac{\lambda^2}{16a_{11}+36\lambda^2\cdot a_{12}+81\lambda^4\cdot a_{22}}f_1(t)f_2(t)\cos(\frac{2\pi x}{a})\cos(\frac{3\pi y}{b})+$$
$$+\frac{\lambda(k_1+\lambda^2k_2)}{h_{11}+\lambda^2h_{12}+\lambda^4h_{22}}f_1(t)\cos(\frac{\pi x}{a})\cos(\frac{\pi y}{b})+\frac{\lambda(2k_1+8\lambda^2k_2)}{h_{11}+4\lambda^2h_{12}+16\lambda^4h_{22}}f_2(t)\cos(\frac{\pi x}{a})\cos(\frac{2\pi y}{b})$$

where $\lambda=a/b$ is the ratio of the plate sides.

Multiply Eq. (14) by $w_{11}(x,y)dxdy$ and $w_{12}(x,y)dxdy$, after substitution of disturbing force equation (18) and expressions Eqs. (21)-(22) in its right and left-hand side respectively, to integrate over the plate surface ($x\in(0,a)$, $y\in(0,b)$). Substituting

$$\xi_1(t)=\frac{f_1(t)}{h},\ \xi_2(t)=\frac{f_2(t)}{h} \tag{23}$$

after the integration, we will obtain differential equations in unknown functions $\xi_1(t)$ and $\xi_2(t)$

$$\begin{aligned}\ddot{\xi}_1+\omega_1^2\xi_1&=-2\beta\dot{\xi}_1+\alpha_1\xi_1^3+\beta_1\xi_1\xi_2^2+\varepsilon P_1\sin\theta_1,\\ \ddot{\xi}_2+\omega_2^2\xi_2&=-2\beta\dot{\xi}_2+\alpha_2\xi_2^3+\beta_2\xi_1^2\xi_2+\varepsilon P_2\sin\theta_2.\end{aligned} \tag{24}$$

where:

$$\omega_1^2=\frac{1}{\rho h}\frac{\pi^4}{a^4}\left(g_{11}+\lambda^2g_{12}+\lambda^4g_{22}-\lambda^2\frac{(k_1+\lambda^2k_2)(e_1+\lambda^2e_2)}{h_{11}+\lambda^2h_{12}+\lambda^4h_{22}}\right) \tag{25a}$$

$$\omega_2^2=\frac{1}{\rho h}\frac{\pi^4}{a^4}\left(g_{11}+4\lambda^2g_{12}+16\cdot\lambda^4g_{22}-4\lambda^2\frac{(k_1+4\cdot\lambda^2k_2)(e_1+\lambda^2e_2)}{h_{11}+4\cdot\lambda^2h_{12}+16\cdot\lambda^4h_{22}}\right) \tag{25b}$$

$$\alpha_1=-\frac{1}{16}\frac{h}{\rho}\frac{\pi^4}{a^2b^2}\left(\frac{\lambda^2}{h_{11}}+\frac{1}{\lambda^2h_{22}}\right),\ \alpha_2=-\frac{1}{16}\frac{h}{\rho}\frac{\pi^4}{a^2b^2}\left(\frac{16\lambda^2}{h_{11}}+\frac{1}{\lambda^2h_{22}}\right) \tag{26}$$

$$\beta_1=\beta_2=-\frac{h}{\rho}\frac{\pi^4}{a^2b^2}\cdot\left(\frac{\lambda^2}{4\cdot h_{11}}+\frac{1}{4\cdot\lambda^2\cdot h_{22}}\right)-$$
$$-\frac{h}{\rho}\frac{\pi^4}{a^2b^2}\cdot\cdot\left(\frac{81}{16}\frac{\lambda^2}{16h_{11}+4\lambda^2\cdot h_{12}+\lambda^4\cdot h_{22}}+\frac{1}{16}\frac{\lambda^2}{16h_{11}+36\lambda^2\cdot h_{12}+81\lambda^4\cdot h_{22}}\right). \tag{27}$$

The Eq. (24) represents differential equations of the forced vibrations of the plate in the two-frequency regime with the frequency given by the Eq. (25).

For the system of forced vibrations of the plate described by Eq. (24) we can suppose [5] asymptotic solution with the boundary (Eq. (19)) and initial conditions (Eq. (20)) in the form of the infinite series [8]:

$$\begin{aligned}\xi_j = \sum_{s=1}^{2} A_j^{(s)} a_s \cos\psi_s + \varepsilon u_j^{(1)}(\tau,\theta_1,\theta_2,a_1,a_2,\psi_1,\psi_2) + \\ + \sum_{s=1}^{2} \varepsilon^2 u_j^{(2)}(\tau,\theta_1,\theta_2,a_1,a_2,\psi_1,\psi_2) + \ldots \ ,\end{aligned} \tag{28}$$

where $\tau=\varepsilon t$ is "slowly-changed time" and $u_j^{(1)}(\tau,\theta_1,a_1,\varphi_1)$, $u_j^{(2)}(\tau,\theta_1,a_1,\varphi_1)$,..... are periodical functions whose arguments are: θ_1 and φ_1 with the period 2π. Amplitude and phase of the solution (Eq. (28)) can be found from the differential equations

$$\begin{aligned}\frac{da_s}{dt} &= \varepsilon A_1^{(s)}(\tau,a_1,a_2,\varphi_1,\varphi_2) + \varepsilon^2 A_2^{(s)}(\tau,a_1,a_2,\varphi_1,\varphi_2) + \ldots.. \ , \\ \frac{d\varphi_s}{dt} &= \omega_s - \frac{p_s}{q_s}\nu_s + \varepsilon B_1^{(s)}(\tau,a_1,a_2,\varphi_1,\varphi_2) + \varepsilon^2 B_2^{(s)}(\tau,a_1,a_2,\varphi_1,\varphi_2) + \ldots... \ ,\end{aligned} \tag{29}$$

where A_1, B_1, A_2, B_2, ...are unknown functions in "slowly-changed time" amplitude and phase. These functions can be determined from the supposed solution (Eq. (28)) in the equation (24) equalizing the coefficients of the same harmonics. Staying on the first approximation, the solution of equation (24) will have the form

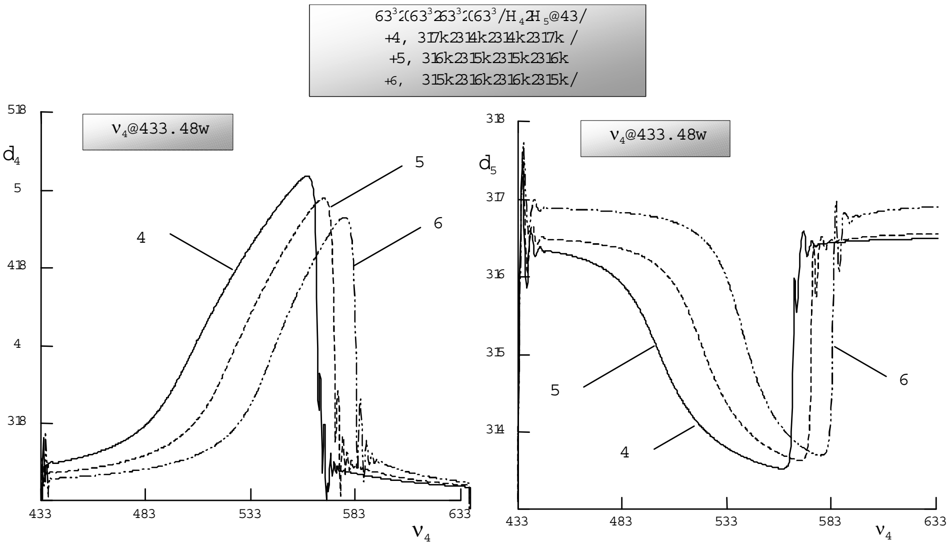
$$\xi_1 = a_1\cos(\nu_1 t + \varphi_1), \xi_2 = a_2\cos(\nu_2 t + \varphi_2)\ , \tag{30}$$

where differential equations in the first approximation will be

$$\begin{aligned}\frac{da_1}{dt} &= -\beta a_1 - \frac{\varepsilon P_1}{\omega_1+\nu_1}\cos\varphi_1\ , \frac{d\varphi_1}{dt} = \omega_1 - \nu_1 - \frac{3}{8}\frac{\alpha_1}{\omega_1}a_1^2 - \frac{1}{4}\frac{\beta_1}{\omega_1}a_2^2 + \frac{\varepsilon P_1}{a_1(\omega_1+\nu_1)}\sin\varphi_1\ , \\ \frac{da_2}{dt} &= -\beta a_2 - \frac{\varepsilon P_2}{\omega_2+\nu_2}\cos\varphi_2\ , \frac{d\varphi_2}{dt} = \omega_2 - \nu_2 - \frac{3}{8}\frac{\alpha_2}{\omega_2}a_2^2 - \frac{1}{4}\frac{\beta_2}{\omega_2}a_1^2 + \frac{\varepsilon P_2}{a_2(\omega_2+\nu_2)}\sin\varphi_2\ .\end{aligned} \tag{31}$$

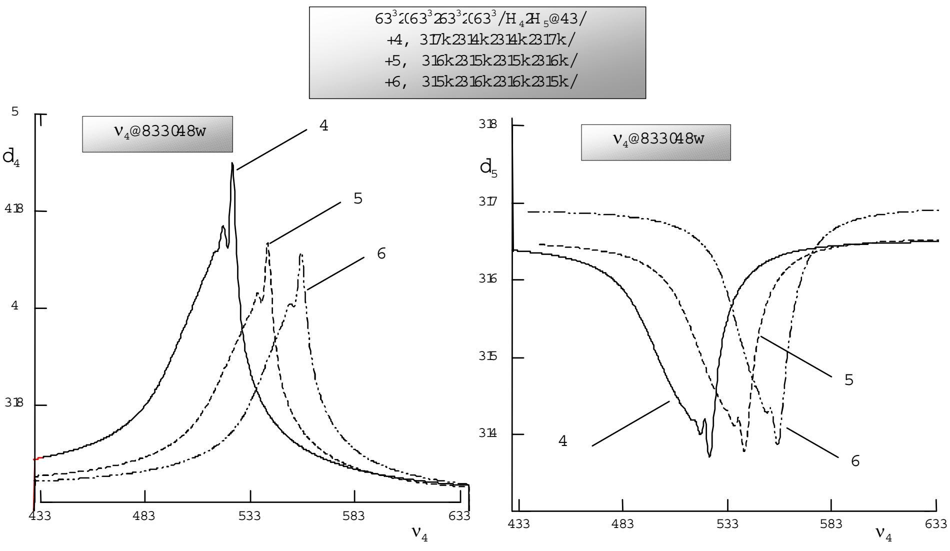
4. NUMERICAL ANALYSIS OF THE FORCED VIBRATIONS OF LAMINATED PLATES UNDER NON-STATIONARY CONDITIONS

The equations (31) are the first approximation differential equations of the asymptotical solution of differential equation (24). Numerical solving of these equations by means of the Runge-Kutta method (the fourth order), gives amplitude frequency characteristics of the two-frequency regime of laminated plate vibrations under non-stationary conditions. The dependence of these curves on changing of same laminate characteristics is given in the next examples.



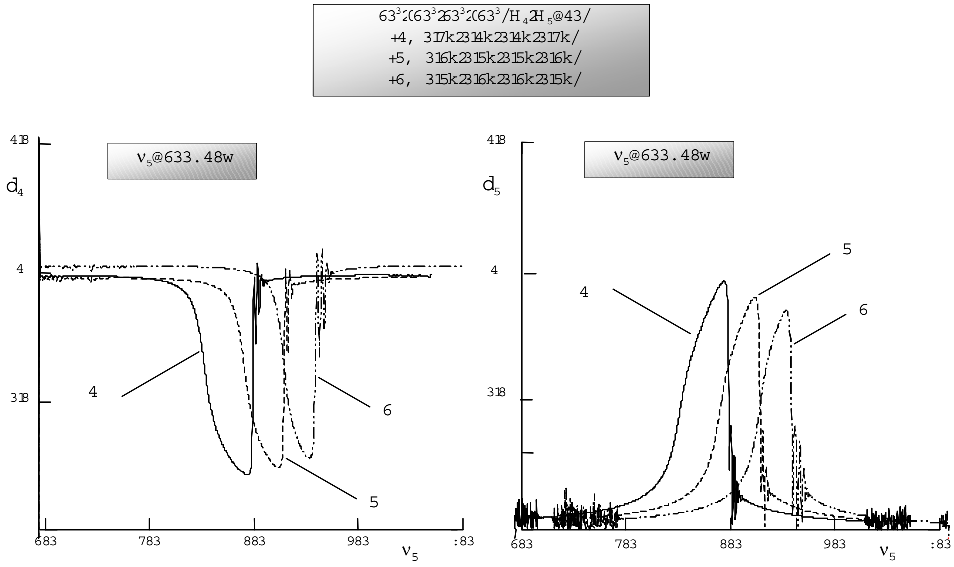
a) b)

Fig. 1. Amplitude-frequency characteristics for different thicknes of lamina
(ν_1 – linear increasing of time, ν_2 – constant)



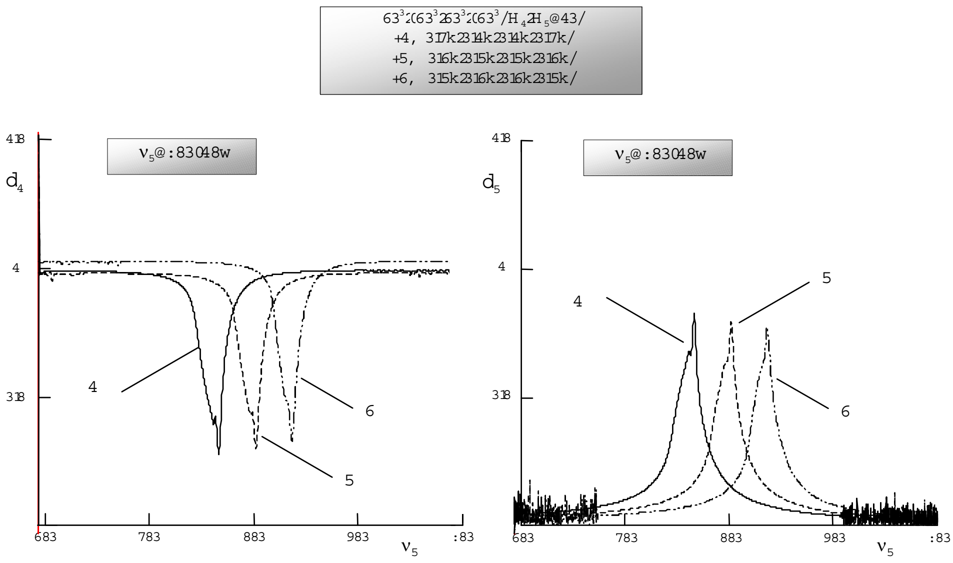
a) b)

Fig. 2. Amplitude-frequency characteristics for different thicknes of lamina
(ν_1 – linear decreasing of time, ν_2 – constant)



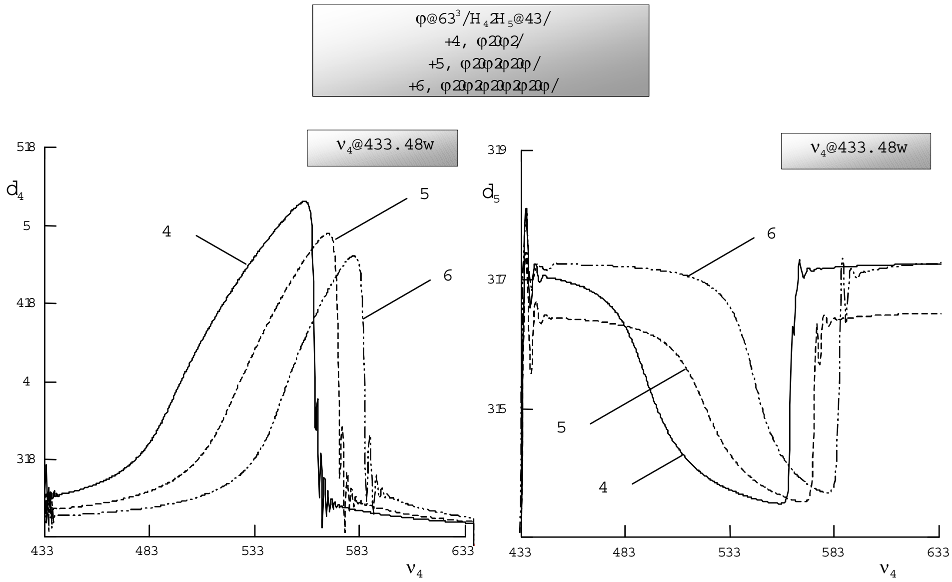
a) b)

Fig. 3. Amplitude-frequency characteristics for different thicknes of lamina
(ν_1 – constant , ν_2 – linear increasing of time)



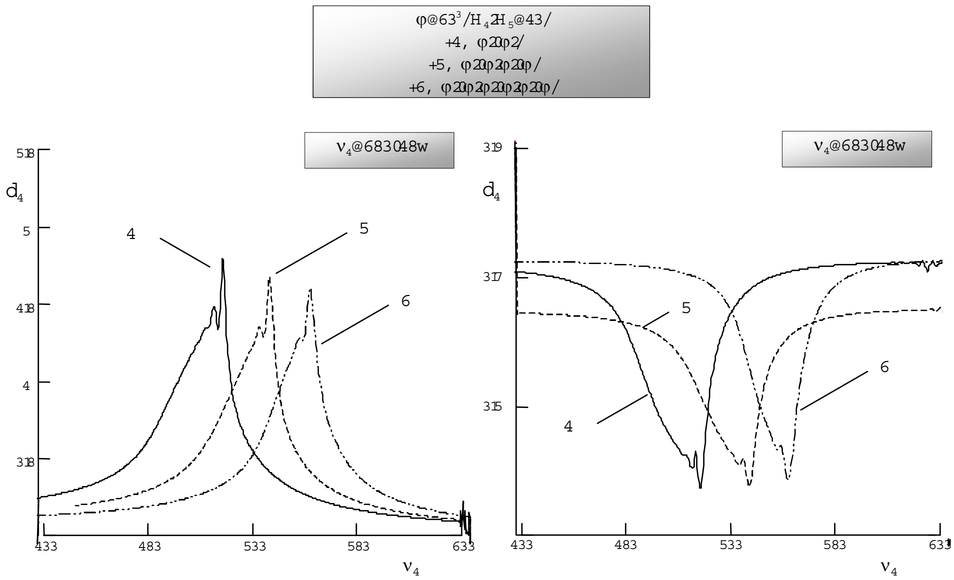
a) b)

Fig. 4. Amplitude-frequency characteristics for different thicknes of lamina
(ν_1 – constant , ν_2 – linear decreasing of time)



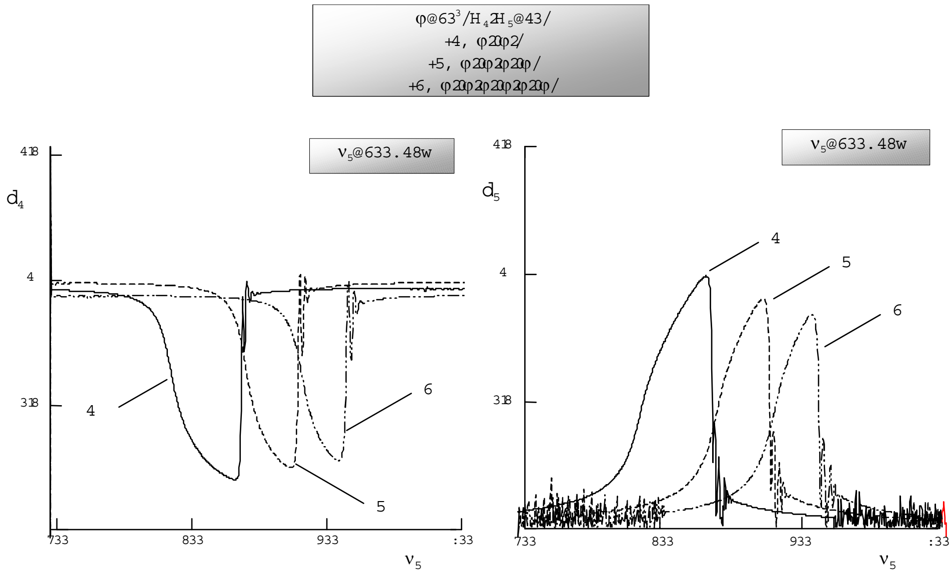
a) b)

Fig. 5. Amplitude-frequency characteristics for different number of lamina (ν_1 – linear increasing of time, ν_2 – constant)



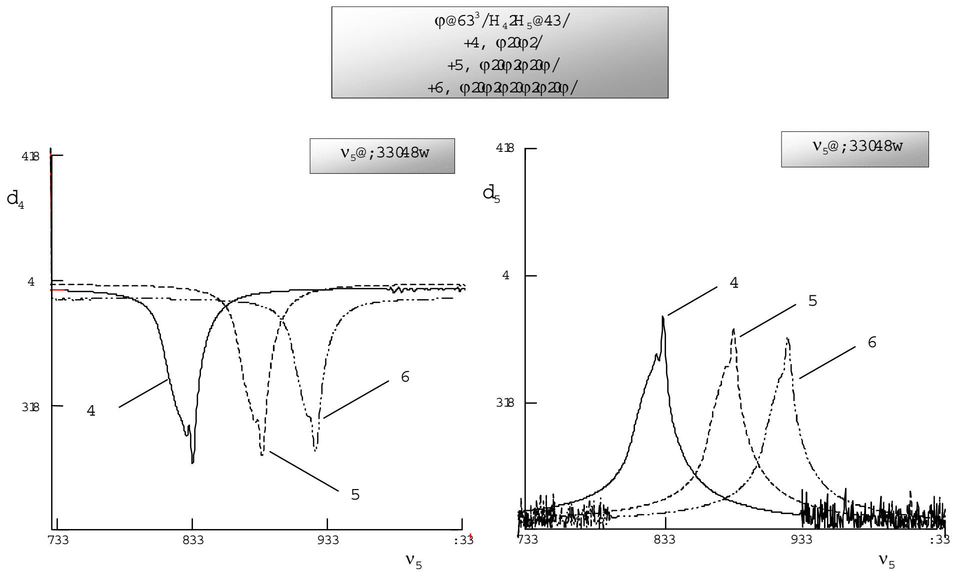
a) b)

Fig. 6. Amplitude-frequency characteristics for different number of lamina (ν_1 – linear decreasing of time, ν_2 – constant)



a) b)

Fig. 7. Amplitude-frequency characteristics for different number of lamina
(ν_1 – constant , ν_2 – linear increasing of time)



a) b)

Fig. 8. Amplitude-frequency characteristics for different number of lamina
(ν_1 – constant , ν_2 – linear decreasing of time)

Amplitude-frequency characteristics of a four-layered laminate ($30^0/-30^0/30^0/-30^0$, $E_1/E_2 = 10$) for different thicknesses of lamina are shown in Figs. 1-4. Amplitude-frequency characteristics at linear increasing and decreasing of external force frequency ν_1 are shown in Fig. 1-2. Amplitude-frequency characteristics at linear increasing and decreasing of external force frequency ν_2 are shown in Fig. 3-4.

Amplitude-frequency characteristics of a laminate ($\varphi = 30^0$, $E_1/E_2 = 10$) for different number of lamina are shown in Figs. 5-8. Amplitude-frequency characteristics at linear increasing and decreasing of external force frequency ν_1 are shown in Fig. 5-6. Amplitude-frequency characteristics at linear increasing and decreasing of external force frequency ν_2 are shown in Fig. 7-8.

5. CONCLUSIONS

On the basis of the analysis of the amplitude-frequency characteristics for the two-frequency regime of laminated plate vibrations under non-stationary conditions we can conclude:

- while increasing the thickness of the inside lamina (and decreasing thickness of the outside lamina), the amplitudes are decreasing,
- while increasing the number of lamina, the amplitudes are decreasing,

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DVOFREKVENTNE NELINEARNE OSCILACIJE ANTISIMETRIČNE UGAONE LAMELASTE PLOČE

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U radu su analizirane dvofrekventne oscilacije ugaone lamelaste ploče slobodno oslonjene na svojim krajevima. Korišćena je klasična Kirhhoff teorija i diferencijalne jednačine Karman-ovog tipa su analizirane koričćenjem Airy-jeve funkcije. Dato je asimptotsko rešenje u prvoj aproksimaciji. Numerički primer obuhvata dvofrekventne oscilacije ploče u nestacionarnom režimu oscilovanja pod dejstvom spoljašnje vremenski zavisne sile. Amplitudno-frekventne i fazno-frekventne karakteristike oscilovanja ploče u nestacionarnim uslovima za različite karakterisitke lamelata su date grafički.

Ključne reči: *Dvofrekventne nelinearne oscilacije, ugaona lamelasta ploča, amplituda, faza, asimptotska metoda Krilov-Bogoljubov-Mitropoljskij.*