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SOME QUATER-SYMMETRIC CONNECTIONS ON KAEHLERIAN MANIFOLDS

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Abstract. *There are two kinds of very naturaly introduced quarter-symmetric connections on both kinds of Kaehlerian spaces, elliptic and hyperbolic. Both of them are constructed over fundamental tensors and both are metric. But one of them is an F-connection (an uncompleted one) and the other one is not. In this paper, we try to construct a complete quarter-symmetric metric connection, which will be an F-connection or a "nearly" F-connection. (MSC 2002: 53A30).*

Key words: *Kaehlerian spaces, tensors, quarter-symmetric metric connections.*

1. INTRODUCTION

Since Hayden has in 1932 has introduced the idea of a metric connection with non-zero torsion, this topic has been studied by many authors. For example, Yano studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. Some other cases of semi-symmetric connections on spaces with F-structures (Kaehlerian and hyperbolic Kaehlerian) have been studied by M. Prvanović and the present author in papers (3), (4). These connections were metric F-connections. Since Gołąb has introduced the idea of a quarter-symmetric connection on any differentiable manifold, this topic has also came into the focus. Mishra and Pandey in (2) have studied quarter-symmetric F-connections in Riemannian manifolds with F-structures and Yano and Imai in (6) studied quarter-symmetric metric connections in Riemannian, Hermitian and Kaehlerian manifolds. The present author considered some curvature conditions of two kinds of quarter-symmetric metric connections on a hyperbolic Kaehlerian space in (5).

The aim of this paper is to give some supplement to the results of paper (5), avoiding much of the curvature theory and to give a possibility to previously defined natural quarter-symmetric connection, which is a metric connection, to be exceptionally an F-connection, or something most close to it.

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2. On a Hyperbolic Kaehlerian Manifold

A hyperbolic Kaehlerian space (manifold) is an even-dimensional pseudo-Riemannian manifold with an F-structure satisfying

$$F_k^i F_j^k = \delta_j^i, \tag{2.1}$$

$$F_{ij} = F_i^k g_{kj} = -F_{ji}, \tag{2.2}$$

$$\tilde{\nabla}_k F_j^i = 0, \tag{2.3}$$

where $\tilde{\nabla}$ stands for the Levi-Civita covariant differentiation operator in the underlying Riemannian manifold. By (2.2), the isomorphism F transforms (sends) any tangent vector into an orthogonal one; as the structure has n linearly independent eigenvectors, they are null or isotropic. That is the reason for being so geometrically different from (elliptic)Kaehlerian spaces, even if differences may formally be very small.

There are tangent vectors of positive scalar square, negative scalar square and null vectors. Null vectors may not be eigenvectors for the structure. There are several kinds of special bases of tangent space, which we can use if we find it convenient.

3. A Quarter-Symmetric Metric Connection

According to Yano and Imai (6), if there is given a quarter-symmetric onnection $\tilde{\nabla}$ with torsion tensor

$$T_{jk}^i = p_j A_k^i - p_k A_j^i \tag{3.1}$$

(p_i being components of a 1-form and A_k^j components of any (1, 1) tensor) and if it should be a metric one

$$\nabla_k g_{ij} = 0, \tag{3.2}$$

then its components will have the following form

$$\Lambda^i_{jk} = \{^{\,i}_{jk}\} - p_k U^{\,i}_{j} + p_j V_k^i - p^i V_{jk}, \tag{3.3}$$

where

$$U_{ij} = \tfrac{1}{2}(A_{ij} - A_{ji}); V_{ij} = \tfrac{1}{2}(A_{ij} + A_{ji}). \tag{3.4}$$

On a hyperbolic Kaehlerian space there are two fundamental tensors; one of them is symmetric and the other one is skew-symmetric. It is very natural to construct a quarter-symmetric metric connection over them.

If $V_{ij} = g_{ij}$ and $U_{ij} = F_{ij}$ then

$$\Lambda^i_{jk} = \{^{\,i}_{jk}\} - p_k F_j^i + p_j \delta_k^i - p^i g_{jk}, \tag{3.5}$$

and these are components of **natural quarter symmetric metric connection**, which has been introduced in (5). The same kind of connection exists in (elliptic) Kaehlerian space (6).

If $V_{ij} = 0$ and $U_{ij} = F_{ij}$ then

$$\Lambda^i_{jk} = \{^i_{jk}\} - p_k F^i_j \tag{3.6}$$

and these are components of **special quarter-symmetric metric connection**, which also has been introduced in (5) and which also exists in (elliptic) Kaehlerian spaces (6).

If $U_{ij} = 0$ and $V_{ij} = g_{ij}$ then we obtain components of semi-symmetric metric connection.

The connection with components (3.6) is an F-connection. The connection with components (3.5) is not an F-connection.

4. F-CONNECTIONS WITH $V_{ij} \neq 0$

It is easy to prove that a semi-symmetric connection is never an F-connection. If we suppose that it is, then

$$p_i F_{kj} - q_j g_{ik} - p_j F_{ki} + q_i g_{kj} = 0, \tag{4.1}$$

where $q_j = p_a F^a_j$. Then, after contracting by F^{jk} we obtain for both cases $p_i = 0$

Now we want to find a connection with coefficients of a type

$$\Lambda^i_{jk} = \{^i_{jk}\} + p_j V^i_k - p^i V_{jk}, \tag{4.2}$$

(V_{jk} is being a symmetric tensor) which is an F-connection. We will calculate separately for hyperbolic and elliptic Kaehlerian space.

If we suppose that the connection with coefficients (4.2) is an F-connection, then, for both cases, there holds

$$-p_i V^s_k F_{sj} - q_j V_{ik} - p_j V^s_k F_{is} + q_i V_{jk} = 0. \tag{4.3}$$

For the hyperbolic Kaehlerian space, after transvection by $F^i_l F^j_t$, we obtain

$$q_l V_{kt} - p_t V_{ik} F^i_l - p_j V_{kl} + p_l V_{kj} = 0. \tag{4.4}$$

After transvection by q^l, we can obtain

$$V_{kt} = \frac{1}{p^2}[p^i V_{ik} p_t - q_t V_{kl} q^l], \tag{4.5}$$

where p stands for the length of generating 1-form p if it is not isotropic.

Applying (4.5)to itself, we obtain

$$V_{kt} = \frac{1}{p^4}[p^i p^a V_{ia} p_k p_t - p^i q^a V_{ia} p_t q_k] - \frac{1}{p^2} V_{ks} q^s q_{t.} \tag{4.6}$$

Then, as the tensor V_{kt} is supposed to be symmetric, there must hold

1. $p_t q_k = p_k q_t$
2. $V_{ka} q^a q_t = V_{ta} q^a q_t$

From 1), there immediately holds $p^2 p_t = 0$ i. e. $p = 0$ and the generator is isotropic. From 2), we get

$$V_{ka}q^a q^k q_t = -p^2 V_{ta} q^a.$$

As the generator should be isotropic, then we get from (4.4)

$$p^l V_{lk} p_t = q_t V_{lk} q^l; \tag{4.7}$$

denote $V_k^l p_l$ by v_k and $V_{lk}q^l$ by u_k. Then

$$p_t v_k = q_t u_k. \tag{4.8}$$

But, from 1) there holds

$$p_t u_k = q_t v_k$$

and, consequently

$$u_k = \alpha v_k;\; q_t = \tfrac{1}{\alpha} p_t.$$

This means that p_t is an eigenvector for the structure, as eigenvalues for the structure are ± 1 then $q_t = \pm p_t$. Then we can rewrite (4.4) in the form

$$p_l (V_{ik} F_i^t \pm V_{kt}) = p_t (V_{ik} F_l^i \pm V_{kl}). \tag{4.9}$$

Let ω_k be such a vector that $p_t \omega^t = 1$Then

$$V_{ik} F_l^i \pm V_{kl} = p_l\ \omega^t (V_{ik} F_t^i \pm V_{kt}).$$

Denote $V_{ik} F_l^i \pm V_{kl}$ by B_{kl} Then

1. $B_{kl} = p_l \omega^t B_{kt}$
2. $B_{kl} F_s^t = \pm B_{ks}$.

As the tensor B depends on the tensor V, as the last one is symmetric and, in some sense, eigenvector for the structure, the only way to fulfill 1) and 2) is

$$V_{ik} = \alpha p_i p_k. \tag{4.10}$$

Then the quarter-symmetric metric connection of form (4.2) degenerates to Levi-Civita connection.

For the elliptic case, there holds from (4.3)

$$q_l V_{tk} + p_l V_{jk} F_t^j = q_t V_{kl} + p_t V_{ik} F_l^i. \tag{4.11}$$

Then,

$$V_{kt} = \frac{1}{p^2}[q_t V_{kl} q^l - p^i V_{ik} p_t]. \tag{4.12}$$

Applying (4.10) to itself twice, we obtain

$$V_{kt} = \frac{1}{p^4}[p^i p^a V_{ai} - p^i q^a V_{ia} q_k p_t] + \frac{1}{p^2} V_{ka} q^a q_t.$$

As V is symmetric, then

$$p_t q_k = p_k q_t. \tag{4.13}$$

After transvection by q^t (as forms p and q are mutually orthogonal) there holds

$$p^2 p_i = 0. \tag{4.14}$$

As the metric tensor of an elliptic Kaehlerian space is positive definite, then $p_t = 0$ and such a connection also degenerates to Levi-Civita connection.

We proved

Theorem1. On a Kaehlerian space, elliptic or hyperbolic, no metric connection with torsion of the form $T^i_{jk} = p_j A^i_k - p_k A^i_j$, where $A_{ij} \neq 0$ is symmetric, is an F-connection.

Then, if we consider the connection

$$\Lambda^i{}_{jk} = \{^{\,i}_{jk}\} - p_k F^i_j + p_j V^i_k - p^i V_{jk}, \tag{4.15}$$

(the tensor V being symmetric), we can expose it in this way

$$\Lambda^i{}_{jk} = L^i_{jk} + p_j V^i_k - p^i V_{jk}, \tag{4.16}$$

where L^i_{jk} are components of a special quarter-symmetric connection (5), which is a metric F-connection, we can repeat the all considerations which lead to the Theorem 1. These considerations do not depend on the fact that Levi-Civita connection is symmetric, but on the fact that it is a metric and F-connection. So, in a very analogous way, we can prove that there holds

Theorem 2. On a Kaehlerian space, elliptic or hyperbolic, no metric connection with coefficients of the form (4.13), where $V_{ij} \neq 0$ is a symmetric tensor, is not an F-connection.

5. THE RECURRENCE OF THE STRUCTURE

As we have no more than just one metric quarter-symmetric F-connection, it naturally raises the question is there any quarter-symmetric metric connection such that the structure is recurrent towards to it.

First, we consider the connection (4.2). Supposing that it gives the recurrence of the structure, we obtain

$$- p_i V_k^s F_{sj} + p^s V_{ik} F_{sj} - p_j V_k^s F_{is} + p^s V_{jk} F_{is} = \gamma_k F_{ij}. \tag{5.1}$$

Contracting (5.1) by F^{ji} for a hyperbolic and elliptic space simultaneously, we obtain

$$p_i V^i_k \pm p^i V_{ik} \mp p_j V^j_k \pm p_j V^j_k = \pm n\gamma_k. \tag{5.2}$$

The left-hand side evidently vanishes for both cases. Then the idea of being recurrent reduces to the idea of being parallel.

If we, however, consider the connection of type (4.13) (or, as well, (4.14)), the covariant derivative of the structure will look as the left-hand side of (5.2). So, we can state

Theorem 3. *On a Kaehlerian space, hyperbolic or elliptic, the structure's covariant derivative never has the form* $\nabla_k F_{ij} = \gamma_k F_{ij}$ *for metric quarter-symmetric connection of types (4.2) or (4.13).*

6. THE NEARLY F-CONNECTIONS

As we have got unsatisfactory answers to our previous questions, we shall consider our natural quarter-symmetric metric connection, whose coefficients are given by (3.6) and to check the possibility that it satisfies the condition

$$\nabla_l \nabla_k F_{ij} - \nabla_k \nabla_l F_{ij} = 0, \tag{6.1}$$

or,equivalently,

$$-R^s_{ikl} F_{sj} - R^s_{jkl} F_{is} = T^s_{kl} \nabla_s F_{ij}. \tag{6.2}$$

A connection satisfying the condition (6.1) (or(6.2)) will be called a nearly F-connection.

We consider the natural quarter-symmetric metric connection since it is the only complete connection of such a kind, generated by both fundamental tensors. First, we shall need some results from the curvature theory of such connections. They have all been proved in (5) and all are valid either for hyperbolic or elliptic Kaehlerian spaces.

$$\begin{aligned} R_{ijkl} = K_{ijkl} - g_{il} p_{kj} + g_{ik} p_{lj} - g_{kj} p_{li} + g_{lj} p_{ki} + p_j p_l F_{ik} + p_i p_k F_{jl} - \\ - p_j p_k F_{il} - p_i p_l F_{jk}, \end{aligned} \tag{6.3}$$

where p_{kj} stands for $\widetilde{\nabla}_k p_j - p_k p_j + p_k q_j + \frac{1}{2} p_a p^a g_{kj}$ (p_j) is a gradient and $q_j = F^s_j p_s$. Further, in (5) it has been proved that

$$p_j p_l = \frac{1}{(n-2)(n-4)} [R_{jk} - (n-3) R_{kj}] F^k_l - \frac{1}{n-4} p_s p^s g_{jl}, \tag{6.4}$$

$$p_j q_k = \frac{1}{(n-2)(n-4)} [R_{jk} - (n-3) R_{kj}] + \frac{1}{n-4} p_s p^s F_{jk}, \tag{6.5}$$

$$p_{kj} = \frac{1}{n-2} [K_{jk} - R_{jk} - g_{jk} \frac{K-R}{2(n-1)} - \frac{1}{(n-2)^2} (R_{kj} - R_{jk}) - \frac{1}{n-4} p_s p^s F_{jk}, \tag{6.6}$$

$$R_{jk} F^{jk} = 2(n-2) p_s p^s, \tag{6.7}$$

$$R_{kj} F^{jk} = -2(n-2) p_s p^s \tag{6.8}$$

We shall interpret (6.2) using (6.3-8). Due to delicateness of this work, we shall operate on the left and right-hand side separately.

On the right-hand side of (6.2), we obtain, for both kinds of spaces

$$\begin{aligned}-2p_ip_kF_{lj}-2p_kp_ig_{lj}-p_kq_jg_{il}-p_kq_jF_{li}+p_kp_jg_{il}+p_jp_kF_{li}+2p_lp_iF_{kj}+\\+2p_lp_ig_{kj}+p_lq_jg_{ik}+p_lq_jF_{ki}-p_jp_lg_{ik}-p_jp_lF_{ki}.\end{aligned} \quad (6.9)$$

Now we shall separate our spaces. We shall consider the hyperbolic case first.
After the contraction by F^{ik} (6.9), we get on the right-hand side for the hyperbolic case

$$(n+2)p_lq_j-(n-2)p_lp_j-q_lp_j-q_lq_j. \quad (6.10)$$

On the left-hand side of (6.2), we obtain by using (6.3-8), for both cases

$$\begin{aligned}&F_{lj}(p_{ki}+p_kq_i)-F_{kj}(p_{li}+p_lq_i)+F_{il}(p_{kj}+p_kq_j)-F_{ik}(p_{lj}+p_lq_j)+\\&+g_{jk}F_i^s(p_{ls}+q_sp_l)-g_{ik}F_j^s(p_{ls}+q_sp_l)+g_{il}F_j^s(p_{ks}+q_sp_k)-g_{jl}F_i^s(p_{ks}+q_sp_k)\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{n-2}[K_{ki}F_{lj}-K_{li}F_{kj}+K_{kj}F_{il}-K_{lj}F_{ik}+K_{ls}F_i^sg_{jk}-K_{ls}F_j^sg_{ik}+K_{ks}F^s{}_jg_{ik}-\\&-K_{ks}F_i^sg_{jl}]+\frac{2(n-4)}{(n-2)^2}(R_{ki}F_{lj}-R_{li}F_{kj}+R_{kj}F_{il}-R_{lj}F_{ik}+R_{ls}F_i^sg_{jk}-R_{ls}F_j^sg_{ik}+\\&+R_{ks}F_j^sg_{il}-R_{ks}F_i^sg_{jl}-\frac{2(n-3)}{(n-2)^2(n-4)}(R_{ik}F_{lj}-R_{il}F_{kj}+R_{jk}F_{il}-R_{jl}F_{ik}+R_{sl}F_i^sg_{jk}-\\&-R_{sl}F_j^sg_{ik}+R_{sk}F_j^sg_{il}-R_{sk}F_i^sg_{jl})+\frac{4p_ap^a}{n-4}(F_{ki}F_{lj}-F_{kj}F_{li}+g_{ik}g_{lj}-g_{il}g_{kj})+\\&+\frac{K-R}{(n-1)(n-2)}(F_{lj}g_{ki}-F_{kj}g_{li}+F_{il}g_{kj}-F_{ik}g_{lj}).\end{aligned} \quad (6.11)$$

For the hyperbolic case, after contraction (6.11) by F^{ik}, we obtain

$$\begin{aligned}&\frac{1}{n-2}((n-3)K_{lj}+K_{ks}F_j^sF_l^k+Kg_{lj})+\frac{2}{n-2}((n-3)R_{ij}+R_{ks}F_j^sF_l^k+Rg_{lj})-\\&-\frac{2(n-3)}{(n-2)^2(n-4)}((n-3)R_{jl}+R_{sk}F_j^sF_l^k+Rg_{lj})-\frac{8p_ap^a}{(n-2)(n-4)}F_{lj.}.\end{aligned} \quad (6.12)$$

Then (6.12) equals to (6.10). Transvecting (6.12) by g^{jl} we obtain

$$2K+4R+\frac{4(n-3)}{(n-2)(n-4)}R.$$

Transvecting (6.10) by g^{jl}, we obtain

$$-(n+1)p_ap^a.$$

Then, for the hyperbolic Kaehlerian space, there holds

$$2\mathrm{K}+4\frac{n^2-7n+11}{(n-2)(n-4)}R=-(n+1)p_ap^a. \quad (6.13)$$

Now, we shall transvect (6.12) by F^{jl}. We obtain

$$4p_a p^a \frac{-n^3+9n^2-27n+20}{(n-2)(n-4)}.$$

Then, for the hyperbolic Kaehlerian space, there holds

$$4p_a p^a \frac{-n^3+9n^2-27n+20}{(n-2)(n-4)} = -(n+3)p_a p^a$$

and, consequently:

$$\mathrm{p_a} p^a = 0, \tag{6.14}$$

$$2K = -4\frac{n^2-7n+11}{(n-2)(n-4)}R. \tag{6.15}$$

So, we have proved

Theorem 4. *If the natural quarter-symmetric metric connection on a hyperbolic Kaehlerian space should be a nearly F-connection, it is necessary to be $p_a\, p^a = 0$ and*

$$K = -2\frac{n^2-7n+11}{(n-2)(n-4)}R.$$

Now we turn our look to the elliptic case. First, we shall transvect right-hand side (6.9) by F^{ik}. We obtain

$$np_l p_j - (n+1)p_l q_j + p_j q_l - q_l q_j. \tag{6.16}$$

Then, we transvect (6.11) by F^{ik} and get

$$\begin{aligned}
&\frac{1}{n-2}((3-n)K_{lj} + K_{ks}F_j^s F_l^k - Kg_{lj}) + \\
&+\frac{2}{(n-2)^2(n-4)}((3-n)R_{lj} - Rg_{lj} + R_{ks}F_j^s F_l^k - 2(n-2)p_a p^a F_{lj}) - \\
&-\frac{2(n-3)}{(n-2)^2(n-4)}((3-n)R_{jl} - Rg_{jl} + R_{sk}F_j^s F_l^k + 2(n-2)p_a p^a F_{lj}) + \\
&+\frac{4np_a p^a}{n-4}F_{lj} - \frac{K-R}{n-1}g_{lj}.
\end{aligned} \tag{6.17}$$

Then (6.17) equals to (6.16). Transvecting (6.17) by g^{jl}, we obtain

$$-\frac{3n-2}{n-1}K + \frac{n^2+2n-4}{(n-2)(n-3)}R.$$

Transvecting (6.16) by g^{jl}, we obtain

$$(n-1)p_a p^a.$$

Then, there must hold

$$(n-1)p_a p^a = \frac{3n-2}{n-1}K + \frac{n^2 = 2n-4}{(n-2)(n-3)}R. \tag{6.18}$$

Transvecting (6.17) by F^{jl}, we obtain

$$4\frac{n^2+2n-4}{n-4}p_a p^a.$$

Transvecting (6.16) by F^{jl}, we obtain $-n\, p_a\, p^a$ Then there holds

$$4\frac{n^2+2n-4}{n-4}p_a p^a = -np_a p^a$$

and, consequently $p_a\, p^a = 0$ As the metric tensor of an (elliptic) Kahlerian space is positively definite, we have proved

Theorem 5. *A natural quarter-symmetric metric connection on an (elliptic) Kaehlerian space cannot be a nearly F-connection.*

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IZVESNE ČETVRT-SIMETRIČNE KONEKSIJE NA KELEROVIM MNOGOSTRUKOSTIMA

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Postoje dve vrste prirodno indukovane četvrt-simetrične koneksije na obema vrstama Kelerovih prostora, eliptičnom i hiperboličnom. Obe vrste koneksija su konstruisane pomoću fundamentalnih tenzora i obe su metričke. Ali jedna od njih je F-koneksija (nekompletna) a druga nije. U ovom radu, pokušavamo da konstruišemo kompletnu četvr-simetričnu koneksiju koja bi bila F-koneksija ili skoro F-koneksija.

Ključne reči: *četvrt-simetrična koneksija, metrička koneksija, F-koneksija, tenzor krivine, hiperbolička struktura.*