SOME MODELS OF CAUSALITY AND
STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We consider some concepts of causality between $\sigma$-algebras and between stochastic processes. Then, we give a generalization of a causality relationship "$G$ is a cause of $E$ within $H$" which was first given by Mykland [4] and which is based on Granger’s definition of causality [1].

In the second part we apply the results to systems of stochastic differential equations. More precisely, we give conditions for weakly uniqueness of the solution of some stochastic differential equations.

Key words: $\sigma$-algebra, causality, weak solution of stochastic differential equation.

1. INTRODUCTION AND NOTATION

The study of Granger causality has been mainly preoccupied with time series. We shall instead concentrate on continuous time processes. Many of systems to which it is natural to apply tests of causality take place in continuous time. For example, this is generally the case within economy.

A $\sigma$-stream is a family $(F_t)_{t \in T}$ of $\sigma$-algebras (over a given set $\Omega$) so that $T$ (the time axis) is a subset of the real numbers and

\[ s \leq t \text{ imply } F_s \subset F_t \text{ for all } s, t \in T \]  \hspace{1cm} (1.1)

Inclusion between $\sigma$-streams is given by

\[ (F_t)_{t \in T} \leq (G_t)_{t \in T} \text{ if } T = T' \text{ and } F_t \supseteq G_t, t \in T \] \hspace{1cm} (1.2)

A probabilistic model for a time-dependent system is described by $(\Omega,F,F,P)$ where $(\Omega,F,P)$ is a probability space and $(F_t)_{t \in T}$ is a "framework" $\sigma$-stream, i.e. $F_t$ are all events in the model up to and including time $t$ (and $F_t$ is a subset of $F$) and every $\sigma$-stream $(G_t)_{t \in T}$ "contained in" $(\Omega,F,F,P)$ satisfies (1.2). Whether or not sup $T = +\infty$ or inf $T = -\infty$, we define

\[ F_{\infty} = V_{t \in T} F_t, \quad F_{-\infty} = \bigcap_{t \in T} F_t. \]

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Given a measurable space \((\Omega, F)\) the \(\sigma\)-algebra induced by the functions \(X_u, u \in U\) is the smallest \(\sigma\)-algebra with respect to which all the \(X_u\) are measurable. The \(\sigma\)-stream induced by the stochastic process

\((X_t)_{t \in T}\) is given by \((F^X_t)_{t \in T}\) where

\[ F^X_t = \sigma\{X_u, u \in T, u \leq t\} \text{ for all } t \in T. \]

A \(\sigma\)-stream may be introduced by several processes, e.g.

\[ F^X_t = F^X_t \cap F^Y_t. \]

A stochastic process \((X_t)_{t \in T}\) is adapted to the \(\sigma\)-stream \((F_t)\) if they have the same time-axis and if all \(X_u, u \leq t\) are \(F\)-measurable, i.e. if

\[ (F^X_t) \subseteq (F_t). \]

The notation \((X_t, F_t)\) means that \((X_t)\) is \((F_t)\)-adapted.

If \(M_i, i = 1, 2, 3\) are sub-\(\sigma\)-algebras of \(F\) in a probability space \((\Omega, F, P)\), conditional independence of \(M_i\) and \(M_2\) given \(M_3\) is denoted by

\[ M_i \perp M_2 | M_3. \]

In a probability space \((\Omega, F, P)\) we define

\[ \overline{F} = \{ A : A \in F, P(A) = 0, 1 \} \text{ and } \overline{M} = M \cap \overline{F}, \]

where \(M\) is sub-\(\sigma\)-algebra of \(F\). Stochastic equivalence (or equality \(P\)-a.s.) between \(\sigma\)-algebras is given by \(M = M'(P \text{-a.s.)} \Leftrightarrow \overline{M} = \overline{M}'\), and note that \(F(X)\) and \(F(Y)\) are equivalent if \(X\) and \(Y\) so are.

Equivalence between \(\sigma\)-streams is given by

\[ (G_t) = (G'_t) (P \text{-a.s.)} \Leftrightarrow \forall t \in T : G_t = G'_t (P \text{-a.s.)} \]

and stochastic processes \((X_t)\) and \((Y_t)\) are equivalent when \(X_t\) and \(Y_t\) are equivalent for all \(t\).

**Definition 1.1.** ([4]) Let the system \((\Omega, F, F_t, P)\) be given, let \(M_t\) be a \(\sigma\)-algebra

\[ M_i \subseteq F_i. \]

A \(\sigma\)-algebra \(M_2\) is a sufficient cause of \(M_1\) at time \(t\) (relative to \((\Omega, F, F_t, P)\)) iff

\[ M_2 \subseteq F_i \text{ and } M_i \perp F_i | M_2. \]

In this way we can describe the causes of single events \((M_1 = \{\Phi, \Omega, A, A^C\})\) and sets of events. Note that if \(M_t \subseteq F_t\) then \(M_t\) is a sufficient cause of \(M_t\) at time \(t\).

The following notion of causality, in terms of Hilbert spaces, was given in [5] and represents a generalization of the definition from [4].

**Definition 1.2.** In a measurable space \((\Omega, F)\) let \((F_t)\), \((G_t)\) and \((H_t)\) be \(\sigma\)-streams of sub-\(\sigma\)-algebras of \(F\), and let \(P\) be a probability measure on \(F\). We say that \((G_t)\) causes \((H_t)\) within \((F_t)\) relative to \(P\) or
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\((H_t) \mathcal{K}(G_t);(F_t);P\)

if
\((G_t) \subseteq (F_t), H_{\infty} \subseteq F_{\infty}\) and if \(G_t\) is sufficient cause of \(H_t\) relative to \((\Omega,F,F_t,P)\) for every \(t \in T\), \(T\) being the time-axis of the \(\sigma\)-streams, i.e. if
\[ H_t \perp F_t | G_t \quad \text{for all} \quad t \in T. \tag{1.3} \]

It is easy to see that (2) may be formulated as
\[ H_u \perp F_t | G_t \quad \text{for all} \quad t, u \in T. \]

The essence of (1.3) is that all information about \((H_t)\) enters the system \((\Omega,F,F_t,P)\) via \((G_t)\).

Definition 1.3. A \(\sigma\)-stream \((H_t)\) is its own cause (within \((F_t)\)) (relative to \(P\)) if
\[(H_t) \mathcal{K}(H_t);(F_t);P.\]

These definitions apply to stochastic processes as if we were talking about the corresponding induced \(\sigma\)-streams. For example, \((X_t,F_t)\) is its own cause if \((F_t)\) is its own cause within \((F_t)\). In addition, \((X_t)\) is caused by itself and by \((Y_t)\) if
\[(F_t^X) \mathcal{K}(F_t^X);(F_t^X).\]

The interpretation of Granger-causality is now that "\(Y\) does not cause \(X\)" if
\[(F_t^X) \mathcal{K}(F_t^X);(F_t^X).\]

Proposition 1.1. [4] In a probability space \((\Omega,F,P)\) let \((F_t)\) and \((G_t)\) be \(\sigma\)-streams and let \(\{(X_t^{(n)})\}\) be a stochastic process satisfying \(X_t^{(n)} \to X,n \to \infty\) in probability, for every \(t \in T\) and
\[ (X_t^{(n)}) \mathcal{K}(G_t);(F) \quad \text{for every} \quad n, \]
\((T\) being the time axis). Then \((X_t) \mathcal{K}(G_t);(F)\) holds for process \((X_t)\).

2. APPLICATION TO THE STOCHASTIC DIFFERENTIABLE EQUATIONS

(I) In this section \(T = [0,t_0], C^d\) is the space of all continuous functions \(T \to R^d\), \(B_d(C^d)\) is the \(\sigma\)-algebra on \(C^d\) making the functions \(\varepsilon_u: C^d \to R, \varepsilon_u(x) = x_u\) measurable for \(u \leq t,\)
\[ B(C^d) = B_{t_0}(C^d) \]
\[ B_d(C^d) = \bigcap_{\alpha \in \Omega} B_{\alpha}(C^d). \]

A causal functional \(a_t\) is a \((B_d(C^d))\)-adapted process on \(C^d\).
A \((d\text{-dimensional})\) stochastic process \((X_t)\) which is continuous (i.e. whose sample-functions are continuous \((P\text{-a.s.})\)), induces a measure \(\mu_X\) on \(B(C^d)\),
\[
\mu_X(B) = P\{X(\omega) \in B\}
\]

\(X(\omega)\) the sample-functions for given \(\omega\).

Let \((\Omega, F, P)\) be a probably space, \(F\)-complete, and \((X_t, F_t)_{t \in T}\) be a continuous \(d\text{-dimensional}\) stochastic process, \(F_0\) complete. Let \(\mu_X\) and \(\mu_W\) be the measures induced by \((X_t)\) and a \(d\text{-dimensional}\) Wiener process \((W_t)\), respectively. Consider the following statements:

a) There is a \((d\text{-dimensional})\) Wiener process \((W_t, F_t)\) and a measurable process \((\alpha_t, F_t)\) satisfying
\[
X_t = \int_0^t \alpha_s ds + W_t \quad \text{(P-a.s.) for every } t \in T
\]
\(\|\cdot\|\) is the Euclidian norm.

b) \(\mu_X << \mu_W\), i.e. \(\mu_X\) is absolutely continuous with respect to \(\mu_W\).

c) \(\int \alpha_s^* \alpha_s ds < \infty \quad \text{(P-a.s.)}\).

d) \((\alpha_t)\) is of the form \((P\text{-a.s.)})\)
\[
\alpha_t = a_t(x) \text{ a.e. in } T
\]
\(\alpha_t\) being a causal functional.

e) \((X_t)\) is its own cause within \((F_t)_{t \in T}\).

Liptser and Shiryaev (1977) studied the relationships between a), b), c) and d) using Girsanov's theorem. This picture is completed by the statement e) as Proposition 2.1 will show. First we define some new statements:

(i) \(b\) and e)
(ii) a) and b) and e)
(iii) a) and c) and e)
(iv) a) and b) and d)
(v) a) and c) and d).

**Proposition 2.1.** The statements (i)-(v) are equivalent. If they apply, the representation (2.1) is unique, i.e. if for every \(t \in T\)
\[
X_t = \int_0^t \beta_s ds + \tilde{W}_t \quad \text{(P-a.s.)}
\]
\((\tilde{W}_t, F_t)\) being a Wiener process and \((\beta_t, F_t)\) a measurable process, then
\[
P(W_t = \tilde{W}_t \text{ for every } t) = 1,
\]
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\[ P(\alpha_i = \beta_i \text{ for almost every } t_i) = 1. \]  

(2.3)

**Proof.** If, for every \( t \),

\[ \int_0^t (\beta_s - \alpha_s) \, ds = W_t - \hat{W}_t \quad (P\text{-a.s.}) \]

the martingale property of \( W_t - \hat{W}_t \) implies (2.2) and (2.3).

The equivalence between iv) and v) follows from Theorem 7.5 and note 7.2.7 in [2] and the implication from iii) to ii) follows from Theorem 7.4 in the same book. As ii) implies i) trivially, it remains to show that i) implies v) and that v) implies iii).

To show that i) implies v) assume first the existence of a \((F_t)\)-adapted \((d\text{-dimensional})\) Wiener process. In the presence of b), Theorem 7.11 from [2] guarantees the representation

\[ X_t = \int_0^t \alpha_s \, ds + W_t \quad (P\text{-a.s.}) \]

for every \( t \in T \), for \( d = 1 \), with \((W_t, F_{t,1}^T)\) as a Wiener-process. \((W_t, F_{t,1}^T)\) being a Wiener process and \((F_{t,1}^T)\) being its own cause within \((F_t)\), \((W_t, F_{t,1})\) is a Wiener process and the implication is proved in this case. In general, we extend the probability space and the framework \(\sigma\text{-stream} to include a Wiener process. As \((W_t)\) in the representation from a) is adapted to \((F_{t,1})\) and therefore to the original framework \(\sigma\text{-stream}, the extension can be abandoned after finding \((W_t)\).

We prove now that v) implies iii). Assume that there are \((d\text{-dimensional})\) Wiener process \((W_t, F_{t,1}^T)\) and causal functional \(a_t\) of the form \(\alpha_t = a_t(x)\) such that

\[ X_t = \int_0^t \alpha_s \, ds + W_t \quad (P\text{-a.s.}) \text{ for every } t \in T, \quad \int_0^t |\alpha_s| \, ds < \infty \quad (P\text{-a.s.}) \text{ and } \int_0^t \alpha_s ^2 \, ds < \infty \quad (P\text{-a.s.}). \]

For \( n = 1, 2, ..., \), for \( x \in \mathcal{C}_d \), set

\[ \tau^{(n)}(x) = \inf \{ t : t \in T \wedge \int_0^t a_s(x) \, ds \geq n \} \text{, or, } \tau^{(n)}(x) = t_0 \text{ if } \int_0^{t_0} a_s(x) \, ds < n. \]

By Lemma 1.11 from [2], \(\tau^{(n)}\) is a stopping time. Thus

\[ a_t^{(n)}(x) = a_t(x) I_{t \leq \tau^{(n)}(x)} \]

is a causal functional \((I\text{ is the indicator function})\). Thus the process

\[ X_t^{(n)} = \int_0^t a_s^{(n)}(X) \, ds + W_t \]

is adapted to \((F_t^{1,T})\). Set

\[ z^{(n)} = \exp \left\{ -\int_0^t a_s^{(n)}(X)^c \, dW_s - \frac{1}{2} \int_0^t a_s^{(n)}(X)^c a_s^{(n)}(X) \, ds \right\} \]

By Corollary 7.2.1 in [3],

\[ E[z^{(n)}] = 1 \]

Accoring to multidimensional Girsanov's theorem it follows that \((X_t^{(n)}, F_t)\) is a Wiener process under measure \(\tilde{P}^{(n)}\), given by
In particular, \((X_t^{(n)})\) is its own cause relative to \(\tilde{P}^{(n)}\), and as \((X_t^{(n)})\) is adapted to \((F_t^X)\),
\[
X_t^{(n)}K_{F_t^X} ; F_t^X ; \tilde{P}^{(n)}.
\]

It is easy to see that
\[
a_t^{(n)}(X) = a_t^{(n)}(X^{(n)}),
\]
thus \(z^{(n)}\) is \(F_{\infty}^X\)-measurable. Now, it follows that
\[
X_t^{(n)}K_{F_t^X} ; F_t^X ; P.
\]

As \(X_t^{(n)}\) converges to \(X_t\) for every \(t\), from Proposition 1.1, it follows that \((X_t)\) is its own cause within \((F_t)\) relative to \(P\).

**II** The stochastic differential equation of unknown \(d\)-dimensional process \(X_t, t \in T\) with the initial value \(\eta\) given by
\[
dX_t = a_t(X)dt + b_t(X)dW_t
\]
\[
X_0 = \eta
\]
is well defined when the following elements are given: dimension \(d\) (that of \((X_t)\) and \((W_t)\)), causal functionals \(a_t\) (\(d\)-dimensional vector) and \(b_t\) (\(d \times d\) matrix) and \(d\)-dimensional distribution function \(F\).

In this case the object \((\Omega,F,F_t^P,W_t,X_t)\) is said to be a weak solution of (2.4) if
i) \((\Omega,F,F_t^P)\) is a probability system with time axis of the form \(T = [0,t_0]\), with \((F_t)\) right continuous and with \(F\) and \(F_0\) complete,
ii) \((W_t,F)\) is a Wiener process,
iii) \((X_t)\) is a continuous adapted process,
iv) \(X_0\) has \(F_\eta\) as its cumulate distribution function,
v) \(\int_0^t |a_t(X)|ds < \infty\) and \(\int_0^t |b_t(X)|ds \leq \infty\) both \(P\)-a.s.,
vi) \(X_t = X_0 + \int_0^t a_t(X)ds + \int_0^t b_t(X)dW_t\) \(\) \(P\)-a.s.

the last integral being a classical stochastic integral over \((W_t)\), which exists since \(b_t\) is causal and because of (2.5).

For given \(d, a_t, b_t\) and \(F_\eta\), the solution of (2.4) is weakly unique if for any two solutions \((\Omega_i,F_i,F_t^i,W_t,X_t^{(i)}), i = 1,2\), of the system, the induced measures \(\mu_{X_1}^{\eta}\) and \(\mu_{X_2}^{\eta}\) coincide.

The Girsanov's theorem is instrumental in showing the following result.

**Proposition 2.2.** If, for every weak solution \((\Omega_i,F,F_t^P,W_t,X_t)\) of (2.2), \(X_t\) is caused by itself and by \(W_t\) within \(F_t\), then the solution is weakly unique.

**Proof.** Let \((\Omega_i,F,F_t^i,W_t,X_t^{(i)}), i = 1,2\), be two weak solutions of (2.2). Without loosing of generality we assume that \(\Omega_1 \cap \Omega_2 = \Phi\) and
\[ \Omega = \Omega_1 \cup \Omega_2 \]

\[ F = \{ A \cup B : A \in F_1, B \in F_2 \} \]

\[ F_t = \{ A \cup B : A \in F_1, B \in F_2 \} \]

\[ P(A \cup B) = \frac{1}{2} [P(A) + P(B), A \in F_1, B \in F_2] \]

\[ W_j(\omega) = \begin{cases} W_1^j(\omega), & \omega \in \Omega_1 \\ W_2^j(\omega), & \omega \in \Omega_2 \end{cases} \]

\[ X_j(\omega) = \begin{cases} X_1^j(\omega), & \omega \in \Omega_1 \\ X_2^j(\omega), & \omega \in \Omega_2 \end{cases} \]

It is easy to see that \((\Omega, F, F_t, P, W_t, X_t)\) is a weak solution of (2.2). Set

\[ j(\omega) = \begin{cases} 1, & \omega \in \Omega_1 \\ 2, & \omega \in \Omega_2 \end{cases} \]

\(X_0\) and \(j\) are independent.

Assume that \(X_t\) is caused by \(F_t^{X,W}\). Then \(F_t^X\) is conditionally independent of \(F_0\) given \(F_0^X\) (since \(W_0 = 0\)), and since \(X_0\) and \(j\) are independent, this implies that \(F_t^X\) is independent of \(j\). Consequently, for \(A \in B(C^d)\), \(P(X \in A \mid j)\) is constant \((P\text{-a.s.})\), and as

\[ P(X \in A \mid j)(\omega) = P(X^i \in A) \]

for almost all \((P\text{-a.s. and } P^i\text{-a.s.)} \omega \in \Omega^j\)

\[ P(X^1 \in A) = P(X^2 \in A) \]

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NEKI MODELI UZROČNOSTI I STOHASTIČKE DIFERENCIJALNE JEDNAČINE

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Ključne reči: σ-algebra, uzročnost, slabo rešenje stohastičke diferencijalne jednačine.