FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 4, N° 17, 2005, pp. 293 - 300

SOME MODELS OF CAUSALITY AND STOCHASTIC DIFFERENTIAL EQUATIONS

UDC 517.9:519.856(045)=20

Ljiljana Petrović

Faculty of Economics, Belgrade, Kamenička 6

Abstract. *We consider some concepts of causality between σ-algebras and between stochastic processes. Then, we give a generalization of a causality relationship "G is a cause of E within H" which was first given by Mykland [4] and which is based on Granger's definition of causality [1].*
In the second part we apply the results to systems of stochastic differential equations. More precisely, we give conditions for weakly uniqueness of the solution of some stochastic differential equations.

Key words: σ-*algebra, causality, weak solution of stochastic differential equation.*

1. INTRODUCTION AND NOTATION

The study of Granger causality has been mainly preocupied with time series. We shall instead concentrate on continuous time processes. Many of systems to which it is natural to apply tests of causality take place in continuous time. For example, this is generally the case within economy.

A σ-stream is a family $(F_t)_{t\in T}$ of σ-algebras (over a given set Ω) so that T (the time axis) is a subset of the real numbers and

$$s \le t \text{ imply } F_s \subset F_t \text{ for all } s, t \in T \tag{1.1}$$

Inclusion between σ-streams is given by

$$(F_t)_{t\in T'} \le (G_t)_{t\in T} \text{ if } T = T' \text{ and } F_t \supset G_t\,,\ t \in T\,. \tag{1.2}$$

A probabilistic model for a time-dependent system is described by (Ω,F,F_t,P) where (Ω,F,P) is a probability space and $(F_t)_{t\in T}$ is a "framework" σ-stream, i.e. F_t are all events in the model up to and including time t (and F_t is a subset of F) and every σ-stream $(G_t)_{t\in T}$ "contained in" (Ω,F,F_t,P) satisfies (1.2). Whether or not $\sup T = +\infty$ or $\inf T = -\infty$, we define

$$F_\infty = V_{t\in T} F_t\,, \qquad F_{-\infty} = \bigcap_{t\in T} F_t\,.$$

Received May 11, 2004

Given a measurable space (Ω,F) the σ-algebra induced by the functions X_u, $u \in U$ is the smallest σ-algebra with respect to which all the X_u are measurable. The σ-stream induced by the stochastic process

$$(X_t)_{t\in T} \text{ is given by } (F_t^X)_{t\in T} \text{ where}$$

$$F_t^X = F\{X_u, u \in T, u \le t\} \text{ for all } t \in T .$$

A σ-stream may be introduced by several processes, e.g.

$$F_t^{X,Y} = F_t^X V F_t^Y .$$

A stochastic process $(X_t)_{t\in T}$ is adapted to the σ-stream (F_t) if they have the same time-axis and if all X_u, $u \le t$ are F-measurable, i.e. if

$$(F_t^X) \le (F_t) .$$

The notation (X_t,F_t) means that (X_t) is (F_t)-adapted.

If M_i, $i = 1,2,3$ are sub-σ-algebras of F in a probability space (Ω,F,P), conditional independence of M_1 and M_2 given M_3 is denoted by

$$M_1 \perp M_2 | M_3 .$$

In a probability space (Ω,F,P) we define

$$\overline{F}^0 = \{A : A \in F, P(A) = 0, \text{ or } 1\} \text{ and } \overline{M} = MV\overline{F}^0$$

where M is sub-σ-algebra of F. Stochastic equivalence (or equality P-a.s.) between σ-algebras is given by $M = M'(P - a.s.) \Leftrightarrow \overline{M} = \overline{M}'$, and note that $F(X)$ and $F(Y)$ are equivalent if X and Y so are.

Equivalence between σ-streams is given by

$$(G_t) = (G_t')(P - a.s.) \Leftrightarrow \forall t \in T : G_t = G_t'(P - a.s.)$$

and stochastic processes (X_t) and (Y_t) are equivalent when X_t and Y_t are equivalent for all t.

Definition 1.1. ([4]) Let the system (Ω,F,F_t,P) be given, let M_1 be a σ-algebra

$$M_1 \subset F_\infty .$$

A σ-algebra M_2 is a sufficient cause of M_1 at time t (relative to (Ω,F,F_t,P)) iff

$$M_2 \subset F_t \quad \text{and} \quad M_1 \perp F_t | M_2 .$$

In this way we can describe the causes of single events ($M_1 = \{\Phi,\Omega,A,A^C\}$) and sets of events. Note that if $M_1 \subset F_t$, then M_1 is a sufficient cause of M_1 at time t.

The following notion of causality, in terms of Hilbert spaces, was given in [5] and represents a generalization of the definition from [4].

Definition 1.2. In a measurable space (Ω,F) let (F_t), (G_t) and (H_t) be σ-streams of sub-σ-algebras of F, and let P be a probability measure on F. We say that (G_t) causes (H_t) within (F_t) relative to P or

$$(H_t)\mathrm{K}(G_t);(F_t);P$$

if
$(G_t) \leq (F_t)$, $H_{<\infty} \subseteq F_{<\infty}$ and if G_t is sufficient cause of H_∞ relative to (Ω,F,F_t,P) for every $t \in T$, T being the time-axis of the σ-streams, i.e. if

$$H_\infty \perp F_t \big| G_t \quad \text{for all} \quad t \in T\ . \tag{1.3}$$

It is easy to see that (2) may be formulated as

$$H_u \perp F_t \big| G_t \quad \text{for all} \quad t,u \in T\ .$$

The essence of (1.3) is that all information about (H_t) enters the system (Ω,F,F_t,P) via (G_t).

Definition 1.3. A σ-stream (H_t) is its own cause (within (F_t)) (relative to P) if

$$(H_t)\mathrm{K}(H_t);(F_t);P\ .$$

These definitions apply to stochastic processes as if we were talking about the corresponding induced σ-streams. For example, (X_t,F_t) is its own cause if (F_t^X) is its own cause within (F_t). In addition, (X_t) is caused by itself and by (Y_t) if

$$(F_t^X)\mathrm{K}(F_t^{X,Y});(F_t)\ .$$

The interpretation of Granger-causality is now that "Y does not cause X" if

$$(F_t^X)\mathrm{K}(F_t^X);(F_t^{X,Y})\ .$$

Proposition 1.1. [4] In a probability space (Ω,F,P) let (F_t) and (G_t) be σ-streams and let $\{(X_t^{(n)})\}$ be a stochastic process satisfying $X_t^{(n)} \to X, n \to \infty$ in probability, for every $t \in T$ and

$$(X_t^{(n)})K(G_t);(F) \ \text{ for every } n,$$

(T being the time axis). Then $(X_t)K(G_t);(F)$ holds for process (X_t).

2. Application to the Stochastic Differential Equations

(I) In this section $T = [0,t_0]$, C^d is the space of all continuous functions $T \to R^d$, $\mathrm{B}_t(C^d)$ is the σ-algebra on C^d making the functions $\varepsilon_u : C^d \to R$,

$$\varepsilon_u(x) = x_u$$

measurable for $u \leq t$,

$$\mathrm{B}(C^d) = \mathrm{B}_{t_0}(C^d)$$

$$\mathrm{B}_+(C^d) = \textstyle\bigcap_{u>t} \mathrm{B}_u(C^d)\ .$$

A causal functional a_t is a $(\mathrm{B}_+(C^d))$ -adapted process on C^d.

A (d-dimensional) stochastic process (X_t) which is continuous (i.e. whose sample-functions are continuous (P-a.s.)), induces a measure μ_X on $B(C^d)$,

$$\mu_X(B) = P\{X(\omega) \in B\},$$

$X(\omega)$ the sample-functions for given ω.

Let (Ω,F,P) be a probably space, F-complete, and $(X_t,F_t)_{t\in T}$ be a continuous d-dimensional stochastic process, F_0 complete. Let μ_X and μ_W be the measures induced by (X_t) and a d-dimensional Wiener process (W_t), respectively. Consider the following statements:

a) There is a (d-dimensional) Wiener process (W_t,F_t) and a measurable process (α_t,F_t) satisfying

$$X_t = \int_0^t \alpha_s ds + W_t \text{ (P-a.s.) for every } t \in T \tag{2.1}$$

$$\int_0^{t_0} |\alpha_s| ds < \infty \quad (P\text{-a.s.})$$

$|.|$ is the Euclidian norm.

b) $\mu_X << \mu_W$, i.e. μ_X is absolutely continuous with respect to μ_W.

c) $\int \alpha_s^* \alpha_s ds < \infty$ (P-a.s.).

d) (α_t) is of the form (P-a.s.)

$$\alpha_t = a_t(x) \text{ a.e. in } T$$

α_t being a causal functional.

e) (X_t) is its own cause within $(F_t)_{t\in T}$.

Lipster and Shiryayev (1977) studied the relationships between a), b) , c) and d) using Girsanov's theorem. This picture is completed by the statement e) as Proposition 2.1 will show. First we define some new statements:

(i) b) and e)
(ii) a) and b) and e)
(iii) a) and c) and e)
(iv) a) and b) and d)
(v) a) and c) and d).

Proposition 2.1. The statements (i)-(v) are equivalent. If they apply, the representation (2.1) is unique, i.e. if for every $t \in T$

$$X_t = \int_0^t \beta_s ds + \hat{W}_t \quad (P\text{-a.s.})$$

$(\hat{W}_t, F_t)$ being a Wiener process and (β_t, F_t) a measurable process, then

$$P(W_t = \hat{W}_t \text{ for every } t) = 1, \tag{2.2}$$

$$P\{\alpha_t = \beta_t \text{ for almost every } t\} = 1 . \quad (2.3)$$

Proof. If, for every t,

$$\int_0^t (\beta_s - \alpha_s) ds = W_t - \hat{W}_t \quad (P\text{-a.s.})$$

the martingale property of $W_t - \hat{W}_t$ implies (2.2) and (2.3).

The equivalence between iv) and v) follows from Theorem 7.5 and note 7.2.7 in [2] and the implication from iii) to ii) follows from Theorem 7.4 in the same book. As ii) implies i) trivially, it remains to show that i) implies v) and that v) implies iii).

To show that i) implies v) assume first the existence of a (F_t)-adapted (d-dimensional) Wiener process. In the presence of b), Theorem 7.11 from [2] guarantees the representation $X_t = \int_0^t \alpha_s ds + W_t$ (P-a.s.) for every $t \in T$, for $d = 1$, with (W_t, F_t^X) as a Wiener-process. (W_t, F_t^X) being a Wiener process and (F_t^X) being its own cause within (F_t), (W_t, F_t) is a Wiener process and the implication is proved in this case. In general, we extend the probability space and the framework σ-stream to include a Wiener process. As (W_t) in the representation from a) is adapted to (F_t^X) and therefore to the original framework σ-stream, the extension can be abandoned after finding (W_t).

We prove now that v) implies iii). Assume that there are (d-dimensional) Wiener process (W_t, F_t^X) and causal functional a_t of the form $\alpha_t = a_t(x)$ such that

$$X_t = \int_0^t \alpha_s ds + W_t \ (P\text{-a.s.}) \text{ for every } t \in T , \ \int_0^{t_0} |\alpha_s| ds < \infty \ (P\text{-a.s.}) \text{ and } \int \alpha_s^* \alpha_s ds < \infty \ (P\text{-a.s.}).$$

For $n = 1,2,...$, for $x \in C^d$, set

$$\tau^{(n)}(x) = \inf\{t : t \in T \wedge \int_0^t a_s^*(x) a_s(x) ds \geq n\} , \text{ or, } \tau^{(n)}(x) = t_0 \text{ if } \int_0^{t_0} a_s^*(x) a_s(x) ds < n .$$

By Lemma 1.11 from [2], $\tau^{(n)}$ is a stopping time. Thus

$$a_t^{(n)}(x) = a_t(x) I_{\{t \leq \tau^{(n)}(x)\}}$$

is a causal functional (I is the indicator function). Thus the process

$$X_t^{(n)} = \int_0^t a_s^{(n)}(X) ds + W_t$$

is adapted to (F_t^X) . Set

$$z^{(n)} = \exp\{-\int_0^{t_0} a_s^{(n)}(X)^* dW_s - \frac{1}{2} \int_0^{t_0} a_s^{(n)}(X)^* a_s^{(n)}(X) ds\}$$

By Corollary 7.2.1 in [3],

$$E\{z^{(n)}\} = 1$$

Accoring to multidimensional Girsanov's theorem it follows that $(X_t^{(n)}, F_t)$ is a Wiener process under measure $\widetilde{P}^{(n)}$, given by

$$d\widetilde{P}^{(n)} = z^{(n)} dP .$$

In particular, $(X_t^{(n)})$ is its own cause relative to $\widetilde{P}^{(n)}$, and as $(X_t^{(n)})$ is adapted to (F_t^X),

$$X_t^{(n)} K F_t^X ; F_t ; \widetilde{P}^{(n)} .$$

It is easy to see that

$$a_t^{(n)}(X) = a_t^{(n)}(X^{(n)}) ,$$

thus $z^{(n)}$ is $F_\infty^{X^{(n)}}$-measurable. Now, it follows that

$$X_t^{(n)} K F_t^X ; F_t ; P .$$

As $X_t^{(n)}$ converges to X_t for every t, from Proposition 1.1, it follows that (X_t) is its own cause within (F_t) relative to P.

(II) The stochastic differential equation of unknown d-dimensional process X_t, $t \in T$ with the initial value η given by

$$dX_t = a_t(X)dt + b_t(X)dW_t \tag{2.4}$$

$$X_0 = \eta$$

is well defined when the following elements are given: dimension d (that of (X_t) and (W_t)), causal functionals a_t (d-dimensional vector) and b_t($d \times d$ matrix) and d-dimensional distribution function F_η.

In this case the object $(\Omega, F, F_t, P, W_t, X_t)$ is said to be a weak solution of (2.4) if

i) (Ω, F, F_t, P) is a probability system with time axis of the form $T = [0, t_0]$, with (F_t) right continuous and with F and F_0 complete,

ii) (W_t, F_t) is a Wiener process,

iii) (X_t) is a continuous adapted process,

iv) X_0 has F_η as its cumulate distribution function,

v) $\int_0^{t_0} | a_s(X)ds | < \infty$ and $\int_0^{t_0} | b_s(X)ds |^2 < \infty$ both P-a.s.), (2.5)

vi) $X_t = X_0 + \int_0^t a_s(X)ds + \int_0^t b_s(X)dW_s$ (P-a.s)

the last integral being a classical stochastic integral over (W_t), which exists since b_t is causal and because of (2.5).

For given d, a_t, b_t and F_η, the solution of (2.4) is weakly unique if for any two solutions $(\Omega^i, F^i, F^i_t, P^i, W^i_t, X^i_t)$, $i = 1,2$, of the system, the induced measures μ_{X^1} and μ_{X^2} coincide.

The Girsanov's theorem is instrumental in showing the following result.

Proposition 2.2. If, for every weak solution $(\Omega, F, F_t, P, W_t, X_t)$ of (2.2), X_t is caused by itself and by W_t within F_t, then the solution is weakly unique.

Proof. Let $(\Omega^i, F^i, F^i_t, P^i, W^i_t, X^i_t)$, $i = 1,2$, be two weak solutions of (2.2). Without loosing of generality we assume that $\Omega^1 \cap \Omega^2 = \Phi$ and

$$\Omega = \Omega^1 \bigcup \Omega^2$$

$$F = \{A \cup B : A \in F^1, B \in F^2\}$$

$$F_t = \{A \cup B : A \in F_t^1, B \in F_t^2\}$$

$$P(A \cup B) = \frac{1}{2}[P(A) + P(B), A \in F^1, B \in F^2]$$

$$W_t(\omega) = \{ \begin{matrix} W_t^1(\omega), & \omega \in \Omega^1 \\ W_t^2(\omega), & \omega \in \Omega^2 \end{matrix}$$

$$X_t(\omega) = \{ \begin{matrix} X_t^1(\omega), & \omega \in \Omega^1 \\ X_t^2(\omega), & \omega \in \Omega^2 \end{matrix} .$$

It is easy to see that $(\Omega, F, F_t, P, W_t, X_t)$ is a weak solution of (2.2). Set

$$j(\omega) = \{ \begin{matrix} 1, & \omega \in \Omega^1 \\ 2 & \omega \in \Omega^2 \end{matrix} .$$

X_0 and j are independent.

Assume that X_t is caused by $F_t^{X,W}$. Then F_∞^X is conditionally independent of F_0 given F_0^X (since $W_0 = 0$), and since X_0 and j are independent, this implies that F_∞^X is independent of j. Consequently, for $A \in B(C^d)$, $P(X \in A \mid j)$ is constant (P-a.s.), and as

$$P(X \in A \mid j)(\omega) = P(X^i \in A)$$

for almost all (P-a.s. and P^i-a.s.) $\omega \in \Omega^i$

$$P(X^1 \in A) = P(X^2 \in A) .$$

REFERENCES

1. Granger, C.W.J. "Investigating Causal Relations by Econometric Models and Cross Spectral Methods", Econometrica, 37, (1969) 424-438.
2. Lipster, R.S., Shiryayev, A.N. "Statistics of Random Processes I, Springer-Verlag, New York, (1977).
3. G. Kallianpur, G. " Stochasting Filtering Theory, Springer-Verlag, New York, (1980).
4. Mykland, P.A. "Statistical Theory of Causality, (to appear).
5. Petrović, Lj. Causality and Stochastic Realization Problem, Publ. Inst. Math. (Beograd) 45 (59), (1989), 203-210.
6. Petrović, Lj. Causality and Markovian Reductions and Extensions of a Family of Hilbert Spaces, Journal of Mathematical Systems, Estimation and Control, (1998), 495-498.

NEKI MODELI UZROČNOSTI I STOHASTIČKE DIFERENCIJALNE JEDNAČINE

Ljiljana Petrović

U radu se razmatraju neki koncepti uzročnosti izmedju σ-algebra i između stohastičkih procesa. Zatim se navodi generalizacija relacije uzročnosti "G je uzrok E u okviru H", koju je prvi dao Mykland [4] i koja se zasniva na Granger-ovoj definiciji uzročnosti [1]. U drugom delu rada rezultati se primenjuju na sisteme stohastičkih diferencijalnih jednačina. Preciznije, daju se uslovi za slabu jedinstvenost rešenja nekih stohastičkih diferencijalnih jednačina.

Ključne reči: σ-*algebra, uzročnost, slabo rešenje stohastičke diferencijalne jednačine..*