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ON THE TWO BODY PROBLEM

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Abstract. *In this paper a new general formula, which show the magnitude of the force, acting between two bodies during the motion on the distance* $\rho(t)$ *between two material points is obtained.*

Key words: *two bodies problem, magnitud of force, Law of gravitation.*

The motion of a system of two bodies, observed as material points, is known in the Celestial mechanics as "the problem of two bodies". Kepler's laws as well as Newton's gravitational force are the ones that relate to the motion of two bodies mutually attracting each other. This is a simple mechanical system of two material points, but its reduction to the Newton's theorems [4] of gravity makes it a significant problem. The main goal is to determine the formula for the force of mutual attraction between bodies. Thus we considered two material points whose masses are m_1 and m_2, which move towards each other so that the distance between their inertia centers is a time function $\rho(t) = \|\mathbf{r}_2 - \mathbf{r}_1\|$, where:

$$\rho(t) = \|\mathbf{r}_2 - \mathbf{r}_1\| = \rho\rho_0; \tag{1}$$

$\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the mass points m_1 and m_2, and $\boldsymbol{\rho} = \boldsymbol{\rho}/\rho$, $\|\rho_0\| = 1$. In accordance with the second and third axiom or laws the motion of two bodies can be written in the following way:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_1, \quad (\ddot{\mathbf{r}}_i := \frac{d^2\mathbf{r}_i}{dt^2}), \tag{2}$$

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_2, \quad (\ddot{\mathbf{r}}_i := \frac{d^2\mathbf{r}_i}{dt^2}), \tag{3}$$

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{4}$$

The derivative of the second order of vector function (1) is

$$\ddot{\boldsymbol{\rho}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = (\ddot{\rho} - \rho\dot{\theta}^2)\boldsymbol{\rho}_0 + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\mathbf{n}_0 = a_\rho\boldsymbol{\rho}_0 + a_n\mathbf{n}_0, \tag{5}$$

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where θ is an angle between vector ρ and some fixed direction; $\mathbf{n}_0 \perp \rho_0$, $\|\mathbf{n}_0\| = 1$; $a_\rho = \ddot{\rho} - \rho\dot{\theta}^2$ is radial acceleration. Substituting derivatives $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$ from equation (2) and (3) into relation (5), according to (4), we obtain:

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{m_1 m_1}{m_1 + m_2}\ddot{\boldsymbol{\rho}},$$

or

$$\frac{m_1 + m_2}{m_1 m_2}\mathbf{F}_2 = (\ddot{\rho} - \rho\dot{\theta}^2)\boldsymbol{\rho}_0 + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\mathbf{n}_0. \tag{6}$$

Magnitude of force with the action in the direction of the distance between two bodies is obtained by scalar multiplication of formula (6) and vector ρ_0. With scalar multiplication of the relation (6) with vector ρ_0, one obtained

$$F_\rho = \frac{m_1 m_2}{m_1 + m_2} a_\rho = M(\ddot{\rho} - \rho\dot{\theta}^2), \tag{7}$$

where

$$M := \frac{m_1 m_2}{m_1 + m_2}.$$

If we take into consideration that $\upsilon_{or}^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2$, the formula (7) can be written in the form

$$F = \chi\frac{m_1 m_2}{\rho}, \tag{8}$$

where

$$\chi = \frac{\dot{\rho}^2 + \rho\rho - \upsilon_{or}^2}{m_1 + m_2}. \tag{9}$$

Thus, formulas (8) or (7) show the magnitude of force, acting between two bodies during the motion depending on the distance $\rho(t)$. In order to make more clear the generalization of these formulas, we present some examples of their use:

1. Two bodies (as material points) are moving along the line (axe) x (e.g. $z = 0$, $y = 0$). Their distance changes according to

$$\rho = x_2 - x_1 = 1 + c\ \sin\omega t. \tag{10}$$

For this example, in formula (7) $\dot{\theta} = 0$, and in formula (8)

$$\rho = -\omega^2 c \sin\omega t = -\omega^2(\rho - l) = -\omega^2(x_2 - x_1 - l),$$

and it follows

$$F = -\frac{m_1 m_2}{m_1 - m_2}\omega^2(\rho - l). \tag{11}$$

It is obvious from here, that for this example

$$c = \frac{m_1 m_2}{m_1 - m_2}\omega^2. \tag{11}$$

2. Bodes of the masses m_1 and m_2, move with respect to each other at the constant distance $\rho = R = const$. Based on the formula (7) and (8), it follows that:

$$F = -M\frac{\upsilon_{or}^2}{R} = -M\frac{R^2\dot{\theta}^2}{R} = -MR\dot{\theta}^2 = \\ -M\frac{4\pi^2}{T}R = -M\frac{4\pi^2 R^3}{T^2R^2} = -f\frac{m_1m_2}{R^2}, \tag{12}$$

where

$$f = \frac{4\pi^2R^3}{(m_1+m_2)T^2}. \tag{13}$$

3. For

$$\rho = ct + R,\ \theta = const. \to F = 0; \tag{14a}$$

$$\rho = gt^2,\ \dot{\theta} = t^{-1}, \to F = Mg; \tag{14b}$$

$$\rho = c_1t^2,\ \dot{\theta} = c_2t^{-1},\ c_1, c_2 = const. \tag{14c}$$

From the condition (14c) it follows that the size of the requested force is inversely proportional to the squared distance ρ^2, namely

$$F = M\frac{C}{\rho^2}. \tag{15}$$

4. Let $\rho(t)$ change according to the Law

$$\rho = a - c\cos\omega t, \qquad \omega = \dot{\theta} = const.$$

Since in that case

$$\ddot{\rho} = \omega^2 c\cos\omega t,$$

it is obtained

$$F = -M(2\rho - a) \tag{16}$$

As much as there is a resemblance of examples (14a) and (14b), that much examples (15) and (16) are alike Kepler's motion and Newton's theorems of gravitation. Let's dedicate more attention to this question. Majority of scientist agree in that Isaac Newton derived the "Law of general Gravitation", based on Kepler's Laws.

In classical and celestial mechanics, the force formula for the force of mutual attraction of bodies is acknowledged, in the form

$$F = f\frac{m_1m_2}{\rho^2}, \tag{17}$$

known as "Newton's Law of universal gravitation". The coefficient f is most frequently called the "universal constant of gravitation". In the expert and teaching literature in physics, it is most often found as "Cavendish constant" $G := f = 6.672 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2}, while in celestial mechanics (see for example [5], p. 536), it is given by

$$f = \frac{4\pi^2a^3}{(m_1+m_2)T^2} = \frac{\mu}{m_1+m_2} \tag{18}$$

where: a is the average distance between center of inertia of celestial bodies (big semiaxis of elliptic trajectory), T is the time of a planet rotation around Sun, and μ is the Gauss constant.

At the first instance our formula (7) or (8) differs much from the formula (17). However, for the various conditions of the change of distance from formula (7) or (8), different formulas of forces (11)-(16), follow.

Formula (17) is obtained as the consequence of the formulas (7) or (8) only from the conditions of Kepler's Laws. In order to prove it, let's write Kepler's Laws of motion of planets around the Sun, using mathematical relations:

$$\rho(t) = \frac{p}{1 + e\cos\theta(t)}, \qquad p = \frac{b^2}{a}; \qquad a, b, e = const.; \tag{I}$$

$$\rho^2\dot{\theta} = C, \qquad C == \frac{2\pi ab}{T} = const. \tag{II}$$

$$a^3 = kT^2, k = const., \tag{III}$$

where p is the parameter of an elliptic trajectory, e is eccentricity, $e \le 1$, a is the large semiaxis of the ellipse, T is the time of rotation of a planet around Sun.

According to the Law (II), it is obtained

$$\ddot{\rho} = \frac{C}{p}\dot{\theta}\, e\cos\theta = \frac{C^2(p-\rho)}{p\rho^3}. \tag{19}$$

Substituting the derivatives $\ddot{\rho}$ and $\dot{\theta}$ in the formula (7), formula (17) is obtained, and it is according to (18), the "Newton Law of gravitation". So, in fact, the formula (17) appears as the consequence of the formula (7) or (8), with the precision of Kepler's Laws for the Sun planetary system.

In addition we can observe that if we fit the valu k in the third Kepler's Laws so that it is the same for all planets and satellites, as well as it is the Gauss' constant μ, then it is obvious in the relation (18), the coefficient of the proportionality f changes from planet to planet.

For the average values ([2] and [3]) of Kepler's motion, the coeficient χ is $3.348253233 \cdot 10^{-22}\ \mathrm{m^2kg^{-1}s^{-2}}$. It is substantially different from the average value of the "gravitationles constant" $f = 6.6724250 \cdot 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$. With help the formula (9) and formula (18) we calculated χ and f for big planets in the Sun system [6].

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O PROBLEMU DVA TELA

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Dokazano je da se opšta formula sile uzajamnog dejstva dva tela može napisati u obliku (6), a njena veličina u obliku (7) ili (8). Pokazuje se da iz tih formula, uz uslove Keplerovih zakona o kretanju planeta Sunčevog sistema proizilazi, može da se dobije Njutnov zakon gravitacije.

Ključne reči: *problem dva tela, veličina sile, Zakon gravitacije.*