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ON THE ROTATION STABILIZATION OF THE UNSTABLE GYROSCOPE CONTAINING FLUID BY ROTATING THE RIGID BODY

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Abstract. *The paper presents the possibility of the stabilization of an unstable rotation of the Lagrange gyroscope containing an ideal fluid by rotating the rigid body. For the counterbalanced rotating rigid body the effect of the stabilization increases in comparison with the unbalanced rigid body.*

Key words: *stabilization of rotation, Lagrange gyroscope containing fluid*

1. INTRODUCTION

An interesting effect of the stabilization in unbalanced gyroscope of Lagrange by the second rotating has been found in the works of Donetsk school of mechanics under supervision of P.V. Kharlamov [1-5]. In S.L. Sobolev's known work [6] it was shown that the Lagrange gyroscope if contains the ideal fluid is rather unstable. The Y.N. Kononov's work [7] shows a possibility of the rotation stabilization of the gyroscope by introduction in a cavity transversal and coaxial partitions. However, in practice it cannot always be carried out.

The possibility of the stabilization by rotating the rigid body in unstable rotation of the Lagrange gyroscope containing an ideal fluid is shown in our study. The equations of the works [8, 9] are foundations of the equations of motion for the considered mechanical system. Some results of the work have been reported at the ICTAM04 [10].

Consider the rotation of the Lagrange gyroscope with the cavity containing an ideal incompressible fluid around a fixed point O_1. The considered gyroscope (body S_1) has the common point O_2 with the second rotating rigid body S^0_2. The body S_1 consists of a rigid body S^0_1 and the ideal fluid contained in the rigid body cavity (Fig. 1).

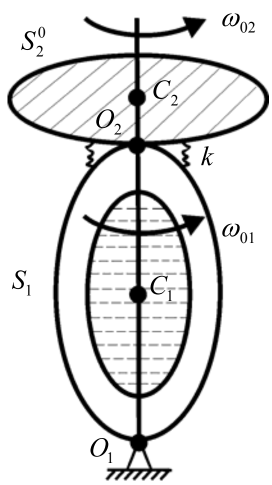

Fig. 1.

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The rigid bodies S_1^0 and S_2^0 are connected in a point O_2 by the elastic restoring spherical hinge with the coefficient of elasticity k ($k > 0$). Let us consider the possibility of stabilization for unstable rotation of a body S_1 by rotating the rigid body S_2^0.

The first rigid body S_1^0 and a fluid are rotating completely with an angular velocity ω_{01} around the axis of geometrical and dynamic symmetry O_1O_2, and the second rigid body S_2^0 - with angular velocity ω_{02} around the axis O_2C_2.

The common point O_2 lies on a straight line O_1C_2, where C_1 and C_2 - are the centers of mass of bodies S_1 and S_2^0 respectively.

The considered system is a special case of the system of the connected rigid bodies with the cavities containing a fluid, investigated in works [8, 9] and therefore the characteristic equation of motion is as follows:

$$\begin{vmatrix} F_1 & \mu + \dfrac{k}{\lambda^2} \\ \mu + \dfrac{k}{\lambda^2} & F_2 \end{vmatrix} = 0 \tag{1}$$

Here

$$F_1 = A_1^{'} + \frac{C_1^{'}}{\lambda} + \frac{a_1^* g - k}{\lambda^2} - (\lambda + \omega_{01}) \sum_{n=1}^{\infty} \frac{E_n}{\lambda + \lambda_n^{'}}, \qquad F_2 = A_2 + \frac{C_2^{'}}{\lambda} + \frac{a_2^* g - k}{\lambda^2},$$

$$A_1^{'} = A_1 + m_2 s_1^2, \quad \mu = s_1 a_2^*, \quad a_1^* = m_1 c_1 + s_1 m_2, \quad a_2^* = m_2 c_2,$$

$$s_1 = O_1O_2, \quad c_i = O_iC_i, \quad C_i^{'} = C_i \omega_{oi} \quad i = (1, 2),$$

m_1 and m_2 - respectively the mass of the body S_1 and the rigid body S_2^0; A_i and C_i –are respectively the equatorial and axial inertia moments of the bodies S_1 and S_2^0 with respect to the point O_i ($i = 1,2$); $\lambda_n^{'} = \tilde{\lambda}_n \omega_{0i}$, $\tilde{\lambda}_n = 1 - \lambda_n / \omega_{01}$.

Coefficient of inertial connection E_n and eigen numbers λ_n are determined from the solution of a corresponding boundary value problem and they are defined only by the geometry of a cavity. Values of the sizes for ellipsoidal, cylindrical and conical cavities are given in [11].

The necessary condition for stability of permanent rotation in the considered system is the following: all roots of the characteristic equation (1) are real.

The equation (1) in case of absence of the relative motion of a fluid ($E_n \equiv 0$, a "frozen" fluid) coincides with the equation obtained and investigated in works [4, 5].

At $k = \infty$ (the cylindrical hinge) the equation (1) is reduced to the equation

$$\tilde{F}_1 + \tilde{F}_2 + 2\mu = 0, \tag{2}$$

where

$$\tilde{F}_1 = A_1^{'} + \frac{C_1^{'}}{\lambda} + \frac{a_1^* g}{\lambda^2} - (\lambda + \omega_{01}) \sum_{n=1}^{\infty} \frac{E_n}{\lambda + \lambda_n^{'}}, \qquad \tilde{F}_2 = A_2 + \frac{C_2^{'}}{\lambda} + \frac{a_2^* g}{\lambda^2}.$$

If elastic restoring moment is absent ($k = 0$) and the center of mass of the second rigid body S_2^0 coincides with the common point O_2 ($c_2 = 0$, $\mu = 0$) the characteristic equation (1) is divided into two independent equations and in this case the possibility of stabilization for unstable rotation of a rigid body with a fluid by rotating rigid body is absent.

As it is known [11] in the majority of practically important cases in the equation (1) it is enough to take into account only the basic tone of the fluid oscillation ($n = 1$). It is always true for ellipsoidal cavities because from an infinite spectrum of the eigen frequencies λ_n the harmonic corresponding to a unique value λ_1 is raised [11].

If we take into account only the first harmonic ($n = 1$) in the equation (1) this equation can be written as a polynomial of the fifth degree

$$a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0, \tag{3}$$

where

$$\begin{aligned}
&a_0 = A_1^* A_2 - \mu^2 > 0, \quad a_1 = (A_1' A_2 - \mu^2)\lambda_1' + A_2 C_1^* + A_1^* C_2 \omega_{02},\\
&a_2 = A_2 C_1' \lambda_1' + g(A_1^* a_2^* + A_2 a_1^*) - (A_1^* + A_2 + 2\mu)k + (A_1' \lambda_1' + C_1^*) C_2 \omega_{02},\\
&a_3 = [g(A_1' a_2^* + A_2 a_1^*) - (A_1' + A_2 + 2\mu)k]\lambda_1' - C_1^* k + g(a_2^* C_1^* + a_1^* C_2 \omega_{02}) + (C_1' \lambda_1' - k) C_2 \omega_{02},\\
&a_4 = (a_2^* g - k) C_1' \lambda_1' + [a_1^* a_2^* g - k(a_1^* + a_2^*)]g + (a_1^* g - k)\lambda_1' C_2 \omega_{02},\\
&a_5 = g[a_1^* a_2^* g - k(a_1^* + a_2^*)]\lambda_1',\\
&A_1^* = A_1' - E_1, \quad C_1^* = C_1' - E_1', \quad C_1' = C_1 \omega_{01}, \quad E_1' = E_1 \omega_{01}.
\end{aligned}$$

Conditions of the reality of roots of the equation of the fifth degree are as follows:

$$\begin{aligned}
&d_1 = M_1^2 - M_1 M_3 > 0,\\
&d_2 = 4 d_1 d_{10} - 9 d_{11} > 0,\\
&d_3 = d_2 h_2 - 2 h_1^2 > 0,\\
&d_4 = d_3 (4 h_1 h_3 - h_2 h_4) - 2(2 d_2 h_3 - h_1 h_4)^2 > 0.
\end{aligned} \tag{4}$$

Here

$$a_0 = M_1 > 0, \quad a_1 = 5M_2, \quad a_2 = 10M_3, \quad a_3 = 10M_4, \quad a_4 = 5M_5, \quad a_5 = M_6;$$

$$d_{10} = 6M_3^2 - 5M_2 M_4 - M_1 M_5, \quad d_{11} = M_2 M_3 - M_1 M_4,$$

$$h_1 = d_1(16\tilde{h}_1 - 15 h_{25}) - 6 h_{23} h_{24}, \quad h_2 = 8 d_1 h_{35} + 48 h_{23} \tilde{h}_2 - 8 h_{24} d_{10},$$

$$h_3 = 6 h_{35} h_{23} - h_{25} d_{10}, \quad h_4 = 8 d_1 h_{35} - 3 h_{23} h_{25},$$

$$\tilde{h}_1 = M_3 M_4 - M_1 M_6, \quad \tilde{h}_2 = 6M_3^2 - 5M_2 M_4 - M_1 M_5,$$

$$h_{23} = M_2 M_3 - M_1 M_4, \quad h_{24} = M_2 M_4 - M_1 M_5,$$

$$h_{25} = M_2 M_5 - M_1 M_6, \quad h_{35} = M_3 M_5 - M_2 M_6.$$

After simple transformations it is possible to show that the system of inequalities (4) is equivalent to inequalities

$$\begin{cases} d_1 > 0,\\ d_2 > 0,\\ \tilde{d}_3 > 0,\\ \tilde{d}_4 > 0, \end{cases} \tag{5}$$

where $d_3 = 2d_1\widetilde{d}_3$, $d_4 = 2d_1^2\widetilde{d}_4$; $\widetilde{d}_3$ and $\widetilde{d}_4$ are polynomials of the 6th and the 8th degree in a_i ($i=\overline{0,5}$), respectively.

Stabilization of the rotation of a rigid body with a fluid can be carried out by the following parameters of the second rigid body: ω_{02}, k, C_2, A_2, m_2, c_2. Since the parameters C_2 and ω_{02} are contained in the coefficients a_i in terms of a product we shall appoint this product through ω_0.

We research the influence of the parameter ω_0 on a possibility of the stabilization. For this purpose we designate

$$a_1 = 5(\widetilde{a}_1\omega_0 + b_1), \quad a_2 = 10(\widetilde{a}_2\omega_0 + b_2), \quad a_3 = 10(\widetilde{a}_3\omega_0 + b_3), \quad a_4 = 5(\widetilde{a}_4\omega_0 + b_4). \tag{6}$$

After substituting the ratio (6) into inequalities (5), we obtain

$$\begin{cases} d_{12}\omega_0^2 + d_{11}\omega_0 + d_{10} > 0, \\ d_{24}\omega_0^4 + d_{23}\omega_0^3 + \ldots + d_{21}\omega_0 + d_{20} > 0, \\ d_{36}\omega_0^6 + d_{35}\omega_0^5 + \ldots + d_{31}\omega_0 + d_{30} > 0, \\ d_{48}\omega_0^8 + d_{47}\omega_0^7 + \ldots + d_{41}\omega_0 + d_{40} > 0, \end{cases} \tag{7}$$

where

$$d_{12} = \widetilde{a}_1^2 > 0, \quad d_{24} = 5\widetilde{a}_1^2(3\widetilde{a}_2^2 - 4\widetilde{a}_1 a_3),$$

$$d_{36} = 28\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4 - 9\widetilde{a}_1^2\widetilde{a}_4^2 - 16\widetilde{a}_1\widetilde{a}_3^3 - 12\widetilde{a}_2^3\widetilde{a}_4 + 8\widetilde{a}_3^2\widetilde{a}_2^2 = d_{33k}k^3 + d_{32k}k^2 + d_{31k}k + d_{30k},$$

$$d_{36} = 72\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4 - 27\widetilde{a}_1^2\widetilde{a}_4 - 32\widetilde{a}_1\widetilde{a}_3^3 - 32\widetilde{a}_2^3\widetilde{a}_4 + 16\widetilde{a}_3^2\widetilde{a}_2^2 = d_{44k}k^3 + d_{42k}k^2 + d_{41k}k + d_{40k},$$

$$\widetilde{a}_1 = A_1^*, \quad 10\widetilde{a}_2 = A_1'\lambda_1' + C_1^* > 0, \quad 10\widetilde{a}_3 = a_1^* g - k, \quad 5\widetilde{a}_4 = (a_1^* g - k)\lambda_1',$$

$$d_{33k} = 2A_1^*/625 > 0, \quad d_{43k} = 2d_{33k} > 0.$$

At $k > ga_1^*$ $\widetilde{a}_3 < 0$, $\widetilde{a}_4 < 0$ and $d_{24} > 0$. Coefficients d_{36} and d_{48} are the cubic polynomials in parameter k with positive coefficients at the higher degrees. Thus, at big enough elastic restoring moment $d_{24} > 0$, $d_{36} > 0$ and $d_{48} > 0$ also there is such a value ω_0 at which the inequalities (7) are valid. Hence, at big enough ω_0 and k, the stabilization for unstable rotation of a rigid body with a fluid is possible.

In the work [5] it is pointed out that the influence of rigidity in the spherical hinge on the effect of stabilization for unbalanced rigid body has a complicated character. Therefore we consider the influence of the elastic restoring moment on a possibility of stabilization for unstable rotation of a rigid body with a fluid. For this purpose we designate

$$a_2 = 10(\widetilde{a}_2 k + b_2), \quad a_3 = 10(\widetilde{a}_3 k + b_3), \quad a_4 = 5(\widetilde{a}_4 k + b_4), \quad a_5 = \widetilde{a}_5 k + b_5. \tag{8}$$

After substitution (8) in inequalities (5) we obtain

$$\begin{cases} d_{11}k + d_{10} > 0, \\ d_{23}k^3 + d_{22}k^2 + d_{21}k + d_{20} > 0, \\ d_{35}k^5 + d_{34}k^4 + \ldots + d_{31}k + d_{30} > 0, \\ d_{47}k^7 + d_{46}k^6 + \ldots + d_{41}k + d_{40} > 0. \end{cases} \tag{9}$$

Here

$$d_{11} = -a_0\tilde{a}_2, \quad d_{23} = -24a_0\tilde{a}_2^3, \quad d_{35} = 160a_0\tilde{a}_2^3(3\tilde{a}_2\tilde{a}_4 - 2\tilde{a}_3^2),$$

$$d_{47} = 128a_0\tilde{a}_2^3(40\tilde{a}_3^3\tilde{a}_5 - 25\tilde{a}_3^2\tilde{a}_4^2 + 27\tilde{a}_2^2\tilde{a}_5^2 + 50\tilde{a}_2\tilde{a}_4^3 - 90\tilde{a}_2\tilde{a}_3\tilde{a}_4\tilde{a}_5),$$

$$10\tilde{a}_2 = -(A_1^* + A_2 + 2\mu) < 0, \quad 10\tilde{a}_3 = -[(A_1^{'} + A_2 + 2\mu)\lambda_1^{'} + C_1^* + \omega_0] < 0, \tag{10}$$

$$5\tilde{a}_4 = -[(C_1^{'} + \omega_0)\lambda_1^{'} + (a_1^* + a_2^*)g] < 0, \quad \tilde{a}_5 = -(a_1^* + a_2^*)g\lambda_1^{'} < 0.$$

From (10) it follows that $d_{11} > 0$, $d_{23} > 0$ and the coefficients d_{35} and d_{47} are respectively the polynomials of the 2-nd and the 4-th degree relative to ω_0 with positive coefficients at the higher degrees. At big enough ω_0 and k inequalities (9) are valid and as it was earlier remarked, the stabilization for unstable rotation of a rigid body with a fluid is possible.

Let us consider a case of the cylindrical hinge ($k = \infty$). In this case instead of the inequalities (9) it is more convenient to use the equation (2) and at $n = 1$ to write down the conditions of the reality of roots of the cubic equation

$$g_0\lambda^3 + 3g_1\lambda^2 + 3g_2\lambda + g_3 = 0$$

as

$$d = 4(g_1^2 - g_0g_2)(g_2^2 - g_1g_3) - (g_1g_2 - g_0g_3)^2 > 0$$

or

$$d_4\omega_0^4 + d_3\omega_0^3 + d_2\omega_0^2 + d_1\omega_0 + d_0 = 0, \tag{11}$$

where

$$d_4 = 3\tilde{a}_2^2 > 0, \quad d_3 = 6\tilde{a}_2(\tilde{a}_2b_1 + b_2) - 4(\tilde{a}_2^3a_0 + a_3),$$

$$d_2 = 3[(b_1^2 - 4a_0b_2)\tilde{a}_2^2 + 2(a_0a_3 + 2b_1b_2)\tilde{a}_2 + b_2^2 - 4b_1a_3],$$

$$d_1 = 6[(a_0a_3b_1 + b_1^2b_2 - 2a_0b_2^2)\tilde{a}_2 + a_0a_3b_2 + b_1b_2^2 - 2a_3b_1^2],$$

$$d_0 = 4(b_1^2 - a_0b_2)(b_2^2 - b_1a_3) - (b_1b_2 - a_0a_3)^2,$$

$$g_0 = A_1^* + A_2 + 2\mu, \quad 3g_1 = (A_1^{'} + A_2 + 2\mu)\lambda_1^{'} + \tilde{C}_1^* + \omega_0,$$

$$\tilde{a}_1 = 1/3, \quad \tilde{a}_2 = 1/3\lambda_1^{'}, \quad b_1 = (A_1^{'} + A_2 + 2\mu)\lambda_1^{'}, \quad b_2 = (a_1^* + a_2^*)g + C_1^{'}\lambda_1^{'}.$$

So as $d_4 > 0$ if we assume that the equation corresponding to inequality (11) has three positive roots and the inequality has the solution

$$\{\omega_1 < \omega_0 < \omega_2\} \bigcup \{\omega_0 < \omega_3\},$$

and if one positive root is ω_0^* then $\omega_0 > \omega_0^*$.

Thus at big enough elastic restoring moment and the big angular velocity rotations of the second rigid body stabilization for unstable rotation of rigid body with a fluid is possible.

For the confirmation of the results of analytical researches, the numerical calculations have been carried out for the ellipsoidal cavity on formulas (7), (9) using the following values of the parameters: $\omega_{02} = 0, 10, 10^2, 10^3$; $k = 0, 1, 10, 10^2, 10^3$; $\omega_{01} = 1 \div 500$; $m_1 = const$; $\beta_1 = 0{,}02 \div 4$ $\beta = c/a$; $A_{01} = C_{01} = 0$. The second rotating rigid body was slightly concave, convex and flat thin circular disk (Fig. 2).

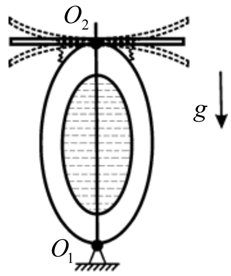

Fig. 2.

The results of the numerical calculations for not free system are presented in fig. 3-6 ($c_2 = 0$, $m_1 = const$, $E_1 \neq 0$). The areas of the stability are dark.

$k = 100, \quad \omega_{02} = 0$

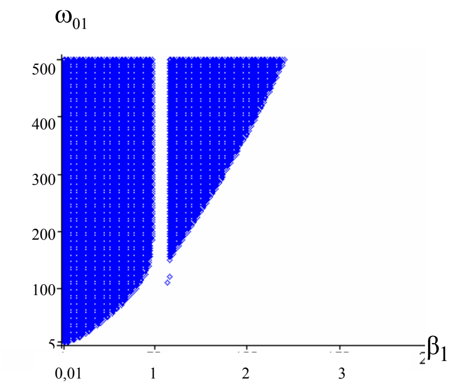

Fig. 3.

$k = 100, \quad \omega_{02} = 0$

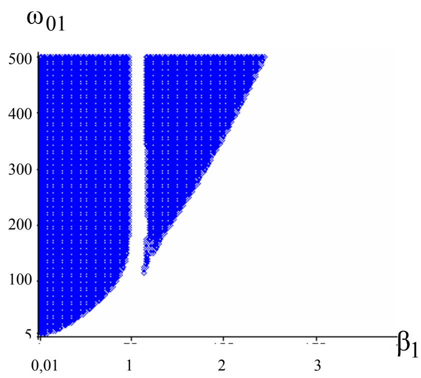

Fig. 4.

$k = 100, \quad \omega_{02} = 1000$

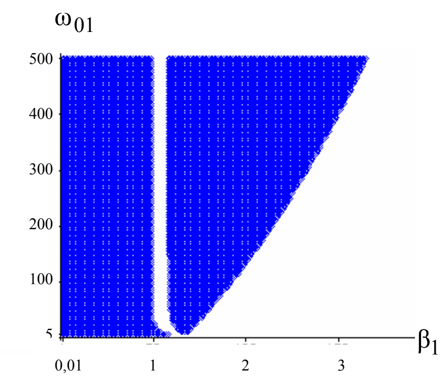

Fig. 5.

$k = 1000, \quad \omega_0 = 1000$

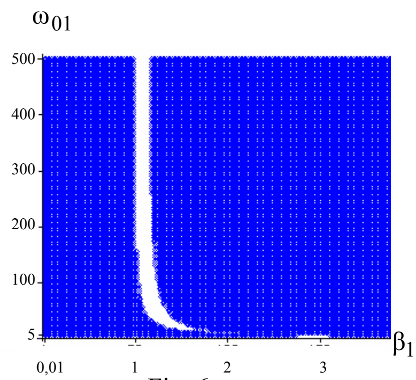

Fig. 6.

Following the analytical and numerical researches, the conclusions are made:

1. An unstable rotation of a rigid body with cavities containing a fluid is possible to stabilize by the rotating rigid body.
2. If elastic restoring moments are absent and the center of mass of the rotating rigid body coincides with the common point of the two rigid bodies, the stabilization will be impossible.
3. The effect similar to the action of restoring moment on the considered system is observed at the big angular velocity of rotation of a rigid body ($\omega_0 > 100$) and at the big elastic restoring moment ($k > 100$).

For the counterbalanced rotating rigid body ($c_2 < 0$) the effect of the stabilization increases in comparison with the unbalanced rigid body ($c_2 \geq 0$).

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O STABILIZACIJI ROTACIJE NESTABILNOG GIROSKOPA SA FLUIDOM PUTEM ROTACIJE KRUTOG TELA

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U radu je prikazana mogućnost stabilizacije nestabilne rotacije Lagranževog giroskopa koji sadrži idealan fluid, putem rotacije krutog tela. U slučaju balansirane rotacije efekat stabilizacije se povećava u poređenju sa neizbalansiranim krutim telom.

Ključne reči: *stabilizacija rotacije, Lagranžev giroskop sa fluidom*