A METHOD FOR NUMERICAL EVALUATING OF INVERSE Z-TRANSFORM

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Abstract. We will discuss the problem of finding the best approximations in the space of real sequences. We introduce orthogonal sequences using Z-Transform and apply it in approximating of inverse Z-Transform. We will illustrate it by some examples.

Key words: Approximation, Orthogonal Functions, Z-Transform

1. INTRODUCTION

The Z-Transform is used to take discrete time domain signals into a complex-variable frequency domain. It plays a similar role to the one the Laplace Transform does in the continuous time domain. Like the Laplace, the Z-Transform opens up new ways of solving problems and designing discrete domain applications. Z-Transform and inverse Z-Transform have applications in numerous sciences: theory of probability, difference equations, signal processing, filter design and so on.

Let \( h = \{h_j\}_{j=0}^\infty \) be an unknown sequence in the space \( l_2 \) whose Z-Transform is a known function \( H(z) \), i.e.,

\[
Z h = \sum_{j=0}^{\infty} h_j z^{-j} = H(z).
\]

The sequence \( h \) is the inverse Z-Transform of \( H(z) \), i.e. \( h = Z^{-1} H(z) \), defined by

\[
h_j = \frac{1}{2\pi i} \oint_{\Gamma} H(z) z^{-1} \, dz \quad (j \in \mathbb{N}_0),
\]

where \( \Gamma \) is a contour in the complex plane containing all poles of \( H(z) \).

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The finding of the inverse Z-Transform is closed with a lot of troubles. We will try to reconstruct this unknown sequence numerically.

Therefore, we will remind on some properties of the Z-Transform and the space $l^2$.

The region of convergence of the Z-Transform of $h$ is the range of values of $z$ for which $H(z)$ is finite. The Z-Transform definition will converge absolutely when the series of real numbers

$$\sum_{j=0}^{\infty} h_j z^{-j}$$

converges. The ratio test for convergence states

$$\lim_{n \to \infty} \left| \frac{h_{n+1} z^{-n}}{h_n z^{-n}} \right| < 1 \quad \Rightarrow \quad |z| > \lim_{n \to \infty} \frac{|h_{n+1}|}{|h_n|} = R.$$ 

Therefore the series converges outside the circle with the centre at origin and radius $R$.

We define a composition and a scalar product in $l^2$ by

$$f \circ g = \{f_j g_j\}_{j \in N}, \quad \langle f, g \rangle = \sum_{j=0}^{\infty} f_j g_j \quad (f, g \in l^2). \quad (1.1)$$

The square of norm of a sequence $f$ in the space $l^2$ is $\|f\|^2 = \langle f, f \rangle$.

A sequence $\{\varphi^{(n)}\}_{n \in N}$ is orthogonal with respect to inner product (1.1) if

$$\langle \varphi^{(m)}, \varphi^{(n)} \rangle = \delta_{m,n} \| \varphi^{(n)} \|^2 \quad (m, n \in N),$$

where $\delta_{m,n}$ is Kronecker delta. We suppose that this sequence is normalized by initial value $\varphi^{(n)}_0 = 1 \quad (n \in N)$.

Let $S^{(n)}$ be the linear over the first $n$ members of orthogonal sequence, i.e.,

$$S^{(n)} = \left\{ \sum_{k=1}^{n} a_k \varphi^{(k)} \mid a_k \in R \quad (k = 1, 2, \ldots, n) \right\}. \quad (1.2)$$

Our purpose is to find approximation with the property

$$\min_{f \in S^{(n)}} \| h - f \| = \| h - h^{(n)} \| \quad (1.3)$$

which we call the best approximation of $h$ in $S^{(n)}$.

If $Zf = F(z)$ and $Zg = G(z)$, then

$$F(\omega)G(z/\omega) = \left( \sum_{j=0}^{\infty} \frac{f_j}{\omega^j} \right) \left( \sum_{k=0}^{\infty} \frac{G_k \omega^k}{z^k} \right) = \sum_{m=0}^{\infty} \frac{1}{\omega^m} \sum_{k=0}^{\infty} \frac{\sum_{j+k+1}^{\infty} f_j G_k}{\omega^{j+k}}.$$ 

Knowing that

$$\frac{1}{2\pi i} \int_{|\omega| = R} \frac{d\omega}{\omega^{j+1}} = \delta_{j,0},$$

we can express $F(\omega)G(z/\omega)$ as

$$\sum_{k=0}^{\infty} \frac{1}{\omega^m} \sum_{k=0}^{\infty} \frac{\sum_{j+k+1}^{\infty} f_j G_k}{\omega^{j+k}}.$$
we conclude that it is valid

\[ Z(f \circ g)(z) = \frac{1}{2\pi i} \oint_{|z| = R} F(\omega)G(z/\omega) \frac{d\omega}{\omega}. \]

Since \( Z(f \circ g) \) is a holomorphic function in the point \( z = 1 \), the scalar product (1.2) can be represented in the following way

\[ \langle f, g \rangle = \frac{1}{2\pi i} \oint_{|z| = 1} F(\omega)G(1/\omega) \frac{d\omega}{\omega} = \sum \text{Res} \frac{F(\omega)G(1/\omega)}{\omega}. \]

Here, we will remind of the basic facts of \( q \)-calculus (see, for example, [1]). So, \( q \)-numbers, \( q \)-factorials and \( q \)-binomials are defined by

\[ [n] = \frac{1 - q^n}{1 - q}, \quad [n]! = [n][n-1] \cdots [1], \quad \left[ \frac{n}{k} \right] = \frac{[n]!}{[n-k]!k!}. \]

2. ORTHOGONAL SEQUENCES

We will start with the sequence \( E = \{e^{(n)}\}_{n \in \mathbb{N}} \), of the sequences defined by

\[ e^{(k)}_j = q^{|j-k|} \quad (j \in \mathbb{N}_0). \]

The sequence \( E \) is fundamental in \( l_2 \). That is why we can express \( \varphi^{(n)} \) by

\[ \varphi^{(n)} = \sum_{k=1}^{n} b_{nk} e^{(k)} \quad \text{where} \quad \varphi_f^{(n)} = \sum_{k=1}^{n} b_{nk} q^{|j-k|}. \]

From the other side, let us denote by

\[ \Phi_n(z) = Z \varphi^{(n)}. \]

\( Z \)-Transform of \( \varphi^{(n)} \). From

\[ Ze^{(k)} = \frac{z}{z - q^k}, \]

we have

\[ \Phi_n(z) = \sum_{k=1}^{n} b_{nk} Ze^{(k)} = z \sum_{k=1}^{n} b_{nk} \phi^{|j-k|}. \]

Since

\[ \Phi_n(z) = Z \varphi^{(n)} = \sum_{j=0}^{n} \varphi^{(n)} \quad \text{and} \quad \lim_{z \to \infty} \Phi_n(z) = \varphi_0^{(n)} = 1, \]

we conclude that \( \Phi_n(z) \) must be a rational function with the monic polynomials of the same degree at numerator and denominator. Because of orthogonality, we have
\[
\Phi_n(q^{-k}) = \sum_{j=1}^{n} q_j^{(n)} q^j = \left< \varphi^{(n)}, e^{(k)} \right> = 0 \quad (k = 1, 2, \ldots, n - 1).
\]

Hence
\[
\Phi_n(z) = \frac{z Q_{n-1}(z; 1/q)}{Q_n(z; q)},
\]
where
\[
Q_n(z; q) = \prod_{k=1}^{n} (z - q^k).
\]

Now, we can expand
\[
\frac{\Phi_n(z)}{z} = \frac{Q_{n-1}(z; 1/q)}{Q_n(z; q)} = \sum_{k=1}^{n} b_{nk} z - q^k,
\]
where
\[
b_{nk} = \frac{Q_{n-1}(q^k; 1/q)}{Q_n(q^k; q)} \quad (k = 1, 2, \ldots, n).
\]

By some evaluating, we have
\[
Q_{n-1}(q^k; 1/q) = \prod_{j=1}^{n-1} (q^k - q^{-j}) = \prod_{j=1}^{n-1} q^{k+j-1} = (-1)^{n-1} q^{-n(n-1)/2} \prod_{j=1}^{n-1} (1 - q^{k+j})
\]
and
\[
Q_n(q^k; q) = \prod_{j=1}^{n} (q^k - q^{-j}) = (-1)^{n-1} q^{-n(n-1)/2} \prod_{j=1}^{n-1} q^{k+n-1} (1 - q^{-j}) \prod_{j=1}^{n-1} (1 - q^j).
\]

At last, the coefficient \(b_{nk}\) is
\[
b_{nk} = (-1)^{n-k} q^{-\frac{n}{2}} \binom{n}{k}^{-1} \binom{n+k-1}{k-1} \quad (k = 1, 2, \ldots, n). \tag{2.3}
\]

The norm of \(\varphi^{(n)}\) can be expressed by
\[
\| \varphi^{(n)} \|_2^2 = \left< \varphi^{(n)}, \varphi^{(n)} \right> = \sum_{|z|<1} \Phi_n(z) \Phi_n(1/z).
\]

Since
\[
\Phi_n(1/z) = -q^{-n} z - \sum_{k=0}^{n-1} \frac{1}{(z - q^k)(z - q^{-k})} \Phi_n(z),
\]
we have
\[
\| \varphi^{(n)} \|_2^2 = -q^{-n} \sum_{x=q}^{x=q} \frac{1}{(z - q^k)(z - q^{-k})},
\]
and finally
\[
\| \varphi^{(n)} \|_2^2 = \frac{q^{n(1-n)}}{1-q^{2n}}. \tag{2.4}
\]
3. APPLICATIONS

Now, we can expand any sequence \( h \) from \( l_2 \) in the series

\[
h = \sum_{k=0}^{\infty} c_k \varphi^{(k)}, \quad \text{where} \quad c_k = \frac{\langle h, \varphi^{(k)} \rangle}{\| \varphi^{(k)} \|^2}.
\]  

(3.1)

From (2.1), we have

\[
\langle h, \varphi^{(k)} \rangle = \left\langle h, \sum_{j=0}^{k} b_{ji} e^{(i)} \right\rangle = \sum_{j=0}^{k} b_{ji} \langle h, e^{(i)} \rangle.
\]

If we denote by \( H(z) = Z h \), it follows

\[
\langle h, e^{(i)} \rangle = \sum_{j=0}^{\infty} h_j (q^i)^j = H(1/q^i).
\]

Hence we can rewrite \( c_k \) in the form

\[
c_k = \frac{1}{\| \varphi^{(k)} \|^2} \sum_{j=0}^{k} b_{ji} H(1/q^i).
\]

(3.2)

The function \( h^{(n)} \) defined by

\[
h^{(n)} = \sum_{k=0}^{n} c_k \varphi^{(k)}
\]

is the best approximation of \( h \) in the space \( l_2 \) with error

\[
\| h - h^{(n)} \|^2 = \| h \|^2 - \sum_{k=1}^{n} \| c_k \varphi^{(k)} \|^2.
\]

(3.3)

According to (1.4), we can evaluate exactly

\[
\| h \|^2 = \langle h, h \rangle = \sum_{|z|<1} \text{Res} \frac{H(z)H(1/z)}{z}.
\]

(3.4)

Example 3.1. Let

\[
H(z) = \frac{z(z + 0.1)}{(z - 0.2)^2(z - 0.3)(z - 0.4)}.
\]

In the Figure 3.1, the exact sequence \( h(j) \) (see M.R. Stojic [3])

\[
h(j) = (75j + 275)(0.1)^j - 400(0.3)^j + 125(0.4)^j
\]

is shown as the continuous line \( h(t) \) and the approximation is drawn by large points.

Applying our method we find the approximation of \( h(j) \), whose relative error is given in Table 3.1.
The relative error is evaluated by the cognition of the exact sequence.

But, very important thing is that we can estimate the error for unknown \( h(j) \) according to formulae (3.3) and (3.4). Especially, for this example, the function \( H(z)H(1/z)/z \) has the poles inside the unit circle in the points 0.2, 0.3 and 0.4. According to (3.4), the square norm of \( h \) is \( ||h||^2 = 3.5722510455544322893 \).

Now, applying (3.3), we can estimate the square norms of errors. So, for \( q=3/4 \), it is \( 4.13047468\times10^{-10} \) and, for \( q = 5/6 \), we get \( 2.3451145\times10^{-12} \).

In the formula (3.3), in the norm of error of approximation the coefficients \( c_k \) and the norms \( ||\varphi^{(k)}|| \) which depend on the parameter \( q \) take part. So, by the suitable choice of \( q \), we can exert an influence to the size of the error of approximation.)
A Method for Numerical Evaluating of Inverse Z-Transform

REFERENCES


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