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A METHOD FOR NUMERICAL EVALUATING OF INVERSE Z-TRANSFORM

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Predrag M. Rajković[1], Miomir S. Stanković[2], Slađana D. Marinković[3]

[1]Department of Mathematics, Faculty of Mechanical Engineering
[2]Department of Mathematics, Faculty of Occupational Safety
[3]Department of Mathematics, Faculty of Electrical Engineering
University of Niš, Serbia and Montenegro

Abstract. *We will discuss the problem of finding the best approximations in the space of real sequences. We introduce orthogonal sequences using Z-Transform and apply it in approximating of inverse Z-Transform. We will illustrate it by some examples.*

Key words: *Approximation, Orthogonal Functions, Z-Transform*

1. INTRODUCTION

The Z-Transform is used to take discrete time domain signals into a complex-variable frequency domain. It plays a similar role to the one the Laplace Transform does in the continuous time domain. Like the Laplace, the Z-Transform opens up new ways of solving problems and designing discrete domain applications. Z-Transform and inverse Z-Transform have applications in numerous sciences: theory of probability, difference equations, signal processing, filter design and so on.

Let $h = \{h_j\}_{j\in N_0}$ be an unknown sequence in the space $\boldsymbol{l_2}$ whose Z-Transform is a known function $H(z)$, i.e.,

$$Zh = \sum_{j=0}^{\infty} h_j z^{-j} = H(z).$$

The sequence h is the inverse Z-Transform of H(z), i.e. $h = Z^{1}H$(z), defined by

$$h_j = \frac{1}{2\pi i}\oint_{\Gamma} H(z) z^{j-1} dz \qquad (j \in N_0).$$

where Γ is a contour in the complex plane containing all poles of $H(z)$.

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The finding of the inverse Z-Transform is closed with a lot of troubles. We will try to reconstruct this unknown sequence numerically.

Therefore, we will remind on some properties of the Z-Transform and the space l_2.

The region of convergence of the Z-Transform of h is the range of values of z for which $H(z)$ is finite. The Z-Transform definition will converge absolutely when the series of real numbers

$$\sum_{j=0}^{\infty} h_j z^{-j}$$

converges. The ratio test for convergence states

$$\lim_{n\to\infty}\left|\frac{h_{n+1}z^{-n-1}}{h_n z^{-n}}\right| < 1 \quad \Rightarrow \quad |z| > \lim_{n\to\infty}\left|\frac{h_{n+1}}{h_n}\right| = R.$$

Therefore the series converges outside the circle with the centre at origin and radius R.

We define a composition and a scalar product in l_2 by

$$f \circ g = \{f_j g_j\}_{j\in N_0}, \qquad \langle f, g\rangle = \sum_{j=0}^{\infty} f_j g_j \quad (f, g \in l_2). \tag{1.1}$$

The square of norm of a sequence f in the space l_2 is $\|f\|^2 = <f,f>$.

A sequence $\{\varphi^{(n)}\}_{n\in N}$ is orthogonal with respect to inner product (1.1) if

$$\left\langle \varphi^{(m)}, \varphi^{(n)} \right\rangle = \delta_{m,n} \, \| \varphi^{(n)} \|^2 \qquad (m,n \in N),$$

where $\delta_{m,n}$ is Kronecker delta. We suppose that this sequence is normalized by initial value $\varphi_0^{(n)} = 1$ $(n\in N)$.

Let $S^{(n)}$ be the linear over the first n members of orthogonal sequence, i.e.,

$$S^{(n)} = \left\{ \sum_{k=1}^{n} a_k \varphi^{(k)} \quad | \; a_k \in R \quad (k = 1,2,...,n) \right\}. \tag{1.2}$$

Our purpose is to find approximation with the property

$$\min_{f\in S^{(n)}} \| h - f \| = \| h - h^{(n)} \| . \tag{1.3}$$

which we call *the best approximation* of h in $S^{(n)}$.

If $Z f = F(z)$ and $Z g = G(z)$, then

$$F(\omega)G(z/\omega) = \left(\sum_{j=0}^{\infty} \frac{f_j}{\omega^j} \right) \left(\sum_{k=0}^{\infty} g_k \frac{\omega^k}{z^k} \right) = \sum_{m=0}^{\infty} \frac{1}{\omega^m} \sum_{k=0}^{\infty} \frac{f_{k+m} g_k}{z^k}.$$

Knowing that

$$\frac{1}{2\pi i} \oint_{|\omega|=R} \frac{d\omega}{\omega^{k+1}} = \delta_{k,0},$$

we conclude that it is valid

$$Z(f\circ g)(z)=\frac{1}{2\pi i}\oint_{|\omega|=R}\mathrm{F}(\omega)\mathrm{G}(z/\omega)\frac{d\omega}{\omega}.$$

Since $Z(f\circ g)$ is a holomorphic function in the point $z = 1$, the scalar product (1.2) can be represented in the following way

$$\langle f,g\rangle=\frac{1}{2\pi i}\oint_{|z|=1}\mathrm{F}(\omega)\mathrm{G}(1/\omega)\frac{d\omega}{\omega}=\sum_{|z_k|<1}\operatorname*{Res}_{\omega=z_k}\frac{F(\omega)G(1/\omega)}{\omega}. \tag{1.4}$$

Here, we will remind of the basic facts of q-calculus (see, for example, [1]). So, q-numbers, q-factorials and q-binomials are defined by

$$[n]=\frac{1-q^n}{1-q},\qquad [n]!=[n][n-1]\cdots[1],\qquad \begin{bmatrix}n\\k\end{bmatrix}=\frac{[n]!}{[n-k]![k]!}.$$

2. ORTHOGONAL SEQUENCES

We will start with the sequence $E=\{e^{(n)}\}_{n\in N}$, of the sequences defined by

$$e_j^{(k)}=q^{jk}\qquad (j\in N_0).$$

The sequence E is fundamental in l_2. That is why we can express $\varphi^{(n)}$ by

$$\varphi^{(n)}=\sum_{k=1}^{n}b_{nk}e^{(k)}\qquad\text{where}\qquad \varphi_j^{(n)}=\sum_{k=1}^{n}b_{nk}q^{jk}. \tag{2.1}$$

From the other side, let us denote by

$$\Phi_n(z)=Z\varphi^{(n)}. \tag{2.2}$$

Z-Transform of $\varphi^{(n)}$. From

$$Ze^{(k)}=\frac{z}{z-q^k},$$

we have

$$\Phi_n(z)=\sum_{k=1}^{n}b_{nk}Ze^{(k)}=z\sum_{k=1}^{n}\frac{b_{nk}}{z-q^k}.$$

Since

$$\Phi_n(z)=Z\varphi^{(n)}=\sum_{j=0}^{\infty}\frac{\varphi_j^{(n)}}{z^j}\qquad\text{and}\qquad \lim_{z\to\infty}\Phi_n(z)=\varphi_0^{(n)}=1,$$

we conclude that $\Phi_n(z)$ must be a rational function with the monic polynomials of the same degree at numerator and denominator. Because of orthogonality, we have

$$\Phi_n(q^{-k}) = \sum_{j=1}^{\infty} \varphi_j^{(n)} q^{jk} = \left\langle \varphi^{(n)}, e^{(k)} \right\rangle = 0 \qquad (k = 1,2,...,n-1).$$

Hence

$$\Phi_n(z) = \frac{z\, Q_{n-1}(z;1/q)}{Q_n(z;q)},$$

where

$$Q_n(z;q) = \prod_{k=1}^{n} (z - q^k).$$

Now, we can expand

$$\frac{\Phi_n(z)}{z} = \frac{Q_{n-1}(z;1/q)}{Q_n(z;q)} = \sum_{k=1}^{n} \frac{b_{nk}}{z - q^k},$$

where

$$b_{nk} = \frac{Q_{n-1}(q^k;1/q)}{Q_n'(q^k;q)} \qquad (k = 1,2,...,n).$$

By some evaluating, we have

$$Q_{n-1}(q^k;1/q) = \prod_{j=1}^{n-1} (q^k - q^{-j}) = \prod_{j=1}^{n-1} \frac{q^{k+j} - 1}{q^j} = (-1)^{n-1} q^{-(n-1)n/2} \prod_{j=1}^{n-1} (1 - q^{k+j})$$

and

$$Q_n'(q^k;q) = \prod_{\substack{j=1 \\ j\neq k}}^{n} (q^k - q^j) = (-1)^{k-1} q^{(k-1)k/2} \prod_{j=1}^{k-1} q^{k(n-k)} (1 - q^j) \prod_{j=1}^{n-k} (1 - q^j).$$

At last, the coefficient b_{nk} is

$$b_{nk} = (-1)^{n-k} q^{-\binom{n}{2} + \binom{k+1}{2} - kn} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix} \quad (k = 1,2,...,n). \tag{2.3}$$

The norm of $\varphi^{(n)}$ can be expressed by

$$\| \varphi^{(n)} \|^2 = \left\langle \varphi^{(n)}, \varphi^{(n)} \right\rangle = \sum_{|z_k|<1} \operatorname*{Res}_{z=z_k} \frac{\Phi_n(z)\Phi_n(1/z)}{z}.$$

Since

$$\Phi_n(1/z) = -q^{-n^2} \frac{z}{(z-q^n)(z-q^{-n})} \frac{1}{\Phi_n(z)},$$

we have

$$\| \varphi^{(n)} \|^2 = -q^{-n^2} \operatorname*{Res}_{z=q^n} \frac{1}{(z-q^n)(z-q^{-n})},$$

and finally

$$\| \varphi^{(n)} \|^2 = \frac{q^{n(1-n)}}{1 - q^{2n}}. \tag{2.4}$$

3. Applications

Now, we can expand any sequence h from l_2 in the series

$$h=\sum_{k=0}^{\infty} c_k \varphi^{(k)}, \qquad \text{where} \qquad c_k=\frac{\left\langle h,\varphi^{(k)}\right\rangle}{\|\varphi^{(k)}\|^2}. \tag{3.1}$$

From (2.1), we have

$$\left\langle h,\varphi^{(k)}\right\rangle=\left\langle h,\sum_{i=1}^{k} b_{ki} e^{(i)}\right\rangle=\sum_{i=1}^{k} b_{ki}\left\langle h,e^{(i)}\right\rangle.$$

If we denote by $H(z)=Z\,h$, it follows

$$\left\langle h,e^{(i)}\right\rangle=\sum_{j=0}^{\infty} h_j (q^i)^j=H(1/q^i).$$

Hence we can rewrite c_k in the form

$$c_k=\frac{1}{\|\varphi^{(k)}\|^2}\sum_{i=1}^{k} b_{ki} H(1/q^i). \tag{3.2}$$

The function $h^{(n)}$ defined by

$$h^{(n)}=\sum_{k=1}^{n} c_k \varphi^{(k)}$$

is the best approximation of h in the space l_2 with error

$$\|h-h^{(n)}\|^2=\|h\|^2-\sum_{k=1}^{n} c_k^2 \|\varphi^{(k)}\|^2. \tag{3.3}$$

According to (1.4), we can evaluate exactly

$$\|h\|^2=\left\langle h,h\right\rangle=\sum_{|z_k|<1} \operatorname*{Res}_{z=z_k}\frac{H(z)H(1/z)}{z}. \tag{3.4}$$

Example 3.1. Let

$$H(z)=\frac{z(z+0.1)}{(z-0.2)^2(z-0.3)(z-0.4)}.$$

In the Figure 3.1, the exact sequence $h(j)$ (*see M.R. Stojic* [3])

$$h(j)=(75j+275)(0.1)^j-400(0.3)^j+125(0.4)^j$$

is shown as the continuous line $h(t)$ and the approximation is drawn by large points.

Applying our method we find the approximation of $h(j)$, whose relative error is given in Table 3.1.

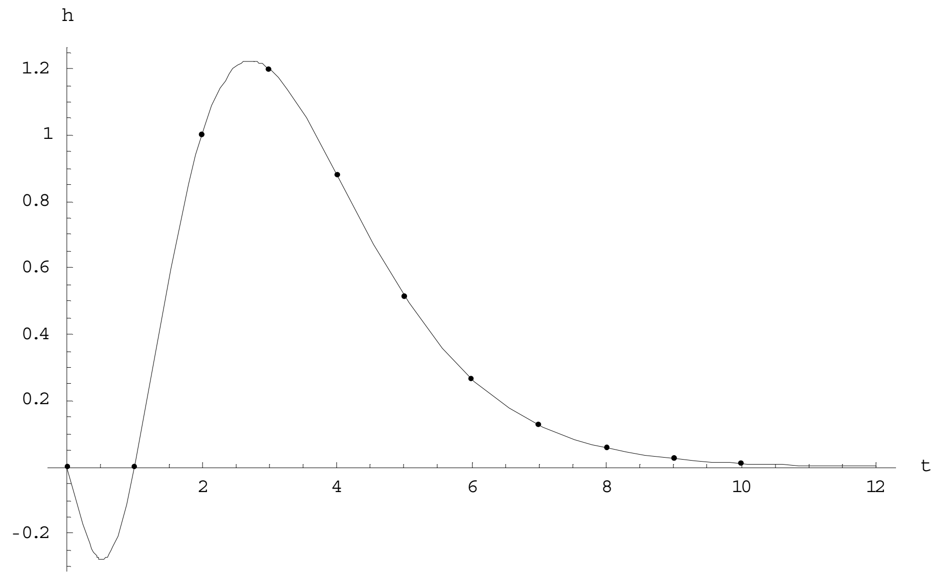

Fig. 3.1.

Table 3.1.

j	approx q=5/6	rel. error	approx q=3/4	rel. error
0	0	0	0	0
1	0	0	0	0
2	0.99999997978	0.202128 (-7)	1.00000000424	0.424853 (-8)
3	1.20000012119	0.100992 (-6)	1.19999994006	0.499529 (-7)
4	0.87999958787	0.468326 (-6)	0.88000049395	0.493950 (-6)
5	0.51600074713	0.144794 (-5)	0.51599759135	0.466792 (-5)
6	0.26679951778	0.180742 (-5)	0.26680658304	0.246741 (-4)
7	0.12755961246	0.303802 (-5)	0.12755192356	0.633148 (-4)
8	0.05791634663	0.598504 (-5)	0.05791519455	0.139072 (-4)
9	0.02538166043	0.181407 (-4)	0.02538820731	0.276083 (-3)
10	0.01085025302	0.488618 (-5)	0.01085445069	0.391762 (-3)

The relative error is evaluated by the cognition of the exact sequence**.**

But, very important thing is that we can estimate the error for unknown $h(j)$ according to formulae (3.3) and (3.4). Especially, for this example, the function $H(z)H(1/z)/z$ has the poles inside the unit circle in the points 0.2, 0.3 and 0.4. According to (3.4), the square norm of h is $||h||^2 =$ 3.5722510455544322893.

Now, applying (3.3), we can estimate the square norms of errors. So, for q=3/4, it is $4.13047468*10^{-10}$ and, for q = 5/6, we get $2.3451145*10^{-12}$.

In the formula (3.3), in the norm of error of approximation the coefficients c_k and the norms $||\varphi^{(k)}||$ which depend on the parameter q take part**.** So, by the suitable choice of q, we can exert an influence to the size of the error of approximation).

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JEDAN METOD ZA NUMERIČKO IZRAČUNAVANJE INVERZNE Z-TRANSFORMACIJE

Predrag M. Rajković, Miomir S. Stanković, Slađana D. Marinković

U radu proučavamo problem nalaženja najbolje aproksimacije u prostoru realnih nizova. Stoga, koristeći Z-transformiju, uvodimo ortogonalne nizove. Ove nizove upotrebljavamo u aproksimiranju inverzne Z-transformacije. Metod ilustrujemo odgovarajućim primerima.