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MODELLING OF LAMINATE COMPOSITES WITH EMBEDDED PIEZOELECTRIC ACTUATORS AND SENSORS

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Abstract. *The increasing need for lightweight structures requires the development of new kinds of materials. The general requirement of those materials is high specific mechanical properties, i.e. the ratio between mechanical properties and mass density. Laminate fiber reinforced materials, which have a very high strength and stiffness in desired directions and lower characteristics in other directions represent a very good choice, and they are already used in various applications like space structures, transportation applications etc. Further behavior improvement of structures made of laminate composites can be achieved by embedding components made of smart material between layers, like piezoelectric patches. In the paper the governing equations of an active layered plate are described based on the Mindlin-Reissner assumption. For a piezoelectric layer the electromechanical-coupled constitutive equations are introduced and the governing equations of piezoelectric actuating loads and sensor output are developed. Then the paper addresses the problem of the finite element formulation of a laminated piezoelectric plate type element.*

1. INTRODUCTION

The increasing need for lightweight structures requires the development of new kinds of materials. A great number of applications, like space structures, transport applications etc., require high specific mechanical properties, i.e. the ratio between mechanical properties and mass density. Laminate fiber reinforced materials, which have a very high strength and stiffness in desired directions and lower characteristics in other directions offer such a combination of characteristics. Their behavior can be further improved by embedding smart material components between layers, with the aim of vibration reduction. Over the last few years the use of piezoelectric materials as actuators and sensors in vibration control has been successfully demonstrated. Therefore, the choice to integrate piezoelectric material in laminate composites with the main idea of obtaining a multifunctional material with sensing and actuating properties besides load carrying function follows.

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The finite element method is the most powerful numerical technique ever developed for solving solid and structural mechanics problems in geometrically complicated regions. The development of the finite element analysis is based on the analytical model. Therefore, the fundamental equations governing the behavior of smart laminates shall be first derived, and then, the finite element formulation of the same problem based on it.

2. CONSTITUTIVE EQUATIONS

We consider a laminate structure with embedded piezoelectric patches covered with electrodes. The analytical model will be developed according to the first order shear deformation theory (FSDT), which takes the Mindlin-Reissner assumptions into account: a straight line perpendicular to the mid-plane before deformation remains straight and inextensible after deformation, but it does not remain necessarily perpendicular to the mid-plane. The displacement field is therefore given in the form:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{1}$$

where u_0 and v_0 are the in-plane displacements, w_0 is the transverse displacement and θ_x and θ_y are the section rotations. The stress field is defined by kinematical relations from the displacement field as:

$$\begin{aligned} & s_{11} = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_x}{\partial x} = s_{11}{}^0 + z\kappa_{11}{}^f; \quad s_{22} = \frac{\partial v_0}{\partial y} + z\frac{\partial \theta_y}{\partial y} = s_{22}{}^0 + z\kappa_{22}{}^f; \quad s_{13} = \frac{\partial w_0}{\partial x} + \theta_x; \\ & s_{12} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) = s_{12}{}^0 + z\kappa_{12}{}^f; \qquad s_{23} = \frac{\partial w_0}{\partial y} + \theta_y \end{aligned} \tag{2}$$

The constitutive equations of a linear piezoelectric material are:

$$\begin{aligned} \mathbf{T} &= \mathbf{C}^E\mathbf{S} - \mathbf{e}^\mathbf{T} E \\ D &= \mathbf{eS} + \boldsymbol{\varepsilon}^\mathbf{S} E, \end{aligned} \quad \text{(3a), or} \quad \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \\ D_3 \end{Bmatrix} = \begin{bmatrix} Q_{11}' & Q_{12}' & Q_{16}' & e'_{31} \\ Q_{12}' & Q_{22}' & Q_{26}' & e'_{31} \\ Q_{16}' & Q_{26}' & Q_{66}' & 0 \\ e'_{31} & e'_{31} & 0 & -\varepsilon'_{33} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{12} \\ -E_3 \end{Bmatrix} \quad \text{(3b)}$$

where $\mathbf{T} = (T_{11}\ T_{22}\ T_{12})$ is the in-plane stress vector, $\mathbf{S} = (S_{11}\ S_{22}\ S_{12})$ is the in-plane strain vector, E the electric field, D the electric displacement, $\mathbf{C}^E$ the piezoelectric material stiffness matrix at E constant, ε^S the dielectric permittivity at **S** constant, **e** is the vector of piezoelectric constants and Q_{ij}' are the reduced material stiffness constants in the reference system of the structure. In the equation (3b) we assume the uniform electric field and displacement across the thickness and aligned on the normal to the mid-plane, $D_1=D_2=0$. The piezoelectric material is polarized in thickness direction and has isotropic piezoelectric properties in the plane, $e'_{31} = e'_{32}$, and $e'_{36} = 0$, which is valid for the most common used piezomaterials. The transverse shear components are excluded from equation (3b) because they are decoupled from the in-plane components.

Taking into consideration the expression for D_3 in (3b) and boundary conditions, the Maxwell's equation for piezoelectric materials ($\nabla D = 0$, because there is no free charge in the piezoelectric material) yields the following expressions for the electric field and electric potential of the k-th piezoelectric layer:

$$\Phi_k = \frac{1}{2}\frac{e'_{31k}}{\varepsilon'_{33k}}\left(\frac{\partial\theta_x}{\partial x}+\frac{\partial\theta_y}{\partial y}\right)\left((z-z_{mk})^2-\left(\frac{h_k}{2}\right)^2\right)+\Delta\Phi_k\left(\frac{z-z_{mk}}{h_k}+\frac{1}{2}\right) \tag{4}$$

$$E_{3k} = -\frac{\Delta\Phi_k}{h_k}-\frac{e'_{31k}}{\varepsilon'_{33k}}\left(\frac{\partial\theta_x}{\partial x}+\frac{\partial\theta_y}{\partial y}\right)(z-z_{mk}) \tag{5}$$

where $\Delta\Phi_k$ is the difference of electric potential between electrodes of the k-th layer (piezolayer), z_{mk} is the distance of the mid-plane of the k-th layer to the mid-plane of the laminate and h_k is the thickness of the k-th layer.

The piezoelectric laminate equations, which relate the in-plane forces $\mathbf{N}=[N_x\ N_y\ N_{xy}]$, moments $\mathbf{M}=[M_x\ M_y\ M_{xy}]$ and transverse shear resultants $\mathbf{Q}=[Q_{xz}\ Q_{yz}]$ to the strains $[\mathbf{S_m}\ \boldsymbol{\kappa}_\mathbf{f}\ \mathbf{S_s}]$ are obtained by integrating stresses over the thickness:

$$\begin{Bmatrix}\mathbf{N}\\ \mathbf{M}\\ \mathbf{Q}\end{Bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{0}\\ \mathbf{B} & \mathbf{D}+\mathbf{D^e} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}\end{bmatrix}\begin{Bmatrix}\mathbf{S_m}\\ \boldsymbol{\kappa}_\mathbf{f}\\ \mathbf{S_s}\end{Bmatrix}+\sum_{k=1}^{n_e}\begin{bmatrix}\mathbf{I_3}\\ z_{mk}\mathbf{I_3}\\ \mathbf{0}\end{bmatrix}\mathbf{e_k}\Delta\Phi_k, \tag{6}$$

where $\mathbf{S_m}=[S_{11}^0\ S_{22}^0\ S_{12}^0]$ are the mid-plane strains, $\boldsymbol{\kappa}_\mathbf{f}=[\kappa_{11}^f\ \kappa_{22}^f\ \kappa_{12}^f]$ are the curvatures and $\mathbf{S_s}=[S_{12}\ S_{13}]$ are the transverse shear strains, n_e is the number of piezoelectric layers, $\mathbf{e_k}=[e_{31}\ e_{31}\ 0]$, **A** is the extensional stiffness matrix, **D** is the bending stiffness matrix, $\mathbf{D^e}$ is the electric-bending coupling stiffness matrix, **B** is the bending-extensional coupling stiffness matrix and **F** is the transverse shear stiffness matrix, whose elements are:

$$(A_{ij},B_{ij},D_{ij},F_{ij})=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}Q'^{(k)}_{ij}\,(1,z,z^2,k_{ij}\frac{C'^{(k)}_{ij}}{Q'^{(k)}_{ij}})dz, \tag{7}$$

where $C_{ij}'^{(k)}$ are the material stiffness constants, $Q_{ij}'^{(k)}$ are the reduced material stiffness constants for the k-th lamina, both in the reference system of the structure and, k_{ij} are the shear correction factors, and D^e_{ij} are given in the form:

$$D^e_{ij}=\begin{cases}0,\text{if } i=3 \text{ or } j=3\\ \displaystyle\sum_{k=1}^{n_e}\frac{e'^2_{31k}}{\varepsilon'_{33k}}\int_{z_k}^{z_{k+1}}z(z-z_{mk})dz,\text{otherwise}\end{cases} \tag{8}$$

3. ACTUATION AND SENSING

We have supposed the isotropic piezoelectric properties in the plane, and therefore the piezoelectric loads are uniform in-plane forces and bending moments acting normally to the contour of the electrode. The expressions for the actuator loads of the k-th layer are obtained by integrating the stresses, which are induced by the electric field:

$$N_{ak} = -\int_{z_k}^{z_{k+1}} e'_{31k} E_3 dz \qquad (9) \qquad\qquad M_{ak} = -\int_{z_k}^{z_{k+1}} z e'_{31k} E_3 dz\,, \qquad (10)$$

and they are included in the eq (6) combined with the eq (8), in the developed form.

If we connect the electrodes of a piezoelectric layer to a charge amplifier, the output voltage will be given in the form:

$$\Phi_{sk} = -\frac{Q_k}{C_s} = -\frac{1}{C_s}\int_{\Omega} D_{3k} d\Omega = -\frac{1}{C_s}\int_{\Omega} (e'_{31k}(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}) - \varepsilon'_{33k}\frac{\Delta\Phi_k}{h_k}) d\Omega\,, \qquad (11)$$

where C_s is the capacitance of the charge amplifier. It can be noticed that electric displacement D_3 depends on the in-plane displacements, but not on the section rotations. We have the opposite situation with the electrical voltage induced by the strains.

4. THE FINITE ELEMENT FORMULATION

The potential energy density of a piezoelectric material is given by equation (12), and the finite element formulation of a piezoelectric laminate can be obtained from the Hamilton's principle (13):

$$E_p = \frac{1}{2}[\{S\}^T\{T\} - \{E\}^T\{D\}] \qquad (12) \qquad\qquad \delta\int_{t_1}^{t_2} (L + W) dt = 0 \qquad (13)$$

where L is the Lagrangian, $L=E_k - E_p$, E_k being the kinetic energy and E_p being the potential energy, W is the virtual work of external forces.

The isoparametric formulation is used, so the interpolation is performed as follows:

$$(x, y, u_0, v_0, w, \theta_x, \theta_y) = \sum_{i=1}^{n_n} N_i \cdot (x_i, y_i, u_{0i}, v_{0i}, w_i, \theta_{xi}, \theta_{y_i})\,, \qquad (14)$$

where N_i are the Lagrangian interpolation functions and on the right hand side of the expression we have the nodal values of variables. The derivatives of the interpolation functions are obtained by means of the Jacobian, since the integration is being performed over the master element. Now, considering the expressions in section 2, eq. (13) gives:

$$
\begin{aligned}
0 = {} & \int_V [-\rho\{\delta u\}^T\{\ddot{\delta u}\} - \{\delta S\}^T[C^E]\{E\} + \{\delta S\}^T[e^T]\{E\} + \{\delta E\}^T[e]\{S\} + \{\delta E\}^T[\varepsilon^S]\{E\} + \{\delta u\}^T\{F_V\}]dV \\
& + \int_\Omega \{\delta u\}^T\{F_\Omega\} + \{\delta u\}^T\{F_P\} - \int_\Omega \delta\Phi q_e d\Omega - \delta\Phi Q = -\{\delta u_i\}^T \int_\Omega \rho h[N]^T[N]d\Omega\{u_i\} - \\
& - \{\delta u_i\}^T \int_\Omega [B_{ip}]^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} [B_{ip}]d\Omega\{u_i\} - \{\delta u_i\}^T \int_\Omega [B_{\gamma_s}]^T[\mathbf{F}][B_{\gamma_s}]d\Omega\{u_i\} - \\
& - \{\delta u_i\}^T \int_\Omega [B_{ip}]^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D^e} \end{bmatrix} [B_{ip}]d\Omega\{u_i\} - \{\delta u_i\}^T \int_\Omega [B_{ip}]^T \begin{bmatrix} \cdots & \{0\} & \cdots \\ \cdots & \{e_k\}z_{mk} & \cdots \end{bmatrix} d\Omega\{\Phi_k\} - \\
& - \{\delta\Phi_k\}^T \int_\Omega \begin{bmatrix} \vdots & \vdots \\ \{e_k\} & \{e_k\}z_{mk} \\ \vdots & \vdots \end{bmatrix} [B_{ip}]d\Omega\{u_i\} + \{\delta\Phi_k\}^T \int_\Omega \begin{bmatrix} \ddots & & 0 \\ & \frac{\varepsilon'_{33k}}{h_k} & \\ 0 & & \ddots \end{bmatrix} d\Omega\{\Phi_k\} + (\{\delta u_i\}^T \int_V \{N\}^T\{F_V\}dV + \\
& + \{\delta u_i\}^T \int_\Omega \{N\}^T\{F_\Omega\}d\Omega + \{\delta u_i\}^T + \{\delta u_i\}^T\{N\}^T\{F_P\}) - (\{\Phi_k\}^T q_e\Omega - \{\Phi_k\}^T Q) = \\
& - \{\delta u_i\}^T \int_\Omega \rho h[N]^T[N]d\Omega\{\ddot{u}_i\} - (\{\delta u_i\}^T \int_\Omega [B_{ip}]^T[\mathbf{C_{ip}}][B_{ip}]d\Omega\{u_i\} - \{\delta u_i\}^T \int_\Omega [B_{\gamma_s}]^T[\mathbf{C_{sh}}][B_{\gamma_s}]d\Omega\{u_i\} - \\
& - \{\delta u_i\}^T \int_\Omega [B_{ip}]^T[\mathbf{C_{em}}][B_{ip}]d\Omega\{u_i\}) - \{\delta u_i\}^T \int_\Omega [B_{ip}]^T[\mathbf{e}]d\Omega\{\Phi_k\} - \{\delta\Phi_k\}^T \int_\Omega [\mathbf{e}]^T[B_{ip}]d\Omega\{u_i\} + \\
& + \{\delta\Phi_k\}^T \int_\Omega [\boldsymbol{\varepsilon}]d\Omega\{\Phi_k\} + \{F_i\} + \{Q_i\}
\end{aligned} \tag{15}
$$

where F_V, F_Ω and F_P represent body, surface and point forces, respectively, $\{u_i\}$ the vector of nodal displacements, $\{\Phi_k\}$, $k = [1, n_e]$, the vector of layers electrical potentials, $[B_{ip}]$, $[B_{\gamma_s}]$ are so defined that $[B_{ip}]\{u\} = \{S_{ip}\}$ and $[B_{\gamma_s}]\{u\} = \{S_{sh}\}$, where $\{S_{ip}\}$ is the vector of in-plane strains, and $\{S_{sh}\}$ the vector of transverse shear strains. The shear locking is avoided by implementing the paradigm of consistency, introduced by Prathap [4], on the transverse shear strains. Upon coupling the matrices the following form of eq (15) is obtained:

$$
\begin{aligned}
[M]\{\ddot{u}\} + [K_{UU}]\{u\} + [K_{U\Phi}]\{\Phi\} &= \{F_i\} \\
[K_{\Phi U}]\{u\} + [K_{\Phi\Phi}]\{\Phi\} &= \{Q_i\}
\end{aligned} \tag{16}
$$

where $[M]$ is the element mass matrix, $[K_{UU}]$, $[K_{U\Phi}]$, $[K_{\Phi\Phi}]$ are the element mechanical stiffness matrix, piezoelectric coupling matrix and dielectric stiffness matrix, respectively and $\{F_i\}$ and $\{Q_i\}$ are the mechanical forces and electric charge.

4. Conclusion

The paper represents a model of a composite laminate with integrated piezoelectric components, based on the first order shear deformation theory and the linear theory of piezoelectricity. The model takes into account a linear variation of electric field over the thickness of a piezoelectric layer. Based on this analytical model, a new finite element formulation has been developed. A quadrilateral finite element is used. Some additional mechanical stiffness could be noticed as a consequence of the piezoelectric characteristic of the material which includes the dependence of electrical field on the section rotations derivatives. The finite element has $5n+n_e$ degrees of freedom (DOF), of which 5 are mechanical DOF at each node (3 translations and 2 rotations), and n_e are electrical potentials of piezoelectric layers in a laminate. The constant difference of electric potentials is assumed over the surface of the piezoelectric layers.

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MODELIRANJE LAMINARIH KOMPOZITNIH MATERIJALA SA INTEGRISANIM PIEZOELEKTRIČNIM AKTUATORIMA I SENZORIMA

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Sve veća potreba za lakim konstrukcijama zahteva razvoj novih vrsta materijala. Od tih materijala se zahtevaju visoka specifična svojstva, tj.visok odnos mehaničkih karakteristika i specifične gustine. Laminarni vlaknasti kompozitni materijali, koji raspolažu visokom čvrstoćom i krutošću u zeljenim pravcima i nižim karakteristikama u drugim pravcima predstavljaju dobar izbor i već se uveliko koriste u različite svrhe, npr. u konstrukcijama za svemirska istraživanja, transportu itd. Viši kvalitet u ponašanju konstrukcija od ovih materijala može se postići integrisanjem aktivnih komponenti, npr. piezo-pločica, izmedju slojeva materijala. U radu su date osnovne jednačine aktivne kompozitne ploče prema predpostavkama Mindlin-Reissner. Za piezoelektrični sloj su date osnovne jednačine spregnutog mehaničkog i električnog polja, kao i jednačine koje definišu pobudu piezoelektričnog aktuatora, odnosno izlaz senzora. Na kraju rada se razmatra formulacija problema primenom metode konačnih elemenata.