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ON A PROCEDURE OF OBTAINING THE EQUATIONS OF MOTION OF THE MATERIAL PARTICLE OVER THE SMOOTH SURFACE

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Abstract. *A new procedure for obtaining the equations of motion of the material point over the smooth surface is proposed. Some stipulations are done as well.*

Key words: *material particle, smooth surface, equations of motion*

INTRODUCTION

In 1976, under the strong impression of Professor V. Vujičić's (Faculty of Natural Sciences and Mathematics, Belgrade) lectures on Analytical Mechanics, I have carried out the procedure (one should say simple and elegant one) which will be the subject of this note, but I have not dared to present it to anybody, although the obtained result was already known and hence justified. Namely, from our (usual, Aristotelian, bivalent) logic it is well known that an incorrect procedure can also lead to the correct result[1], and here is a question on a procedure one step of which (it will be explicitly pointed out in the next section) seemed unsufficiently grounded[2]. And when soon after that I noticed this very step in a paper [6] and in a collection of problems [7] (although in some different consideration), I had not the courage to refer only to this reasoning, bearing in mind the

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[1] Let us just remember the high school example that from the wrong premises that 1=2 and hence 2=1 follows, after the addition, an indisputable identity 3=3 !

[2] Anyway, the moto:

"... two wrongs do make a right
in California"
G. Strang [1973]

"... two rights make a right even
in California"
R.L. Taylor [1989]

can serve as a confirmation that this hesitation of mine is not groundless (in essence a part of the correspondence or, why not say, the "retorts" interchanged between the cited authors) put by J.C. Simo and M.S. Rifai, very recognised authors in the finite elements method area, at the begining of their paper *"A class of mixed assumed strain methods and the method of incompatible modes"* (prepared for *International Journal for Numerical Methods in Engineering*) which, thanks to Professor M. Berković (Mathematical Faculty, Belgrade), I had an opportunity to see in the manuscript form in the early nineties.

precaution − several times mentioned at the very suggestive lectures by Professor M. Leko (Faculty of Natural Sciences and Mathematics, Belgrade) − that *"analogy has not the power of the proof"*.

However, having seen just recently in the reference [2] (obtained due to the courtesy of Dr D. Radojević from the Mathematical Institute in Belgrade) that the above mentioned step is in essence used in the procedure of the derivation of the differential equations of motion of a scleronomic holonomic system from D'Alembert's principle, I have dared to attempt to give, by this note, a contribution to the "collection" of different derivations of the differential equation of the material point over the smooth surface.

DIFFERENTIAL EQUATIONS OF MOTION IN INDEPENDENT COORDINATES OF THE MATERIAL POINT OVER THE SMOOTH SURFACE

It is well known that the Lagrange equations of the first kind fot the material point of unique mass moving, in the absence of any forces, over the smooth unmoving surface

$$f(x^1, x^2, x^3) = 0 \tag{1}$$

can be written in the form[3]

$$\ddot{x}^i + \overset{g}{\Gamma}{}^i_{jk}\, \dot{x}^j \dot{x}^k = \lambda g^{ij} \frac{\partial f}{\partial x^j} \quad , \tag{2a}$$

i.e., using the notion of an absolute derivative, in the form

$$\frac{\overset{g}{D} v^i}{Dt} = \lambda g^{ij} \frac{\partial f}{\partial x^j} \;\; (Dt = dt); \tag{2b}$$

x^i are some curvilinear coordinates in the three-dimensional Euclidean space and the covariant coordinates of the metric tensor of this space in the system x^i are determined by

$$g_{ij} = \delta_{k\ell} \frac{\partial z^k}{\partial x^i} \frac{\partial z^\ell}{\partial x^j}, \tag{3}$$

where z^i are the rectangular Cartesian coordinates connected with the curvilinear coordinates by the coordinate transformation of the form[4]

$$z^i = z^i(x^j) \quad ; \tag{4}$$

$\overset{g}{\Gamma}{}^i_{jk}$ are Christoffel symbols of the second kind with respect to the metric tensor g_{ij} ; λ is the Lagrange multiplicator and v^i are the contravariant coordinates of the velocity of the point under consideration.

On the other hand, it is known that the Lagrange equations of the second kind for the

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[3] Einstein's summation convention for diagonally repeated indices is used; Latin indices have the range {1,2,3}, while Greek indices will have the range {1,2}.

[4] This transformation is supposed to be sufficiently smooth and that $|\partial z^i / \partial x^j| \neq 0$ in the corresponding domain of the space under consideration!

motion of the point over the surface (1) can be written in the form (cf. for example with the expressions on p. 216 in [3])

$$\ddot{u}^{\alpha} + \overset{a}{\Gamma}{}^{\alpha}_{\beta\gamma}\,\dot{u}^{\beta}\dot{u}^{\gamma} = 0 \quad , \tag{5a}$$

i.e.

$$\frac{\overset{a}{D} v^{\alpha}}{Dt} = 0, \tag{5b}$$

where $\overset{a}{\Gamma}{}^{\alpha}_{\beta\gamma}$ are Christoffel symbols of the second kind with respect to the surface metric tensor $a_{\alpha\beta}$, determined with

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial u^{\alpha}}\frac{\partial x^j}{\partial u^{\beta}}, \tag{6}$$

and u^{α} are the surface curvilinear coordinates (so called Gaussian parameters); the contravariant vector $v^{\alpha} = du^{\alpha}/dt$ determines the velocity of the point on this surface (s. e.g. [5], p. 208).

It seems very reasonable that the following question arised: can equations (5) be derived from equations (2) using an operator capable to "drag along" the equations established with respect to the space to the ones connected with the corresponding subspace?

Of course, if a contravariant form of equations (2) is in question, the following system of the *double character* ([5], p. 186-187) would impose itself

$$\frac{\partial x^i}{\partial u^{\alpha}}. \tag{7}$$

But, when contravariant equations (2) are in question, it is not strange that there is a hesitation to apply, as an operator, the system of inverse quantities

$$\frac{\partial u^{\alpha}}{\partial x^i}! \tag{8}$$

Namely, when we speak of a surface in the three-dimensional Euclidean space, defined by the parametric equations of the form

$$x^i = x^i(u^{\alpha}), \tag{9}$$

it is well known that from these three equations it is not possible to determine, in a unique way, two quantities u^{α} in function of three quantities x^i and hence it is not possible to speak of the derivatives of form (8)[5] !

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[5] However, the following relations are quoted in [6]

$$y^i = y^i(x^j) \quad (i = 1,\dots,m;\ j = 1,\dots,n),$$

but immediately after them the inverse ones are written

$$x^j = x^j(y^i),$$

although $n<m$!

However, since the very partial derivatives of form (8) have been recently noticed on p. 15 in [2], as well as the expression of Christoffel symbols of the second kind with respect to the Euclidean space by these symbols for a surface – without discussing the question of the inversion of the corresponding coordinate transformations – it seems that there is no reason any more to steer clear of the old idea to apply operator (8) to equations (2).

In essense we start from the supposition that, except the curvilinear coordinates x^i, another curvilinear coordinates u^i are introduced in the Euclidean space by the following relations

$$u^i = u^i(x^j) \tag{10}$$

which are supposed to be enough smooth and that

$$\left|\frac{\partial u^i}{\partial x^j}\right| \neq 0 \tag{11}$$

in a certain domain, i.e. that an inverse transformation exists too

$$x^i = x^i(u^j). \tag{12}$$

If, in addition, we suppose[6] that the curvilinear coordinate u^3 is expressed by the curvilinear coordinates x^i using the equation of constraint (1), i.e.

$$\left.\begin{aligned} u^1 &= u^1(x^i) \\ u^2 &= u^2(x^i) \\ u^3 &= f(x^i) = 0 \end{aligned}\right\}, \tag{13}$$

which means that thus chosen curvilinear coordinates u^i identically satisfy the equation of constraint, i.e.

$$f(x^i) = f[x^i(u^j)] = 0, \tag{14}$$

it is clear that the position of the point on surface (1) will be determined only by two *independent coordinates*, u^1 and u^2, since, according to $(13)_3$, $u^3 = 0$ permanently during the motion of the point over the surface. In this case transformations (12) are reduced to

$$x^i = x^i(u^\alpha). \tag{15}$$

Without a loss of generality, we shall suppose that the family of the coordinate u^3-lines is *orthogonal* to the family of the coordinate surfaces $u^3 = const$.

Let us concern with the following question: can operator (8) "transport" absolute differential (2b) from the space into the subspace? By the composition of system (8) with the left side in (2a) we obtain

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On the other side, on p. 214 in [3] it was *first* supposed that the inversion of the corresponding transformation is enabled and *just after* that the equation of the constraint, fixing one of the variables, is taken into acount!

[6] Cf. with the procedure on p. 214 in [3].

$$\begin{aligned}
\frac{\partial u^\alpha}{\partial x^i}\left(\ddot{x}^i + \overset{g}{\Gamma}{}^i_{jk}\,\dot{x}^j\dot{x}^k\right) &= \frac{\partial u^\alpha}{\partial x^i}\left(\frac{dv^i}{dt} + \overset{g}{\Gamma}{}^i_{jk}\,v^j v^k\right) \\
&= \frac{d}{dt}\left(\frac{\partial u^\alpha}{\partial x^i}v^i\right) - v^i\,\frac{d}{dt}\frac{\partial u^\alpha}{\partial x^i} + \overset{g}{\Gamma}{}^i_{jk}\,\frac{\partial u^\alpha}{\partial x^i}v^j v^k \\
&= \frac{dv^\alpha}{dt} - v^i\,\frac{\partial^2 u^\alpha}{\partial x^i\partial x^j}v^j + \overset{g}{\Gamma}{}^i_{jk}\,\frac{\partial u^\alpha}{\partial x^i}v^j v^k \\
&= \dot{v}^\alpha - \frac{\partial^2 u^\alpha}{\partial x^i\partial x^j}\frac{\partial x^i}{\partial u^\beta}\frac{\partial x^j}{\partial u^\gamma}v^\beta v^\gamma + \overset{g}{\Gamma}{}^i_{jk}\,\frac{\partial u^\alpha}{\partial x^i}\frac{\partial x^j}{\partial u^\beta}\frac{\partial x^k}{\partial u^\gamma}v^\beta v^\gamma \\
&= \dot{v}^\alpha + \left[\overset{g}{\Gamma}{}^i_{jk}\,\frac{\partial u^\alpha}{\partial x^i}\frac{\partial x^j}{\partial u^\beta}\frac{\partial x^k}{\partial u^\gamma} - \frac{\partial^2 u^\alpha}{\partial x^i\partial x^j}\frac{\partial x^i}{\partial u^\beta}\frac{\partial x^j}{\partial u^\gamma}\right]v^\beta v^\gamma \\
&= \ldots \quad .
\end{aligned} \tag{16}$$

However, by the same *formalism* as on p. 128 in [7][7] we can conclude that the expression in the brackets represents Christoffel symbols of the second kind with respect to the metric tensor $a_{\alpha\beta}$, so it is possible to write

$$\ldots = \dot{v}^\alpha + \overset{a}{\Gamma}{}^\alpha_{\beta\gamma}\,v^\beta v^\gamma, \tag{17}$$

and this, *by definition*, represents an absolute derivative of the vector v^α with respect to this metric

$$\ldots = \frac{\overset{a}{D}v^\alpha}{Dt}. \tag{18}$$

On the other side, if we perform the composition of the right side of equations (2a) with system (8), we shall obtain

$$\frac{\partial u^\alpha}{\partial x^i}\lambda g^{ij}\frac{\partial f}{\partial x^j} = \lambda g^{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial f}{\partial x^j} = 0, \tag{19}$$

because a dot product of the mutually *orthogonal* vectors is in question. Namely, the quantity $\partial u^3/\partial x^i = \partial f/\partial x^i$, as a gradient of the scalar function $u^3(x^i)$, is orthogonal to the surface $u^3=0$, while, for example, the quantity $\partial u^1/\partial x^i$, as a gradient of the function $u^1(x^i)$, is orthogonal to the surface $u^1 = 0$; however, in this surface the coordinate lines u^3 lie and hence the normals to the surface $u^1 = 0$ will be orthogonal to the tangents of the u^3-lines as well, but these tangents, due to the previous supposition concerning the u^3-lines orthogonality, are in the direction of the normals to the surface $u^3 = 0$.

Hence, *formally* applying the operator $\partial u^\alpha/\partial x^i$ to equations (2) relating to the space we come to requested equations (5b) relating to the subspace (surface) and due to the tensor

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[7] Let us note that an *orthogonal* system of the curvilinear coordinates x^i was in question in [7] !

character[8] of these equations they will rest equal to zero in any other coordinates on the surface while the previous special choice of the system u^i had only the aim to perform an easier derivation of the conclusion that the differential equation of the motion with respect to the general coordinates on the surface can be, in the absence of active forces, written in the form of expression (5b).

CONCLUDING REMARKS

Aware of the fact that, by its method of drawing conclusions and perceiving some incompleteness, this note goes just along the edge of a rigorous reasoning characteristic for a severe analytical procedure, the undersigned of these lines dare to present a belief that the greatest contribution of this note is perhaps the fact that in the search for[9] a procedure − one should say more simple but in essence a *formal* one − for the derivation of equations of motion of the material point over the smooth surface the presence of a *formal step* in the usual procedures[10] for obtaining these equations was indirectly pointed out[11]. This formality, in essence, needs one more consistent derivation to be performed from the subspace (a non-Euclidean one) point of view and hence this means necessity to redefine the operations of covariant (absolute) differentiation, as mentioned in [9] in [10].

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[8] This way of concluding is frequently encountered in literature: *"It is easy to check that the equations ... are tensorial, in the form independent of the choice of the coordinates, invariant under the arbitrary coordinate transformations and hence they may be written with respect to some system of curvilinear koordinates."* ([4], p. 42).

[9] It should be noted that the seeking for a space where some equations, expressions, … , solutions will be the simplest (*"In essence we are looking for a such space ... where the equations ... have the simplest form."*;[2], p. 2) sometimes transforms to the seeking of a space where the solution exists (let us remember the seeking of the quadratic equations solutions in the set of complex numbers when it was impossible to do that in the set of real numbers!).
It is perhaps the moment to point out the usual procedure in the curvilinear finite elements introduction: *"The basic idea serving as a fund for the introduction of the elements with curvilinear contours lies in the mapping of the curvilinear element from the system of global coordinates in an element of the simple form in some local coordinate system where all considerations concerning an element with rectilinear contours are valid."* ([8], p. 29). However, it seems that the thorough attention was not paid to the *very physical* justification of such a mapping! Namely, there is always a fear of uncritical application of some particular procedures in some particular space (where some solutions are searched for) although they do not suit for space in question.
Finally, let us cite Truesdell's words: *"However, there are indications that the entire approach through the principle of virtual work ought properly to be regarded in terms of a principle of invariance."* (p. 15 in [1]) and we can conclude that the approach using the principle of virtual work should be considered as a formal one in non-Euclidean spaces!

[10] See for example p. 216 in [3] where it was said that the absolute derivatives in the subspace were introduced *"in full analogy"* with these derivatives in the embedded spaces!

[11] It should be honestly remarked that this statement in a way puts in doubt the expressions from which follows *"the deepness of the Lagrange conceptions in Mechanics who without knowledge of the today usual notions in Differential Geometry came to the most general differential equations of motion"* ([2], p. 15)!

REFERENCES

1. C. Truesdell, *Invariant and complete stress functions for general continua*, Arch. Rational Mech. Anal., 4, (1959), 1-29.
2. R. Stojanović, *Application of Tensor Calculus and Differential Geometry in Mechanics*, Faculty of Natural Sciences and Mathematics, Belgrade, 1960. *(in Serbian)*
3. T. Anđelić, R. Stojanović, *Rational Mechanics*, Zavod za izdavanje udžbenika, Belgrade, 1965. *(in Serbian)*
4. R. Stojanović, *Introduction in Nonlinear Mechanics of Continua*, Zavod za izdavanje udžbenika, Belgrade, 1965. *(in Serbian)*
5. T.P. Anđelić, *Tensor Calculus*, Naučna knjiga, Belgrade, 1967. *(in Serbian)*
6. N. Bokan, *Some properties of fundamental bipoint tensor*, Matematički vesnik, 8 (23), 1971, 367-371.
7. M.D. Leko, M. Plavšić, *Solved Problems in Tensor Calculus*, Građevinska knjiga, Belgrade, 1973. *(in Serbian)*
8. N. Hajdin, M. Sekulović, *Finite Element Method and Its Application in Mechanics of Solids*, Proc. 14th Yugoslav Congress of Theoretical and Applied Mechanics, Portorož, 1978. *(in Serbian)*
9. Z. Drašković, *Again on the absolute integral*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 2, 8, (1998), 649-654.
10. Z. Drašković, *Contribution to the discussion on absolute integration of differential equations of geodesics in non-Euclidean space*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 3, 11, (2001), 55-70.

O POSTUPKU DOBIJANJA JEDNAČINA KRETANJA MATERIJALNE TAČKE PO GLATKOJ POVRŠI

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Predlaže se novi postupak dobijanja jednačina kretanja materijalne tačke po glatkoj površi. Dati su takodje neki doprinosi.

Ključne reči: *materijalna tačka, glatka površ, jednačine kretanja*