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Invited Paper

LYAPUNOV AND NON-LYAPUNOV STABILTY THEORY: LINEAR AUTONOMOUS AND NON-AUTONOMOUS SINGULAR SYSTEMS

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Abstract. *Singular systems are those in which the dynamics are governed by a combination of algebraic and differential equations. The complex nature of singular systems causes many difficulties in the analytical and numerical studies of such systems, particularly when there is a need for their control. In that sense the question of their stability deserves great attention. A particular class of these systems operates in free as well as in forced regime. A brief survey of the results concerning their stability in the sense of Lyapunov and finite and practical stability are presented as the basis for their high quality dynamical investigation.*

Key words: *Singular systems, Lyapunov stability, Finite and Practical Stabilty*

1. INTRODUCTION

Singular systems are those in which dynamics are governed by a combination of algebraic and differential equations. In that sense, the algebrac equations represent the constraints to the solution of the differential part.

These systems also known as descriptor, semi-state or generalized systems, arise naturally as linear approximations of linear and non-linear system models in many applications.

2. PRELIMINARIES AND NOTATION

Consider linear singular systems represented, by:

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$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t)\ ,\ \ \mathbf{y}(t) = C\mathbf{x}(t), \mathbf{x}(t_o) = \mathbf{x}_0\ , \tag{1}$$

or:

$$E\,\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t)\ ,\ \ \mathbf{y}(t) = C\,\mathbf{x}(t),\ \ \mathbf{x}(t_0) = \mathbf{x}_0\ \ , \tag{2}$$

with the matrix E possibly singular, where $\mathbf{x}(t) \in \Re^n$ is a generalized state-space vector and $\mathbf{u}(t) \in \Re^m$ is a control variable.

Matrices **A** and **B** are of the appropriate operating in a free regime and system given by eq. (2) is operating in a forced regime, i.e. some external force is applied on it. In other words, system given by eq. (1) is autonomous while that given by eq. (2) is not.

It should be stressed that, in the general case, the initial conditions for an autonomus, and a system operating in the forced regime need not be the same.

The complex nature of singular systems causes many difficultes *in analytical and numerical treatment* that do not appear when systems in the normal form are considered. In this sense questions of existence, solvability, uniqueness, and smothness are present which must be solved in satisfactory manner.

The survey of updated results for singular systems and a broad bibliography can be found in *Campbell* (1980, 1982), *Lewis* (1986), *Debeljkovic et al.* (1996.a, 1996.b, 1998) and in the two special issues of the journal *Circuits, Systems and Signal Procesing* (1986, 1989).

3. Stabilty in the sense of Lyapunov

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as Lyapunov stability, finite time stabilty, practical stability, technical stabilty and Bounded input bounded output stability. The first part of this section is concerned with the stability of the equlibrium points in the sense of Lyapunov stability of *linear autonomous and non-autonomous singular systems.* In the second part of the paper the basic results in the area of finite and practical stabilty are presented.

3.1. Linear autonomous singular systems

Stability definitions

Definition 1. Eq.(1) is exponentially stable if one can find two positive constants α, β such that for every solution of Eq.(1), $\|\mathbf{x}(t)\| \le \alpha \|\mathbf{x}_0(t)\| e^{-\beta t}$, *Pandolfi* (1980).

Definition 2. The system given by Eq.(1) will be termed *asymptotically stable* if and only if, for all consistent initial conditions $\mathbf{x}_0$, $\mathbf{x}(t) \to \mathbf{0}$ *as* $t \to \infty$, *Owens, Debeljkovic* (1985).

Definition 3. The system given by Eq. (1) is *asymptotically stable* if all roots of det (sE - A) , *i.e.* all finite eigenvalues of this matrix pencil, are in the open left - half complex plane, and system under consideration is *impulsive free* if there is no $\mathbf{X}_0$ such that $\mathbf{x}$(t) exibits discontinuous behavior in the free regime, *Lewis* (1986).

Definition 4. The system given by Eq. (1) is called *asymptotically stable* iff all finite eigenvalues λ_i, i = 1, …,n_1, of the matrix pencil (λE −A) have negative parts, *Muller* (1993).

Definition 5. The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of system given by Eq. (1) is said to be *stable* if for every $\varepsilon > 0$, and for any $t_0 \in J$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ hold for all $t \geq t_0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, where J denotes time interval such that $J = [t_0, +\infty)$, $t_0 \geq 0$, *Chen, Liu* (1997).

Definition 6. The equilibrium point $\mathbf{x} = \mathbf{0}$ of a system given by Eq. (1) is said to be *unstable* if there exist a $\varepsilon > 0$, and $t_0 \in J$, for any $\delta > 0$, such that there exists a $t^* \geq t_0$, for which $\|\mathbf{x}(t^*, t_0, \mathbf{x}_0)\| \geq \varepsilon$ holds, although $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, *Chen, Liu* (1997).

Definition 7. The equilibrium point $\mathbf{x} = \mathbf{0}$ of a system given by Eq. (1) is said to be *attractive* if for every $t_0 \in J$, there exists an $\eta = \eta(t_0) > 0$, such that $\lim_{t\to\infty} \mathbf{x}(t,t_0,\mathbf{x}_0) = 0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \eta$, *Chen, Liu* (1997).

Definition 8. The equilibrium point $\mathbf{x} = \mathbf{0}$ of a singular system given by Eq. (1) is said to be *asymptotically stable* if it is stable and attractive, *Chen, Liu* (1997).

Lemma 1. The equilibrium point $\mathbf{x} = \mathbf{0}$ of a linear singular system given by Eq. (1) is *asymptotically stable* if and only if it is *impulsive - free*, and $\sigma(E,A) \subset C^-$ *Chen, Liu* (1997).

Lemma 2. The equilibrium point $\mathbf{x} = \mathbf{0}$ of a system given by Eq. (1) is *asymptotically stable* if and only if it is *impulsive-free*, *and* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, *Chen, Liu* (1997).

Stability theorems

Theorem 1. Eq. (1), with $A = I$, I being the identity matrix, is *exponentially stable* if and only if the eigenvalues of E have non positive real parts, *Pandolfi* (1980).

Theorem 2. Let I_Ω be the matrix which represents the operator on $\Re^n$ which is the identity on Ω and the zero operator on Λ.

Eq. (1), with $A = I$, is stable if an $n \times n$ matrix P exist, which is the solution of the matrix equation:

$$E^{\mathrm{T}} P + PE = -I_\Omega, \tag{3}$$

with the following properties:

$$P = P^T,$$
$$P\,\mathbf{q} = \mathbf{0},\ \mathbf{q} \in \Lambda \tag{4}$$
$$\mathbf{q}^T P\,\mathbf{q} > 0,\ \mathbf{q} \neq \mathbf{0},\ \mathbf{q} \in \Omega.,$$

where:

$$\Omega = W_k = \aleph(I - EE^D),\ \Lambda = \aleph(EE^D), \tag{5}$$

where W_k is the subspace of consistent intial conditions.

$\aleph$ denotes the kerrnel or null space of the matrix ( ).

Theorem 3. The system given by Eq. (1) is *asymptotically stable* if and only if:

a) A is invertible

b) exist a positive-definite, self-adjoint operator P on $\Re^n$ exist, such that:

$$A^T PE + E^T PA = -Q \tag{6}$$

where Q is self-adjoint and positive definite in the sense that:

$$\mathbf{x}^T(t)Q\mathbf{x}(t) > 0 \text{ for all } \boldsymbol{x} \in W_{k^*} \setminus \{\mathbf{0}\}, \tag{7}$$

Owens, Debeljkovic (1985).

Theorem 4. The system given by Eq. (1) is *asymptotically stable* if and only if:

a) A is invertible

b) a positive-definite, self-adjoint operator P exist, such that:

$$\mathbf{x}^T(t)\,(A^T PE + E^T PA)\,\mathbf{x}(t) = -\mathbf{x}^T(t)I\mathbf{x}(t),\ \forall\ \mathbf{x} \in W_{k^*}. \tag{8}$$

Owens, Debeljkovic (1985).

Theorem 5. Let (E,A) be regular and (E,A,C) be observable.

Then (E,A) is *impulsive free* and *asymptotically stable* if and only if a positive definite solution P to:

$$A^T PE + E^T PA + E^T C^T CE = 0, \tag{9}$$

exist and if P_1 and P_2 are two such solutions, then $E^T P_1 E = E^T P_2 E$, *Lewis* (1986).

Theorem 6. If there are symmetric matrices P, Q satisfying:

$$A^T PE + E^T PA = -Q \tag{10}$$

and if:

$$\mathbf{x}^T E^T PE\,\mathbf{x} > 0\ \forall \mathrm{x} = S_1\mathbf{y}_1 \neq 0, \tag{11}$$

$$\mathrm{x}^T Q\,\mathbf{x} \geq 0\ \forall \mathrm{x} = S_1\mathbf{y}_1, \tag{12}$$

then the system described by Eq. (1) is *asymptotically stable* if:

$$\operatorname{rank}\begin{bmatrix} sE - A \\ S_1^T Q \end{bmatrix} = \mathrm{n}\ \forall s \in \mathbf{C}, \tag{13}$$

and marginally stable if the condition given by Eq. (12) does not hold, *Muller* (1993).

Theorem 7. The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of a system given by Eq. (1) is *asymptotically stable,* if an $n \times n$ symmetric positive definite matrix P exist, such that along the solutions of system, given by Eq. (1), the derivative of function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$, is a negative definite for the variates of $E\boldsymbol{x}$, *Chen, Liu* (1997).

Theorem 8. If an n x n symmetric, positive definite matrix P exists, such that along with the solutions of system, given by Eq. (1), the derivative of the function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$ i.e. $\dot{V}(E\boldsymbol{x})$ is a positive definite for all variates of $E\boldsymbol{x}$, then the equilibrium point $\mathbf{x} = \mathbf{0}$ of the system given by Eq. (1) is *unstable, Chen, Liu* (1997).

Theorem 9. If an n x n symmetric, positive definite matrix P exists, such that along with the solutions of system, given by Eq. (1), the derivative of the function $V(E\boldsymbol{x}) = (E\boldsymbol{x})^T P(E\boldsymbol{x})$ i.e. $\dot{V}(E\boldsymbol{x})$ is negative semidefinite for all variates of $E\boldsymbol{x}$, then the equilibrium point $\mathbf{x} = \mathbf{0}$ of the system, given by Eq. (1), is *stable, Chen, Liu* (1997).

Theorem 10. Let (E,A) be regular and (E,A,C) be impulse observable and finite dynamics detectable. Then (E,A) is stable and impulse-free if and only if a solution (P,H) to the generalized *Lyapunov equations* (*GLE*) exists.

$$A^T P + H^T A + C^T C = 0\,, \tag{14}$$

$$H^T E = E^T P \geq 0\,, \tag{15}$$

Takaba et al. (1995).

The system, given by Eq. (1), is equivalent to:

$$E_1\, \dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + B_1 \mathbf{u}(t)\,, \tag{16a}$$

$$E_2\, \dot{\mathbf{x}}_2(t) = A_2 \mathbf{x}_2(t) + B_2 \mathbf{u}(t)\,, \tag{16b}$$

where $\mathbf{x}^T = [\mathbf{x}_1^T \quad \mathbf{x}_2^T]$.

Lemma 3. The system, given by Eq. (1), is *asymptotically stable* if and only if the *"slow" sub - system*, Eq. (16a) is asymptotically stable, *Zhang et al.* (1998a)

Lemma 4. $\mathbf{x}_1 \neq \mathbf{0}$ is equivalent to $E^{h+1}\mathbf{x} \neq \mathbf{0}$, *Zhang et al.* (1998a).

Define Lyapunov function as:

$$V(E^{h+1}\mathbf{x}) = \mathbf{x}^T (E^{h+1})^T P E^{h+1} \mathbf{x}\,, \tag{17}$$

where: $P > 0$, $P \in \mathbf{R}^{nxn}$ satisfying: $V(E^{h+1}\mathbf{x}) > 0$ if $E^{h+1}\mathbf{x} \neq \mathbf{0}$, when $V(0) = 0$.

From Eq. (1) and Eq. (16), bearing in mind that $EA = AE$, one can obtain:

$$(E^h)^T A^T P E^{h+1} + (E^{h+1})^T P A E^h = -(E^{h+1})^T W E^{h+1} \tag{18}$$

where $W > 0$, $W \in \Re^{n\times n}$.

Eq. (18) is said to be *Lyapunov equation* for a system given by Eq. (1).

Denote with r :

$$r = \deg\det(sE - A) = rank\, E_1\,. \tag{19}$$

Theorem 11. The system, given by Eq. (1), is *asymptotically stable* if and only if for any matrix $W > 0$, Eq. (18) has a solution $P \geq 0$ with a positive external exponent r, *Zhang et al.* (1998a).

Theorem 12. The system, given by Eq. (1), is *asymptotically stable* if and only if for any given $W > 0$ the Lyapunov Eq. (18) has the solution $P > 0$, *Zhang et al* (1998a).

3.2. Linear non-autonomous singular systems

In the sequel, the *generalized Lyapunov equations* given by *Bender* (1987) are further studied for continuous-time singular systems. Under a rank condition, the stability of continuous-time singular systems is related to the uniqueness of the solutions of the Lyapunov equations, provided that the systems are controlable. Furthermore, under certain conditions, the *controllability Grammians* obtained from the *Lyapunov equations* are guaranteed to be positive definite. All the results are valid for both impulsive and non-impulsive singular systems. However, for time-invariant systems with a *regular pencil* ($sE - A$), all these definitions reduce down to two definitions of controllability at infinity. *These are analogous to the difference between controllability and reachability.*

The parameters of *the Laurent expansion* of the generalized resolvent matrix $(sE-A)^{-1}$ are a very useful tool for analyzing singular systems. This is because they separate the subspace spanned by solutions in the eigenspace associated with finite eigenvalues of the pencil ($sE - A$) from the subcpace spanned by solutions associated with infinite eigenvalues. The infinite-eigenspace solutions can be termed as a "impulsive" solutions in a continuous-time system.

The Laurent parameters can thus be used to split the system, given by Eq, (2) into causal (*nonimpulsive*) and noncausal (*impulsive*) subsystems.

The *Laurent parameters*, also known as fundamental matrices, have played an important part in the analysis of singular systems.

Bender (1987) introduced the *reachability Grammians* and associated them with Lyapunov-like equations without the non-impulsive or causality restriction.

Suppose that ($sE - A$) is a regular pencil. The system, given by Eq. (2) is denoted by (E, A, B). It is known that the *Laurent parameters* $\{\phi_k, -\mu \le k < \infty\}$ specify the unique series expansion of the resolvent matrix about $s = \infty$.

$$(sE-A)^{-1} = s^{-1}\sum_{k=-\mu}^{\infty}\phi_k s^{-k}, \mu \ge 0 \tag{20}$$

valid in some set $0 < |s| \le \delta$, $\delta > 0$.

The positive integer μ is the nilpotent index.

Two square invertible matrices U and V exist such that (E, A, B) is transformed to the Weierstrass canonical form:

$$\overline{E} = U^{-1}EV^{-1}, \ \overline{A} = U^{-1}AV^{-1}, \tag{21}$$

$$\overline{B} = U^{-1}B, \ \overline{C} = CV^{-1}, \tag{22}$$

with:

$$s\overline{E} - \overline{A} = \begin{bmatrix} sI-J & 0 \\ 0 & sN-I \end{bmatrix}, \ \overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ \overline{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T, \tag{23}$$

where J and N are in the Jordan canonical form and N is nilpotent.

Also, the corresponding Laurent parameters in Weierstrass form are:

$$\overline{\phi}_k = V\phi_k U = \begin{cases} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix}, k \ge 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix}, k < 0 \end{cases} \tag{24}$$

Remark 1. If E is nonsingular, the singular system, given by Eq. (2) can be premultiplied by E^{-1} to derive an equivalent state-space system. In this case the following simplifications occur:

$$\phi_0 = I, \ \ U = E, \ \ V = I, \ \ J = E^{-1}A, \ \ B_1 = E^{-1}B, \ C_1 = C, \tag{25}$$

and N, B_2 and C_2 do not exist (i.e., N is a zero-dimensional matrix).

In this case the eigenvalues of the pencil ($sE - A$) are the eigenvalues of $E^{-1}A$ and are obviously finite. If $E = I$, eq. (2) is already in the Weierstrass canonical form and one can have:

$$U = I, J = A, B_1 = B. \tag{26}$$

We now summarize some useful propreties of the *Laurent parameters:*

$$E\,\phi_k - A\,\phi_{k-1} = \phi_k E - \phi_{k-1} A = \delta_{0k} I, \tag{27}$$

$$\phi_0 E \phi_0 = \phi_0 \;,\; \phi_{-1} A \phi_{-1} = -\phi_{-1} \tag{28}$$

$$\phi_k = \begin{cases} (\phi_0 A)^k \phi_0, & k \geq 0 \\ (-\phi_{-1} E)^{-k-1} \phi_{-1}, & k < 0 \end{cases} \tag{29}$$

$$E\,\phi_k A = A\,\phi_k E, \forall k \tag{30}$$

$$\phi_k E \phi_j = \phi_j E \phi_k = \phi_k A \phi_j = \phi_j A \phi_k \tag{31}$$

if: $k < 0, j \geq 0$

$$\left.\begin{array}{l} (-\phi_{-1}E)^{\mu} = (-E\phi_{-1})^{\mu} = 0 \\ (-\phi_{-1}E)^{\mu-1} \neq 0, (-E\phi_{-1})^{\mu-1} \neq 0 \end{array}\right\} \tag{32}$$

That is, H_F is the subspace spanned by causal solutions and H_I is the subspace spanned by noncausal or *"infinite frequency"* or *"impulsive"* solutions.

Note that if E is nonsingular, $H_F = \Re^n$, $H_I = 0$, $\phi_0 = I$, $\phi_0 E = E = E\phi_0$, and $\phi_{-1} = \phi_{-1}A = A\phi_{-1} = 0$.

The solution of a singular system can be expressed directly in terms of the Laurent parameters.

$$\mathbf{x} = \phi_0 E\,\mathbf{x} - \phi_{-1} A x(t) =$$

$$= \left(e^{\phi_0 A t}\mathbf{x}_0 + \int_0^t e^{\phi_0 A(t-\tau)} \phi_0 B\mathbf{u}(\tau)d\tau \right) - \left((-\phi_{-1}E)^m \mathbf{x}^{(m)}(t) + \sum_{k=0}^{m-1} (-\phi_{-1}E)^k \phi_{-1} B\mathbf{u}^{(k)}(t) \right) \tag{33}$$

$$y(t) = C\,(\phi_0 E - \phi_{-1}A)x(t)\,, (34)$$

where, $i \geq 0$ and $m \geq 0$.

As indicated by the property of Eq. (33), the *Laurent parameters* can be used to separate the causal solution subspace from the noncausal solution subspace.

Definition 9. If the integral exists, the *causal continuous-time singular system reachability Grammian* is:

$$G_c^{cr} = \int_0^\infty \phi_0 e^{A\phi_0 t} BB^T e^{\phi_0^T A^T t} \phi_0^T dt\,. \tag{35}$$

Bender (1987).

The noncausal continuous-time singular system *reachability Grammian* is:

$$G_{nc}^{cr} = -\sum_{k=-\mu}^{-1} \phi_k BB^T \phi_k^T\,. \tag{36}$$

The continuous-time singular system reachability Grammian is:

$$G^{cr} = G_c^{cr} + G_{nc}^{cr}\,. \tag{37}$$

If the integral does not exist, only G_{nc}^{cr} is defined, Bender (1987).

The columns of $\phi_0 E G_c^{cr} E^T \phi_0^T = G_c^{cr}$ span the causal reachable subspace, and the columns of G_{nc}^{cr} span the noncausal reachable subspace, which is the subspace "reachable at ∞".

By the same argument the columns of G^{cr} span the reachable subspace for the entire system.

Theorem 13.

i) If G_c^{cr} exists, it satisfies $\phi_0 (E G_c^{cr} A^T + A G_c^{cr} E^T) \phi_0^T = -\phi_0 B B^T \phi_0^T$. (38)

ii) G_{nc}^{cr} always exists and satisfies $\phi_{-1} (E G_{nc}^{cr} E^T - A G_{nc}^{cr} A^T) \phi_{-1}^T = \phi_{-1} B B^T \phi_{-1}^T$. (39)

iii) Suppose the range of R^c (*see Apendix B*) contains the range of $\phi_0 E$ (i.e.,the pair (J, B_1) is reachable). Then if all finite eigenvalues of the pencil $(sE - A)$ have real part less than zero, Eq. (38) has a symmetric solution G_c^{cr} which satisfies: $\mathbf{x}^T G_c^{cr} \mathrm{x} > 0$ for all x such that:

$$\mathbf{x} = E^T \phi_0^T \mathbf{w} \neq \mathbf{0}. \tag{40}$$

Furthermore, $\phi_0 E G_c^{cr} E^T \phi_0^T$ is unique.

Conversely, if Eq. (38) has a symmetric solution, then $\phi_0 E G_c^{cr} E^T \phi_0^T$ is unique and all finite eigenvalues of the pencil $(sE - A)$ have real part less than zero.

iv) If the rank of R_{nc} contains the range of $\phi_{-1} A$ (i.e., if the pair (N, B_2) is reachable), then Eq. (39) has a symmetric solution G_{nc}^{cr} satisfying: $\mathbf{x}^T G_{nc}^{cr} \mathbf{x} < 0$, for all x such that:

$$\mathbf{x} = A^T \phi_{-1}^T \mathbf{w} \neq 0. \tag{41}$$

Furthermore, $\phi_{-1} A G_{nc}^{cr} A^T \phi_{-1}^T$ is unique, *Bender* (1987).

Definition 10. A singular system is *asymptotically stable* if and only if its slow subsystem (I, J, B_1, C_1) is asymptotically stable. The slow subsystem is *controllable*, or equivalently, the descriptor system is *R-controllable*, if and only if:

$$\mathrm{rank}[B_1, J B_1, \ldots, J^{n_1-1} B_1] = n_1, \tag{42}$$

where $n_1 = \mathrm{degree}(\det(sE - A))$ is the dimension of the slow subsystem.

The fast subsystem is *controllable* if and only if:

$$\mathrm{rank}[B_2, N B_2, \ldots, N^{\mu-1} B_2] = n - n_1. \tag{43}$$

Dai (1989).

The *controllability* of a singular system implies both its slow and fast subsystems are *controllable*.

Definition 11. For the continuous-time descriptor system (E,A,B,C), the *slow controllability Grammian* is:

$$G_s^c = \int_0^\infty \phi_0 e^{A\phi_0 t} B B^T e^{\phi_0^T A^T t} \phi_0^T dt, \tag{44}$$

provided that the integral exists.

The *fast controllability Grammian* is:

$$G_f^c = \sum_{k=-\mu}^{-1} \phi_k BB^T \phi_k^T . \tag{45}$$

The *controllability Grammian* is:

$$G^c = G_s^c + G_f^c , \tag{46}$$

Zhang et al. (1988b).

In Weierstrass canonical form, given by Eq. (21-22), the corresponding Grammians of G_s^c and G_f^c are denoted by $\bar{G}_s^c$ and $\bar{G}_f^c$ respectively.

From Eq. (21-22), it can be easily shown that:

$$\bar{G}_s^c = V G_s^c V^T , \ \bar{G}_f^c = V G_f^c V^T . \tag{47}$$

Proposition 1.

i) $\phi_0 E G_s^c E^T \phi_0^T = G_s^c ,$ (48)

ii) $\phi_{-1} A G_f^c A^T \phi_{-1}^T = G_f^c .$ (49)

Theorem 14.

i) G_s^c satisfies

$$G_s^c A^T \phi_0^T + \phi_0 A G_s^c = -\phi_0 B B^T \phi_0^T . \tag{50}$$

ii) G_f^c uniquely satisfies:

$$G_f^c - \phi_{-1} E G_f^c E^T \phi_{-1}^T = \phi_{-1} B B^T \phi_{-1}^T . \tag{51}$$

iii) If the system, given by Eq. (2), is *asymptotically stable,* then the slow subsystem *is controllable if and only* if Eq. (50) has the unique solution $G_s^c \geq 0$ which satisfies:

$$\text{rank}(G_s^c) = \text{degree } (\det(sE - A)). \tag{52}$$

iv) The fast subsystem is *controllable if and only if*:

$$\text{rank}(G_f^c) = n - \text{degree}(\det(sE - A)). \tag{53}$$

v) If the system, given by Eq. (2), is *asymptotically stable*, then system given by Eq. (2), *is controllable if and only if* :

$$G^c = G_s^c + G_f^c > 0, \tag{54}$$

Zhang et al (1988b).

Remark 2. If E is nonsingular, then $\phi_0 = I$ and $\phi_{-1} = 0$.

In this case, the *controllability Grammian* G^c becomes:

$$G^c = \int_0^\infty e^{At} BB^T e^{A^T t} dt . \tag{55}$$

It can be seen that G^c satisfies:

$$G^c A^T + A G^c = -B B^T . \tag{56}$$

Therefore, normal systems and singular systems have unified Grammian form and Lyapunov equations, *Zhang et al* (1988b).

4. NON – LYAPUNOV STABILTY

Boundedness properties of system response i.e. the solution of system models, are very important form the engineering viewpoint. Realizing this fact numerous definitions of the so-called technical and practical stability were introduced. These definitions were essentially based on predefined boundaries for perturbation of initial conditions and allowable perturbation of the system response. This means that one is not only interested in system stability in the sense of Lyapunov but also in bound of systems trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state space which are defined *a priori* in the given problem. Besides, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval.

4.1. Finite and practical stabilty

Our primary interest is to investigate boundedness properties of systems used in suitable canonical form, i.e.:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t)\ , \tag{57a}$$

$$0 = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t)\ . \tag{57b}$$

The boundedness properties of solutions of (1) can be expressed in the equivalent form as constraints on solutions of (57), if the transformation from (1) to (57) is nonsingular.

Thus, we will present our problem for the singular systems given in the both forms.

Before formulating the problem we first introduce the set:

$$S_G(\rho) = \{\mathbf{x}(t) \in \Re^n : \|\,\mathbf{x}(t)\,\|_G^2 < \rho,\ \ G = G^T > 0\}\ . \tag{58}$$

Systems governed by (1) or by (57) will be considered over time interval $T = [0,\tau]$, where quantity τ may be either a positive real number or symbol $+\infty$, so that the finite and practical stability can be treated simultaneously. It is obvious that $T \in \Re^n$.

Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded. Let index "*a*" stands for the set of all allowable states of system and index "*i*" for the set of all allowable initial states of the system, such that $S_i \subset S_a$.

As the system considered is time invariant it is sufficient to consider its solutions $\mathbf{x}$ as functions of only current time t and initial value $\mathbf{x}_0$ at the initial moment $t_0 = 0$, i.e. the adopted notation is as $\mathbf{x}(t,\mathbf{x}_0)$. In an abbreviated form the value of solution $\mathbf{x}$ at the moment t will be written as $\mathbf{x}(t)$.

Stability definitions

Definition 12. System (1) is practically stable w.r.t. (T, i, a, G) if and only if $\mathbf{x}_0 \in W_{k^*}$, satisfying $\|\,\mathbf{x}_0\,\|_G^2 < i$, implies $\|\,\mathbf{x}(t)\,\|_G^2 < a, \forall t \in T$, *Debeljkovic, Owens* (1985).

Here $G = E^TPE$ with $P = P^T > 0$, is an arbitrary specified matrix and W_{k^*} is subspace of consistent initial conditions.

Definition 13. A solution $\mathbf{x}(t, \mathbf{x}_0)$ of the system (57) is (T, i, a, G) - bounded if and only if $\mathbf{x}_0 \in m$ and $\| \mathbf{x}_0 \|_G^2 < i$, implies $\| \mathbf{x}(t, \mathbf{x}_0) \|_G^2 < a$ on T, *Debeljkovic et al.* (1993).

Definition 14. A solution $\mathbf{x}(t, \mathbf{x}_0)$ of the system (57) is (T, i, a, G) - unbounded if and only if there exists a $(t^*, \mathbf{x}_0) \in T \times m$, such that $\| \mathbf{x}_0 \|_G^2 < i$, implies $\| \mathbf{x}_1(t^*, \mathbf{x}_0) \|_G^2 \geq a$, *Debeljkovic et al.* (1993).

Definition 4.4. A solution $\mathbf{x}(t, \mathbf{x}_0)$ of the system (57) is (T, i, a_1, a_2) - bounded if and only if $\mathbf{x}_0 \in m$ and $\| \mathbf{x}_{10} \|^2 < i$, $\| \mathbf{x}_{20} \|^2 < i a_2 / a_1$, implies $\| \mathbf{x}_1(t, \mathbf{x}_0) \|^2 < a_1$ and $\| \mathbf{x}_2(t, \mathbf{x}_0) \|^2 < a_2$ on T, *Debeljkovic et al.* (1993).

Definition 15. A solution $\mathbf{x}(t, \mathbf{x}_0)$ of the system (57) is (T, i, a_1, a_2) – unbounded if and only if there exists a $(t^*, \mathbf{x}_0) \in T \times m$, such that $\| \mathbf{x}_{10} \|^2 < i$ and $\| \mathbf{x}_{20} \|^2 < i a_2 / a_1$, implies $\| \mathbf{x}_1(t^*, \mathbf{x}_0) \|^2 \geq a_1$ or $\| \mathbf{x}_2(t^*, \mathbf{x}_0) \|^2 \geq a_2$, *Debeljkovic et al.* (1993).

Stability theorems

Theorem 15. The system governed by (1) is finite time stable or practically stable w.r.t. (T, i, a, G) if the following condition is satisfied:

$$\ln \frac{a}{i} > \Lambda_{\max}(M) t, \quad \forall t \in T, \tag{59}$$

where:

$$\Lambda_{\max}(M) = \max\left(\mathbf{x}^T M \mathbf{x} : \quad \mathbf{x} \in W_k \setminus \{\mathbf{0}\}, \quad \mathbf{x}^T E^T P E \mathbf{x} = 1 \right). \tag{60}$$

$$M = A^T P E + E^T P A, \tag{61}$$

Debeljkovic, Owens (1985).

In the sequel we present a new result, explaining idea of introducing (60) in dynamical analysis of linear singular systems over finite time interval.

Using:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t), \quad P = P^T > 0, \tag{62}$$

as Lyapunov function for the system (1) on the subspace if consistent initial conditions W_k, it is obvious that computing its time derivative, along the trajectories of (1) yields:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \mathbf{x}^T(t)(A^T P E + E^T P A)\mathbf{x}(t) \\ &\leq \lambda_{\max}(A^T P E + E^T P A)\mathbf{x}^T(t)\mathbf{x}(t) \leq \lambda_{\max}(\square)\mathbf{x}^T(t)\mathbf{x}(t) \end{aligned} . \tag{63}$$

On the other side, following the basic ideas of finite time stability concepts,

$$\frac{dV(\mathbf{x}(t))}{dt} = \frac{d(\mathbf{x}^T(t) E^T P E \mathbf{x}(t))}{dt} \leq \lambda_{\max}(\square)\mathbf{x}^T(t)\mathbf{x}(t), \tag{64}$$

or:

$$\int_0^{t_0} \frac{d(\mathbf{x}^T(t) E^T P E \mathbf{x}(t))}{\mathbf{x}^T(t) E^T P E \mathbf{x}(t)} \leq \int_0^{t_0} \lambda_{\max}(\square) dt, \tag{65}$$

one should integrate the previous inequality, to obtain final result.

But it is obvious that the solution of integral, on the left side of (65), in general can not be solved.

A simple numerical example shows necessity to formulate system matrix eigenvalue in the manner as it has been done by (60).

Let us adopt:

$$E = \begin{bmatrix} 0 & e_1 \\ 0 & e_2 \end{bmatrix}. \tag{66}$$

Then the integral has the form:

$$\int_0^{t_0} \frac{(e_1^2 + e_2^2) dx_2^2}{x_1^2 + x_2^2}. \tag{67}$$

If one puts:

$$\begin{aligned} x_2^2(t) &= f(t), \\ x_1^2(t) &= g(t) - f(t), \end{aligned} \tag{68}$$

so one can write:

$$\int_0^{t_0} \frac{df(t)}{g(t)}, \tag{69}$$

functions $f(t)$ and $g(t)$ being independent, so it can be easily seen that the solution of (67), in general, does not exist.

But it is interesting to point out, that in some particular cases, solution can be found.

Lets have this choice:

$$\begin{aligned} x_2^2(t) &= x^{m+1}(t), \\ x_1^2(t) &= (ax^n(t) + b)^{-p} - x^{m+1}(t), \end{aligned} \tag{70}$$

so, (65) yields to:

$$\int_0^{t_0} \frac{d(x^{m+1}(t))}{(ax^n(t)+b)^{-p}} = \int_0^{t_0} (m+1) x^m(t) (ax^n(t)+b)^p \, dx(t), \tag{71}$$

what correspond to integral of differential binomial, having closed solution if and only if:

i) p is integer or:

ii) $\dfrac{m+1}{m}$ is integer

or:

iii) $\dfrac{m+1}{m} + p$ is integer .

This discussions shows reasonableness of introducing the largest system eigenvalue as it has been done in (60).

5. Conclussion

To assure *asymptotical stability for linear singular systems* it is not enough only to have the eigenvalues of matrix pair (E, A) in the left half complex plane, but also to provide an impulse-free motion of the system under consideration.

Some different approaches have been shown in order to construct Lyapunov stability theory for a particular class of linear singular systems operating in free and forced regimes.

Basic definitions and some theorems concerning *finite and practical stabilty* of linear singular systems are also presented. It has been shown that only particular choice of maximal system eigenvalue can leed to desired results.

APPENDIX A - *NOTATIONS*

With $\aleph(F)$ and $\Re(F)$ we will denote the kernel (null space) and range on any operator F, respectively, i.e.:

$$\aleph(F) = \{\mathbf{x}: F\mathbf{x} = 0, \ \forall\ \mathbf{x} \in \mathbf{R}^n\}, \tag{A1}$$

$$\Re(F) = \{\mathbf{y} \in \Re^m,\ \mathbf{y} = F\mathbf{x},\ \mathrm{x} \in \mathbf{R}^n\}, \tag{A2}$$

with:

$$\dim \aleph(F) + \dim \Re(F) = n\,. \text{ (A3)}$$

APPENDIX B - REACHABILITY GRAMMIANS

We begin this section by defining the *reachable subspace* in terms of the Laurent parameters.

We follow the deveopment of *Lewis* (1985).

We shall define the reachable subspace in terms of the following *reachability matrices*:

$$R_c = (\phi_0 B \cdots \phi_{n-1} B), \tag{B1}$$

$$R_{nc} = (\phi_{-\mu} B \cdots \phi_{-1} B), \tag{B2}$$

and:

$$R = (R_{nc}\ R_c). \tag{B3}$$

The subscript c implies that the columns of R_c span the reachable part of the causal solution subspace, and the subscript nc implies that the columns of R_{nc} span the reachable part of the noncausal solution subspace.

Definition B1. For a continuous-time singular system, the *causal reachable subspace* is the space spanned by the columns of R_c, *the noncausal reachable subspace* is the space spanned by the columns of R_{nc}, and the reachable subspace is the space spanned by the columns of R, Lewis (1985).

Remark B1:

1) If the reachable subspace defined here for the continuus-time system, given by Eq. (2) is equal to $\Re^n$, the singular system is "controllable" in the sense of Cobb (1984). That is a $(\mu - 1)$ - times continuously differentiable input $\mathbf{u}(t)$ exist which will steer the descriptor vector $\mathbf{x}(t)$ from any initial condition in the range of $\phi_0 E$ to any arbitrary location in the descriptor space $\Re^n$ in finite time.

This is an extension of (and if $E = I$ is equivalent to) the usual definition of reachability for state-space systems.

2) If and only if the causal subsystem is reachable, i.e., if the pair (J, B_1) is reachable,

do the columns of R_c span the range of $\phi_0 E$.

That is, the columns of R_c span the causal solution subspace.

3) If and only if the noncausal subsystem is reachable, i.e., if the pair (N,B_2) is reachable, do the columns of R_{nc} span the range of $\phi_{-1}A$.

That is, the columns of R_{nc} span the noncausal solution subspace.

REFERENCES

1. Bender, D. J., "Lyapunov - Like Equations and Reachability / Observability Gramians for Descriptor Systems", *IEEE Trans. Automat. Cont.*, **AC-32** (4) (1987) 343–348.
2. Campbell, S. L., *Singular Systems of Differential Equations*, Pitman, Marshfield, Mass., (1980).
3. Campbell, S. L., *Singular Systems of Differential Equations II*, Pitman, Marshfield, Mass., (1982).
4. Chen,C., Y. Liu, "Lyapunov Stability Analysis of Linear Singular Dynamical Systems", *Proc. Int. Conference on Intelligent Processing Systems, Beijing, (China), October 28 - 31,* (1997) 635–639.
5. *Circuits, Systems and Signal Processing*, Special Issue on Semi state Systems, **5** (1) (1986).
6. *Circuits, Systems and Signal Processing*, Special Issue: Recent Advances in Singular Systems, **8** (3) (1989).
7. Dai, L., *Singular Control Systems*, Springer Verlag, Berlin, (1989).
8. Debeljković, D. Lj., D. H. Owens, "On Practical Stability", *Proc. MELECON Conference*, Madrid (Spain), October (1985) 103–105.
9. Debeljković, D. Lj., V. B. Bajić, Z. Gajić, B. Petrović, "Boundedness and Existence of Solutions of Regular and Irregular Singular Systems", *Publications of Faculty of Electrical Eng. Series: Automatic Control*, Belgrade (YU), (1) (1993) 69–78.
10. Debeljkovic, D. Lj., S. A. Milinkovic, M. B. Jovanovic, *Application of Singular Systems Theory in Chemical Engineering*, MAPRET Lecture – Monograph, 12th International Congress of Chemical and Process Engineering, CHISA 96, Praha, Czech Republic (1996).
11. Debeljkovic, Lj. D., S. A. Milinkovic, M. B. Jovanovic, *Continuous Singular Control Systems,* GIP Kultura, Beograd, (1996.a).
12. Debeljkovic, Lj .D., S. A. Milinkovic, M. B. Jovanovic, Lj. A. Jacic, *Discrete Singular Control System,* GIP Kultura, Beograd, (1998).
13. Lewis, F. L., "Fundamental, Reachabilty and Observabilty Matrices for Descriptor Systems ", *IEEE Trans. Automat. Cont*., **AC-30** (45) (1985) 502–505.
14. Lewis, F. L., "A Survey of Linear Singular Systems", *Circ. Syst. Sig. Proc.*, **5** (1) (1986) 3–36.
15. Lewis, F. L., "Recent Work in Singular Systems", *Proc. Int. Symp. on Sing. Syst.*, Atlanta, GA (1987) 20–24.
16. Muller, P.C., "Stability of Linear Mechanical Systems with Homonymic Constraints", Appl. Mech. Rev. (11), part 2, November (1983), 160 - 164.
17. Owens, D. H., D. Lj. Debeljković, "Consistency and Liapunov Stability of Linear Decsriptor Systems: a Geometric Approach", IMA Journal of Math. Control and Information, (1985), No.2, 139 - 151.
18. Pandolfi, L., "Controllability and Stabilization for Linear Systems of Algebraic and Differential Equations", *JOTA*, **30** (4) (1980) 601–620.
19. Takaba, K., N. Morihira, T. Katayama, "A Generalized Lyapunov Theorem for Descriptor System", *Systems &Control Letters,* (**24**) (1995) 49 - 51.
20. Wu, H., K. Muzukami, "Lyapunov Stability Theory and Robust Control of Uncertain Descriptor Systems", *Int. J. Science,* Vol.26, (10), (1995), 1981 - 1991.
21. Zhang, Q., G. Dai, J. Lam, L. Q. Zhang, M. De La Sen, "Asymptotical Stability and Stabilization of Descriptor Systems", *Acta Automatica Sinica,* Vol. 24 (2), (1998a), 208 - 211.
22. Zhang, L., J. Lam, Q. Zhang, "New Lyapunov and Riccati Equations for Descriptor Systems: Continuous - Time Case ", *Proc. Fifth ICARCV 98,* Singapore, December (1998b) TP1.5 - 965 - 969.

TEORIJA LJAPUNOVLJEVE I NELJAPUNOVLJEVE STABILNOSTI: LINEARNI AUTONOMNI I NEAUTONOMNI SINGULARNI SISTEMI

Dragutin Lj. Debeljković

Singularni sistemi su oni sistemi čija dinamika zadovoljava sistem kombinovanih algebarskih i diferencijalnih jednačina. Složena priroda singularnih sistema je uzrok mnogih teškoća u analitičkim i numeričkim studiranjima takvih sistema, posebno kada je neophodno upravljanje. U tom smislu pitanje njihove stabilnosti privlači posebnu pažnju. Posebna klasa tih sistema se nalazi kako u slobodnom tako i u prinudnom režimu. Kratak pregled tih rezultata koji se tiču njihove stabilnosti u smislu Ljapunovljeve konačne i praktične stabilnosti predstavljaju osnovu za njihovu bolju kvalitativnu dinamički analizu.

Ključne reči: *singularni sistemi, Ljapunovljeva stabilnost, konačna i praktična stabilnost*