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COMPRESSIBLE CHANNEL FLOW OVER A PERMEABLE WALL

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Abstract. *The problem of 2-D compressible gas flow through a channel with one permeable wall, which makes the part of the contour of a porous body, is treated in the paper as the problem of strong interaction between the channel flow and the flow through the porous body. Simplified equations governing both flows are solved by using matching conditions on the permeable wall, whereby the need for using empirically defined slip boundary conditions by Beavers and Joseph is eliminated. Slip velocity is found to increase down the channel attaining its maximum value, equal to the value for an incompressible fluid, at the exit cross section of the channel. Exact expressions for the friction coefficient and the relative increase of the mass flow rate due to the slip are derived, and better agreement with the existing experiments is shown to take place than by using the concept of slip boundary conditions.*

1. INTRODUCTION

It is well known in the classical Fluid Mechanics that viscous fluid adheres to the boundaries of the fluid flow, thus equating its velocity with the velocity of these boundaries. Since the form of the space occupied by the fluid, stationary or not, is usually prescribed in advance, fluid velocity at the boundaries is also a known quantity which serves as a boundary condition (no-slip boundary condition) in the solution of Navier-Stokes, or some other, approximate equations describing the fluid flow. There are, however, some specific problems of Fluid Mechanics, technically important enough, in which some slip of the fluid over the fluid boundaries may occur. Typical examples of this kind represent (a) rarified gas flow in which the free molecular path is not negligibly small with respect to a reference length scale, and (b) liquid and gas flow over a porous surface which constitutes the integral part of a contour of a porous body, through which the fluid flow takes place also. In both of these two cases slip velocity must be somehow modeled, most frequently in an empirical way, in order to get the boundary conditions, necessary for the solution of fluid flow equations.

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Interestingly enough, slip velocity in both of these two cases, although they are physically quite different, is modeled in such a way that it is proportional to the velocity gradient at the contour. Since in this paper we will be considering the fluid flow over a permeable wall, we will state the result of modeling the slip velocity for this flow as conjectured by Beavers & Joseph [1]:

$$u_o = -\frac{k}{\mu}\frac{\partial p}{\partial x} + \frac{\sqrt{k}}{\alpha}\frac{\partial u}{\partial x}\bigg|_{y=0} \tag{1}$$

where u_o is the slip velocity over the contour along which x axis is directed, p is the pressure, u is the velocity component in the direction of x, y is the coordinate in the direction perpendicular to the wall, k is the permeability of the porous material, μ is the viscosity of the fluid, and α is the so-called slip coefficient to be determined experimentally. Experiments conducted with different liquids [1] and with air [2] have verified the conjecture (1) and have enabled the determination of the slip coefficient for various porous materials. It is shown that slip coefficient primarily depends on the structure of the material, rather than on its porosity and the kind of fluid.

A new attempt for the solution of fluid flow over porous walls was performed in [3]. Both the fluid flow above the wall and the fluid flow inside the porous body were treated separately, with the slip velocity as an arbitrary quantity, not known beforehand. Formal solutions obtained in this way are then matched by equating the shear stresses on the contour of the body emanating from both flows. In this way the slip velocity can be obtained without any previous assumptions and both solutions can be completed. The attempt was tested on the example of an incompressible 2-D flow through a channel with one wall contiguous to a porous block, whereby the flow through both the channel and the block was exposed to the same pressure difference – the problem which was experimentally explored in [1] and [2]. The results obtained show that the thickness of the block plays a very important role in this problem – the effect that, naturally, could not be encountered by the empirical model (1). It turns out that this model is justifiable for a block of infinite thickness only. For a block of finite thickness the concept of Beavers and Joseph is incorrect in that the slip coefficient cannot be defined. Comparison of the obtained results with the experimental data shows excellent qualitative agreement, while the quantitative compa-rison of both results enables a very reliable determination of the effective viscosity for various porous materials.

This paper represents a relatively simple extension of the results obtained in [3] to the compressible flow. Compressible flows in channels with one porous wall find very useful applications in so-called aerostatic slider bearings and in self-acting slider bearings. Method for calculating fluid flow over porous surfaces, developed in here and in [3], points out to the possibility of applying it for the solution of problems of fluid flow around bodies coated with thin blocks of porous material, aimed at the flow control in the boundary layer which is formed around the body for high Reynolds numbers in this case.

2. PROBLEM STATEMENT AND GOVERNING EQUATIONS

We consider the problem depicted in Fig. 1. h_o is the height of the channel and l_o is its length, while the corresponding dimensions of the porous block are h and l_o. Compressible fluid flows through both the channel and the block from left to right, whereby the pressure in the inlet cross section is p_i and in the exit cross section is p_e. Momentum equation by means of which we describe both flows, supposedly isothermal, can be concisely written as [4]:

$$\frac{\rho}{s^2}(\vec{V}\cdot\nabla)\vec{V}+B\frac{\mu}{k}\vec{V}=-\nabla p+\tilde{\mu}\Delta\vec{V}$$

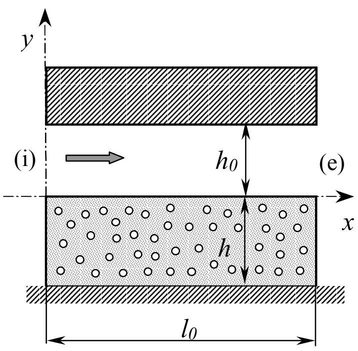

Fig. 1. Channel flow with one porous wall from the inlet cross section (*i*) to the outlet cross section (*e*)

where, in addition to denotations already used before, $\vec{V}$ is the velocity vector of a 2-D flow in Cartesian coordinates, ρ is the fluid density, s is the porosity, $\tilde{\mu}$ is effective viscosity, and B is a binary coefficient, which may attain the values 0 or 1. For the channel flow (free fluid!) B=0. Then, we have automatically s=1 and $\tilde{\mu}=\mu$, so that the above equation reduces to the well known Navier-Stokes equation. For B=1 momentum equation for the flow in a porous medium is obtained. In this equation the terms immediately on the left and on the right from the equality sign are due to the well known Darcy model. If the viscous terms on the right are added we get the Darcy-Brinkman model of flow, while all the terms in the equation, with the inertia (first term) included make the so-called Darcy-Brinkman-Lapwood model of flow through porous media. For a compressible flow, which is considered in here, the continuity equation of the form $\nabla\cdot(p\vec{V})=0$, the same for both flows, should be joined to the momentum one. For the analysis to follow it is convenient to right down the forgoing equations in nondimensional form by using the following characteristic scales, equal for both flows, for various independent and dependent variables: h_o for all lengths, u_e - average velocity at the exit of the channel, for all velocities, and p_e and ρ_e- the pressure and the density at the exit of the channel, respectively, for the pressure and the density. If we for simplicity keep the same notations for dimensional as well as for the corresponding nondimensional quantities, the above equations will attain the following form:

$$\frac{\tilde{\mathrm{Re}}}{s^2}p(\vec{V}\cdot\nabla)\vec{V}+B\sigma^2\vec{V}=-\frac{\tilde{\mathrm{Re}}}{\gamma\mathrm{M}_e^2}\nabla p+\Delta\vec{V} \qquad (2)$$

In there $\tilde{\mathrm{Re}}=\rho_e u_e h_o/\tilde{\mu}$ is the Reynolds number defined by means of the effective viscosity, $\mathrm{M}_e=u_e\Big/\sqrt{\gamma p_e/\rho_e}$ is the Mach number, $\sigma=h_o\Big/\sqrt{\tilde{k}}$ where $\tilde{k}=k\tilde{\mu}/\mu$ is the effective permeability of the porous block, and γ is the ratio of specific heats. In order to simplify these equations we will now suppose that $h_0 \ll l_o$, and we will introduce a small parameter $\varepsilon = h_o / l_o$. Then, it is natural to expect that in both the channel and the block the variations of all physical quantities in the direction of x will be considerably slower, than in the direction of y, and also that the transverse velocity component will be much smaller than the longitudinal one. In order to make these assumptions mathematically explicit we will introduce a slow coordinate $\xi = \varepsilon x$ and a small transverse velocity: $v = \varepsilon V(\xi, y)$, so that $V = O(1)$. Also, taking care about the conditions under which the experiments in [1] and [2] are conducted, we will suppose the following order of magnitude of the governing parameters present in (2):

$$\tilde{\mathrm{Re}} = O(1),\ \mathrm{M}_e = O(\sqrt{\varepsilon}),\ \sigma = O(1),\ s = O(1).$$

Then, the equations will take the form:

$$\begin{gathered} \frac{\varepsilon \tilde{\mathrm{Re}}}{s^2} p(uu_\xi + Vu_y) + B\sigma^2 u = -\tilde{\lambda}^{-1} p_\xi + u_{yy} + O(\varepsilon^2) \\ p_y = O(\varepsilon^2), \quad (pu)_\xi + (pV)_y = 0 \end{gathered} \tag{3}$$

where $\tilde{\lambda} = \tilde{\mu} l_o u_e / p_e h_o^2$ is an $O(1)$ parameter, which plays an important role in the solution of this problem. It is to be noted that it is linearly proportional to the mass flow rate through the channel, and that when the flow through the channel is considered separately ($B = 0$) it becomes: $\lambda = \mu l_o u_e / p_e h_o^2$. First order equations that follow from (3) obviously have the form:

$$u_{yy} - B\sigma^2 u = \tilde{\lambda}^{-1} p', \quad (pu)_\xi + (pV)_y = O \tag{4}$$

These equations are to be solved separately for the flow in the channel and for the flow in the block, respecting the following boundary conditions:

– for the channel: $u = U_o(\xi),\ \ V = V_o(\xi)$ for $y = 0$, and $u = V = 0$, for $y = 1$, where $U_o(\xi)$ is an arbitrary and so far an unknown slip velocity on the porous wall of the channel, and $V_o(\xi)$ is a possible transverse velocity component, which would be responsible for the mass transfer between the channel and the block, and:

– for the block: the same as for the channel for $y = 0$, and $u = V = 0$, for $y = -H$, where $H = h/h_o$. To these boundary conditions an equality of normal and shear stresses on the porous wall of the channel should be joined. This leads to the equality of the pressure in the channel and in the block in all cross sections, and to: $\mu(u_y)_{y=o,B=o} = \tilde{\mu}(u_y)_{y=o,B=1.}$ The problem posed in such a way is relatively easily solvable by simple analitical methods.

3. Solution of the Problem and Discussion

As said already, for the flow in the channel: $B = 0$ and $\tilde{\lambda} = \lambda$. The solution of the momentum equation in (4), by using the corresponding boundary conditions, is obtained as:

$$u = U_o(1 - y) - \frac{p'}{2\lambda} y(1 - y) \tag{5}$$

while the integration in y of the continuity equation from 0 to 1 gives:

$$pV_o(\xi) = \frac{1}{2}(pU_o)' - \frac{1}{12\lambda}(pp')' \tag{6}$$

The necessary expression for the shear stress on the porous wall is:

$$\frac{\tau_w}{\varepsilon\, p_e} = -\lambda\,(U_o + \frac{p'}{2\lambda}) \tag{7}$$

The corresponding expressions for the flow in the block are obtained in exactly the same way (the integration of the continuity equation in this case is performed between -H and 0), and read:

$$u = \left[\frac{U_o}{\mathrm{th}(\sigma H)} + \frac{\mathrm{ch}(\sigma H)-1}{\tilde{\lambda}\sigma^2\,\mathrm{sh}(\sigma H)}\,p'\right]\mathrm{sh}(\sigma y) + (U_o + \frac{p'}{\tilde{\lambda}\sigma^2})\,\mathrm{ch}(\sigma y) - \frac{p'}{\tilde{\lambda}\sigma^2} \tag{8}$$

$$pV_o(\xi) = -\frac{\mathrm{ch}(\sigma H)-1}{\sigma\,\mathrm{sh}(\sigma H)}(pU_o)' - \frac{1}{\tilde{\lambda}\sigma^3}\left[2\frac{\mathrm{ch}(\sigma H)-1}{\mathrm{sh}(\sigma H)} - \sigma H\right](p\,p')' \tag{9}$$

$$\frac{\tau_w}{\varepsilon\, p_e} = \frac{\tilde{\lambda}\sigma}{\mathrm{th}(\sigma H)}U_o + \frac{\mathrm{ch}(\sigma H)-1}{\sigma\,\mathrm{sh}(\sigma H)}\,p' \tag{10}$$

By equating the shear stresses we now get the relation:

$$(\frac{\tilde{\lambda}\sigma}{\mathrm{th}(\sigma H)} + \lambda)U_o + (\frac{\mathrm{ch}(\sigma H)-1}{\sigma\,\mathrm{sh}(\sigma H)} + \frac{1}{2})p' = 0 \tag{11}$$

If we multiply it by p and differentiate it, and then combine it with an equation which is obtained by elimination of $V_o(\xi)$ from the previous equations, we will obtain a system of homogeneous equations for $(pp')'$ and $(pU_o)'$. This system posseses the mathematically trivial solutions only: $(pp')' = 0$, and $(pU_o)' = 0$. These solutions lead to: $V_o(\xi) = 0$, which means that to this order of approximation there is no mass transfer between the channel and the block.

Integration of the equation for the pressure with the employment of the boundary conditions: $p(0) = P = p_i / p_e > 1$ and $p(1) = 1$ gives the following pressure distribution in both the channel and the block: $p^2 = P^2 - (P^2 - 1)\xi$. Then it follows from (11):

$$pU_o = \frac{P^2-1}{4}\,\frac{2\,(\mathrm{ch}(\sigma H)-1) + \sigma\,\mathrm{sh}(\sigma H)}{\sigma\,\mathrm{ch}(\sigma H)(\tilde{\lambda}\sigma + \lambda\,\mathrm{th}(\sigma H))} \tag{12}$$

Since there is no mass transfer between the channel and the block, the mass flow rate in the channel is constant, so that an additional condition must be satisfied: $\int_0^1 pu\,\mathrm{d}y = 1$. Its application leads to:

$$6\lambda pU_o - pp' = 12\lambda\,, \tag{13}$$

which in the combination with the previously derived expression for U_o finally provides the desired expression for the slip velocity: $pU_o = U_o^{(o)}$ where:

$$U_o^{(o)} = \frac{6\left\{2\left[\mathrm{ch}(\sigma H)-1\right] + \sigma\,\mathrm{sh}(\sigma H)\right\}}{3\left\{2\left[\mathrm{ch}(\sigma H)-1\right] + \sigma\,\mathrm{sh}(\sigma H)\right\} + \sigma\,\mathrm{ch}(\sigma H)\left[\tilde{\mu}\sigma/\mu + \mathrm{th}(\sigma H)\right]} \tag{14}$$

is exactly the slip velocity in the case of flow of an incompressible fluid, as shown in [3]. Thus, the nondimensional slip velocity for a compressible flow is not an "absolute" constant as in incompressible flow case, but it increases along the channel inversely proportionally to the pressure, attaining its maximum value at the exit cross section. This value corresponds to the slip velocity of an incompressible fluid. For $\sigma\to\infty$, i.e. for an absolutely inpermeable block, it is naturally obtained: $U_o^{(o)} \to 0$, while for a fully permeable block: $\sigma\to 0$, we get $U_o^{(o)} \to 2$, i.e. in this case the maximum slip velocity is twice as much as the average velocity at the exit cross section of the channel.

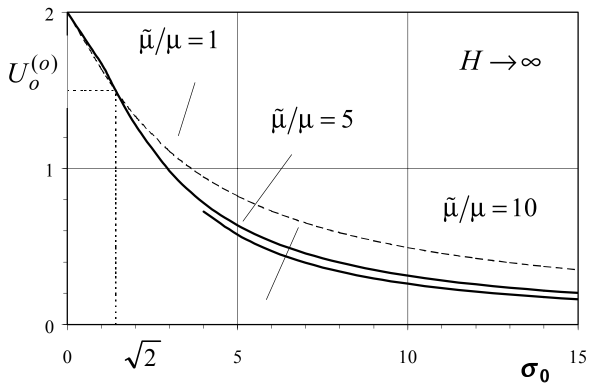

Fig. 2. Dependence of the maximum slip velocity $U_o^{(o)}$ on the parameters σ_o and $\tilde{\mu}/\mu$ for a porous block of infinite thickness.

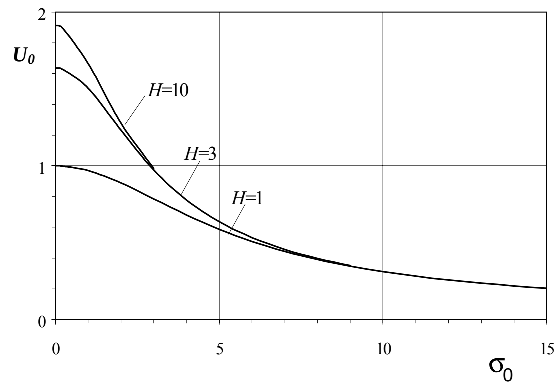

Fig. 3. Dependence of the maximum slip velocity $U_o^{(o)}$ on the parameters σ_o and $\tilde{\mu}/\mu = 5$ on the porous block of finite thickness.

In Fig. 2 we show how the maximum slip velocity (14) depends on $\sigma_o = \sigma(\tilde{\mu}/\mu)^{1/2}$ for $H\to\infty$ and for different ratios of the effective viscosity and the fluid viscosity, which is according to its definition always greater than one. It is noticed that all curves pass through the point: $\sigma_o = \sqrt{2}$, $U_o^{(o)} = 3/2$. We will show a little later that shear stress on the permeable wall in this point is zero, and that a purely Darcy flow takes place in the block. In Fig. 3 we show the dependence of the maximum slip velocity on σ_o for a finite height H and for an arbitrarily chosen value of the parameter $\tilde{\mu}/\mu = 5$. It is noticed that the effect of the height of the block is pronounced for relatively small values of σ_o only, i.e. in channels of extremely small heights – so called microchannels and/or for blocks made of extremely permeable materials.

If the value of λ in the classical case in which there is no-slip on the walls of the channel, i.e. for $U_o^{(o)} = 0$ is denoted by λ_o, we get from (13): $\lambda_o = (P^2 - 1)/24$. From here we may obtain in a simple manner the classical and the well known relation between the pressure drop and the mass flow rate through the channel [5]. This relation incorporates also the classical formula for the friction coefficients $f_o = 24/\mathrm{Re}$, where $\mathrm{Re} = \rho_e u_e h_o/\mu$ is the Reynolds number defined via fluid viscosity. If the definition of λ_o is now inserted in (13), a formula for the relative increase of the mass flow rate through the channel, caused by the slip on the porous wall, can be easily derived. This increase was the primary subject of measurements conducted in (1) and (2). Expressed by $U_o^{(o)}$ this increase reads:

$$\frac{\lambda}{\lambda_o} - 1 = \frac{\dot{M}}{\dot{M}_o} - 1 = \frac{U_o^{(o)}}{2 - U_o^{(o)}} \tag{15}$$

where $\dot{M}_o$ and $\dot{M}$ are the corresponding mass flow rates. Also, from the same relation one can get the correction of the friction coefficient due to the slip:

$$f = f_o(1 - U_o^{(o)}/2) \tag{16}$$

Relying on their model (1), Beavers and Joseph [1] also derived a formula for the relative increase of the mass flow rate. If a comparison of their formula with our formula (15) is made, one may conclude that both formulas coincide for $H \to \infty$ only, and reduce to each other if $\alpha = (\tilde{\mu}/\mu)^{1/2}$. For a block of finite height their forms do not coincide, so that one cannot recognize the value of the slip coefficient α by comparing both expressions, which means that in this case their model is not applicable.

We have made a further comparison of our result (16) with the measurements in [2]. The measurements were conducted by using the porous materials with the comercial names Foam metal A (5 series of measurements) and Foam metal B (3 series of measurements). By using the expression (16) we managed to get the following average values of the ratio of the effective viscosity and the fluid viscosity: for Foam metal A - $\tilde{\mu}/\mu = 4.662$, and for Foam metal B - $\tilde{\mu}/\mu = 5.022$. In both cases maximum deviation of the experimental results from the average values was 0.2%, which convincingly speaks in favor of the theory presented here. Obviously, the theoretical results obtained in here can be very successfully and relyably utilized for the experimental determination of the effective viscosity for various porous media. In contrast to the incompressible flow case [3] in which the corresponding solution was an exact solution of the governing equations, the solution obtained here is an approximate, first order solution. Thus, it makes sense to find some more, higher order approximations in order to get more exact results. This is, however, beyond the scope of this work.

4. Conclusions

The following conclusions can now be made, based on the analysis performed in the paper.

a) Compressible flow in a channel with one permeable wall can be successfully treated as the problem of strong interaction between the fluid and the porous medium, i.e. by simultaneous solution of the equations describing the fluid flow in the channel and the fluid flow in the porous block, exactly as in the incompressible flow case [3]. At that the slip velocity is considered as a quantity which is not known in advance and which can be determined by matching the shear stresses on the porous boundary. This way, the need for using the empirically defined boundary condition by Beavers & Joseph is eliminated.

b) The Beavers and Joseph condition is shown to be correct for the blocks of infinite height only. For the blocks of finite height the condition is erroneous and the idea of the slip coefficient has no meaning.

c) It is our strong opinion that the method applied in this paper can be utilized for the solution of some other problems in techniques, as for example for the solution of boundary layer flow over porous surfaces, whereby the porous surface would be used to control the flow.

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STRUJANJE STIŠLJIVOG GASA KROZ KANAL SA POROZNIM ZIDOM

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Ravansko strujanje stišljivog gasa kroz kanal sa jednim poroznim zidom, koji čini deo konture nekog poroznog tela, tretira se u radu kao problem jake interakcije izmedju strujanja u kanalu i strujanja kroz porozno telo. Pojednostavljene jednačine kojima se opisuju oba strujanja rešene su korišćenjem uslova spajanja na poroznom zidu, pri čemu se eliminiše potreba za korišćenjem uslova klizanja fluida koji su empirijski definisali Beavers i Joseph. Pronadjeno je da brzina klizanja raste u pravcu strujanja i da dostiže svoju maksimalnu vrednost u izlaznom preseku kanala. Ta vrednost je jednaka onoj koja se ima kod strujanja nestišljivog fluida. U radu su izvedene tačne vrednosti koeficijenta trenja i relativnog povećanja masenog protoka kroz kanal, izazvanog proklizavanjem fluida duž poroznog zida. Dobijeni rezultati bolje se slažu sa postojećim eksperimentima, nego u slučaju korišćenja graničnog uslova klizanja.