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VIBRATIONS OF THE ROTOR WITH NON-LINEAR PROPERTIES

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Abstract. *The aim of the paper is to determine the approximate analytic solution of the eq.(1) and to analyze it. The elliptic-Krylov-Bogolubov method [7] will be extended and adopted for solving the differential equations with complex function. First, the generating solution of the strong differential equation without small non-linearities will be obtained. Based on that solution the trial solution of the eq.(1) will be formed. The developed procedure will be applied for solving of the differential equation where the small function is dependent on the time derivative of the complex function. The obtained results will be compared with numeric one.*

INTRODUCTION

The problem of rotor vibrations exists for a long time. Recently, the investigations are more intensive due to the fact that rotor velocity is very high and the vibration problem seems to be dominant. Besides, it is a requirement for increasing of the efficiency of the rotating machines, where the rotors are the basic working parts, i.e., the decreasing of the vibrations as they represent the energy losses in the system.

Usually the rotor systems are considered as linear systems [1], [2] or systems with small non-linearities [3], [4]. Unfortunately, the results obtained using such a model does not give satisfactory explanation for the phenomena which appear in the system. Sometimes, the qualitative results are acceptable, but the quantitative results are incorrect. It requires the rotor to be considered as a strong non-linear complex system which enables the real properties of the rotor to be described in the more correct manner. Some results in analyzing of strong non-linear rotors are presented in the papers [5], [6]. The rotor is analyzed as a shaft disc system. The disc is settled in the middle of the shaft. The shaft is supported at the both ends with rigid bearings. The mass of the shaft is negligible in comparison to the mass of the disc. The elastic properties of the shaft are non-linear. The gyroscopic force of the system has to be taken into consideration.

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The mathematical model of the rotor is

$$a\ddot{z} + b_1 z + b_3 z(z\bar{z}) - ig\dot{z} = \varepsilon f(z, \dot{z}, cc), \tag{1}$$

where z is the complex deflection function, $\bar{z}$ is a complex conjugate function, a, b_1, b_3 and g are constants, $i = \sqrt{-1}$ is the imaginary unit, εf is a small non-linear function of the complex function and its time derivative and the complex conjugate functions cc.

The aim of the paper is to determine the approximate analytic solution of the eq.(1) and to analyze it. The elliptic-Krylov-Bogolubov method [7] will be extended and adopted for solving the differential equations with complex function. First, the generating solution of the strong differential equation without small non-linearities will be obtained. Based on that solution the trial solution of the Eq.(1) will be formed. The developed procedure will be applied for solving of the differential equation where the small function is dependent on the time derivative of the complex function. The obtained results will be compared with numeric one.

GENERATING SOLUTION

Let us consider the case when the small non-linearity is negligible. The differential equation of motion is

$$a\ddot{z} + b_1 z + b_3 z(z\bar{z}) - ig\dot{z} = 0 . \tag{2}$$

The exact analytic solution of the equation is

$$z = (A + iB)\, exp[i(\omega t + \alpha)] cn(\omega_1 t + \beta, m), \tag{3}$$

where cn is the Jacobi elliptic function [8] with argument $\omega_1 t + \beta$ and modulus m, and A, B, α and β are constants dependent on the initial conditions $z(0) = z_0$ and $\dot{z}(0) = \dot{z}_0$. For the arbitrary values of A, B, α and β it is

$$z_0 = (A + iB)\, exp(i\alpha) cn(\beta, m), \tag{4}$$

and

$$\dot{z}_0 = (A + iB)\, exp(i\alpha)[i\omega cn(\beta, m) - \omega_1 sn(\beta, m) dn(\beta, m)], \tag{5}$$

where sn and dn are also Jacobi elliptic functions [8].

The solution (3) has two unknown frequencies ω and ω_1 and a value of the modulus m which are obtained by substituting (3) and its first and second time derivatives into (2) and equating the terms with the same order of the elliptic functions. It is

$$\omega = \frac{g}{2a}, \quad \omega_1 = \sqrt{\frac{b_1}{a} + \omega^2 + b}, \quad m = \frac{b}{2\omega_1^2}, \tag{6}$$

where

$$b = \frac{b_3}{a}(A^2 + B^2). \tag{7}$$

Analyzing the obtained results it is evident that the frequency ω does not depend on the initial conditions and also on the value of coefficient of non-linearity. It depends on the gyroscopic parameter g. Due to non-linearity b_3 the frequency ω_1 depends on the initial values of amplitudes of vibration A and B. The modulus (6) of the elliptic function cn also depends on the initial amplitudes. The values of ω, ω_1 and m are not functions of α and of the initial phase angle β.

The other form of the solution (3) is

$$z = \sqrt{A^2 + B^2}\, cn(\omega_1 t + \beta, m)\, exp[i(\omega t + \alpha + arctan\frac{B}{A})] \tag{8}$$

Comparing with the polar form of the complex function it means that the radius is

$$\rho = \sqrt{A^2 + B^2}\, cn(\omega_1 t + \beta, m)\,, \tag{9}$$

and the argument is

$$\theta = \omega t + \alpha_1\,, \tag{10}$$

where $\alpha_1 = \alpha + arctan(B / A)$. Analyzing the relations (9) and (10) it has been seen that the distance between the rotor center and a fixed initial position varies periodical due to the cn elliptic function. The amplitude of vibration depends on the initial values A and B. For $\sqrt{A^2+B^2}$ higher, the amplitude of vibration is larger. The period of amplitude variation depends on the modulus m, i.e. on the value $\sqrt{A^2+B^2}$. Namely, the period of vibration is $4K(m)/\omega_1$. For higher values of the initial term $\sqrt{A^2+B^2}$ the value of m is smaller and ω_1 is higher. The period of vibration is decreasing tending to $2\pi / \sqrt{\omega^2 + b_1 / a}$. The cn function tends to a harmonic cosines function. For the same reasons the frequency of vibration increases for increasing of the value of $\sqrt{A^2+B^2}$. The θ - t relation (10) describes the variation of the angle position in time. It is a linear time function.

For the initial values $A = 0.1$, $B = 0.3$, $\alpha = 0.785$, $\beta = 0.4$ and the parameter values $a = b_1 = b_3 = g = 1$ the solution (8) in x-y plane is plotted (Fig.1) where x and y are the real and imaginary parts of the complex function z, respectively.

Let us consider some special cases.

For $g = 0$, the frequency is zero and the motion is represented with an one frequency solution

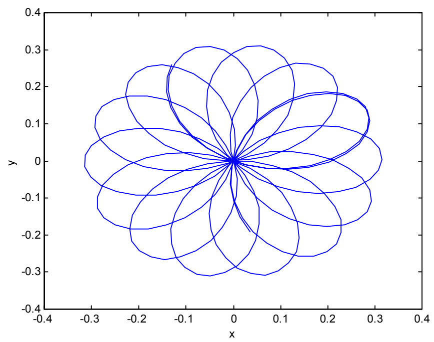

Fig.1. The orbital motion of the rotor center

$$z = (A + iB)\, exp(i\alpha)\, cn(\omega_1 t + \beta, m)\,, \tag{11}$$

where $\omega_1=\sqrt{\frac{b_1}{a}+b}$. The angle position of the rotor center does not change in time. It is fixed in the space and depends on the initial position. The distance of rotor center is a periodical time variable function. The orbital motion is along a line in angle direction α

For the case when the non-linearity is neglected i.e., $b_3 = 0$ the modulus m is zero and the cn function transforms to a harmonic cosines function. The solution has the form

$$z=(A+iB)\,exp[i(\omega t+\alpha)]\,cos(\omega_1 t+\beta)\,,$$

and the variation of the distance and of the angle position of the rotor center is

$$\rho=\sqrt{A^2+B^2}\,cos(\omega_1 t+\beta)\,,\ \theta=\omega t+\alpha+arctan\frac{B}{A}\,.$$

The period of amplitude vibration is $2\pi/\sqrt{\omega^2+b_1/a}$ .The angle position variation is a linear time variable function.

TRIAL SOLUTION

Using the generating solution (3) the trial solution for the eq.(1) is introduced as

$$z=[A(t)+iB(t)]\,exp[i\omega t+i\alpha(t)]\,cn[\psi(t)+\beta(t),m(t)]\,,\tag{12}$$

where the functions A, B, α and β are time dependent. The function $\psi=\int_0^t\omega_1 dt$ is also time dependent as the frequency ω_1 depends on A and B. The same is for the modulus of the Jacobi elliptic function m. To solve the differential equation (1) the following constraints have to be imposed:

1. The solution (12) has to satisfy the differential equation (1).
2. The first time derivative of the solution (12) has the form as the time derivative of the generating solution, i.e., it is

$$\dot{z}=[A(t)+iB(t)]i\omega\,exp[i\omega t+i\alpha(t)][i\omega cn-\omega_1 sndn]\,,\tag{13}$$

   where $cn = cn[\psi,m(t)]$, $sn = sn[\psi,m(t)]$, $dn = dn[\psi,m(t)]$.
3. The relationship between the frequency and parameters A and B must have the same form for the trial solution as for the generating solution (6).
4. The relationship between the modulus of the Jacobi elliptic function and the parameters A and B is time dependent and has the form (6).

Due to the constraint 2. the following relation exists

$$\begin{aligned}&[\dot{A}(t)+i\dot{B}(t)]exp(i\omega t)\,exp[i\alpha(t)]cn+[A(t)+iB(t)]i\dot{\alpha}(t)\,exp(i\omega t)\,exp[i\alpha(t)]cn+\\&[A(t)+iB(t)]\dot{\beta}(t)\,exp(i\omega t)\,exp[i\alpha(t)]cn_\psi+[A(t)+iB(t)]\dot{m}(t)\,exp(i\omega t)\,exp[i\alpha(t)]cn_m=0,\end{aligned}\tag{14}$$

where $(\bullet)_\psi=\partial/\partial\psi$ is the first derivative with respect to the argument and $(\bullet)_m=\partial/\partial m$ is the derivative with respect to the modulus of the Jacobi elliptic function.

Substituting the solution (12), the first (13) and the time derivative of (13) into (1) the following differential equation is obtained

$$\begin{aligned}
&[\dot{A}(t)+i\dot{B}(t)]\,exp(i\omega t)\,exp[i\alpha(t)][i\omega cn+\omega_1(t)cn_\psi]+\\
&[A(t)+iB(t)]i\dot{\alpha}(t)\,exp(i\omega t)\,exp[i\alpha(t)][i\omega cn+\omega_1(t)cn_\psi]+\\
&[A(t)+iB(t)]\,exp(i\omega t)\,exp[i\alpha(t)]\dot{\omega}_1(t)cn_\psi+\\
&[A(t)+iB(t)]\,exp(i\omega t)\,exp[i\alpha(t)]\dot{\beta}(t)[i\omega cn_\psi+\omega_1(t)cn_{\psi\psi}]+\\
&[A(t)+iB(t)]\,exp(i\omega t)\,exp[i\alpha(t)]\dot{m}(t)[\,i\omega cn_m+\omega_1(t)cn_{\psi m}]=\frac{\varepsilon}{a}f,
\end{aligned}\tag{15}$$

where $(\bullet)_{\psi\psi}=\partial^2/\partial\psi^2$, and $(\bullet)_{\psi m}=\partial^2/\partial\psi\partial m$.

Separating the real and imaginary terms in the relation (14) and the eq.(15) the following four first order differential equations are obtained

$$\dot{A}(t)cn-B(t)\dot{\alpha}(t)cn+A(t)\dot{\beta}(t)cn_\psi+A(t)\dot{m}(t)cn_m=0\,,\tag{16}$$

$$\dot{B}(t)cn+A(t)\dot{\alpha}(t)cn+B(t)\dot{\beta}(t)cn_\psi+B(t)\dot{m}(t)cn_m=0\,,\tag{17}$$

$$\begin{aligned}
&\dot{A}(t)\omega_1(t)cn_\psi-\dot{B}(t)\omega cn+A(t)\dot{\omega}_1(t)cn_\psi-\\
&[A(t)\omega cn+B(t)\omega_1(t)cn_\psi]\dot{\alpha}(t)+[A(t)\omega_1(t)cn_{\psi\psi}-B(t)\omega cn_\psi\,]\dot{\beta}(t)+\\
&[A(t)\omega_1(t)cn_{\psi m}-B(t)\omega cn_m]\dot{m}(t)=\frac{\varepsilon}{a}Re(f\,exp[-i\omega t-i\alpha(t)]),
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\dot{A}(t)\omega cn+\dot{B}(t)\omega_1(t)cn_\psi+[A(t)\omega_1(t)cn_\psi-B(t)\omega cn]\dot{\alpha}(t)+\\
&B(t)\dot{\omega}_1(t)cn_\psi+[A(t)\omega cn_\psi+B(t)\omega_1(t)cn_{\psi\psi}]\dot{\beta}(t)+\\
&[A(t)\omega cn_m+B(t)\omega_1(t)cn_{\psi m}]\dot{m}(t)=\frac{\varepsilon}{a}Im(f\,exp[-i\omega t-i\alpha(t)]).
\end{aligned}\tag{19}$$

These equations correspond to the eq.(1) and the task of finding the solution $z(t)$ is transformed into finding four functions $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$ and also the functions $m(t)$ and $\omega_1(t)$ which are functions of $A(t)$ and $B(t)$. The solving procedure of the system of equations (16)-(19) is not an easy task. Using the averaging procedure suggested by Coppola and Rand [9] the time unknown time variable functions are obtained.

EXAMPLE

Let us consider the case when an additional small damping force acts. The mathematical model of the system is

$$a\ddot{z}+b_1z+b_3z(z\bar{z})-ig\dot{z}=-\varepsilon\dot{z},\tag{20}$$

where $\varepsilon \ll 1$ is the damping coefficient which has a small value.

Using the previous procedure the differential equation (20) is transformed to a system of four first order differential equations

$$\dot{A}(t)\omega_1(t)cn_\psi - \dot{B}(t)\omega cn + A(t)\dot{\omega}_1(t)cn_\psi -$$
$$[A(t)\omega cn + B(t)\omega_1(t)cn_\psi]\dot{\alpha}(t) + [A(t)\omega_1(t)cn_{\psi\psi} - B(t)\omega cn_\psi]\dot{\beta}(t) + \tag{21}$$
$$[A(t)\omega_1(t)cn_{\psi m} - B(t)\omega cn_m]\dot{m}(t) = \frac{\varepsilon\omega}{a}(A\omega cn - B\omega_1 sndn),$$

$$\dot{A}(t)\omega cn + \dot{B}(t)\omega_1(t)cn_\psi + [A(t)\omega_1(t)cn_\psi - B(t)\omega cn]\dot{\alpha}(t) +$$
$$B(t)\dot{\omega}_1(t)cn_\psi + [A(t)\omega cn_\psi + B(t)\omega_1(t)cn_{\psi\psi}]\dot{\beta}(t) + \tag{22}$$
$$[A(t)\omega cn_m + B(t)\omega_1(t)cn_{\psi m}]\dot{m}(t) = \frac{\varepsilon\omega}{a}(A\omega_1 sndn + B\omega cn),$$

where

$$\dot{\alpha} = \frac{\dot{A}B - A\dot{B}}{A^2 + B^2}, \; \dot{\beta} = -\frac{A\dot{A} + B\dot{B}}{A^2 + B^2}\frac{cn}{cn_\psi} - \dot{m}\frac{cn_m}{cn_\psi}, \tag{23}$$

$$\dot{m} = \frac{\dot{\omega}_1}{\omega_1}[1 - \frac{b_3}{a\omega_1^2}(A^2 + B^2)], \;\; \dot{\omega}_1 = \frac{b_3}{a\omega_1}(A\dot{A} + B\dot{B}). \tag{24}$$

Introducing the functions (23) into (21) and (22) the averaging operation over the period 4K of Jacobi elliptic function is done. Integrating the equations the approximate functions A(t) and $B(t)$ are obtained. Substituting these functions into relation (23) and integrating the approximate functions $\alpha(t)$ and $\beta(t)$ are obtained. According to (6) the time variable frequency ω_1 and the modulus m are denoted. The approximate solution (12) is plotted in x-y plane.

It can be concluded that due to the damping force the functions $A(t)$ and $B(t)$ decrease in time. It causes the decrease of the amplitude of vibration. At the same time the frequency of vibration ω_1 decreases and the modulus of the Jacobi elliptic function increases. It causes the period of vibration to increase in time. Comparing the values of the amplitude of the system without damping with those with damping it can be concluded that the amplitude of vibration is for the whole time period is lower for the damping case than for the case without damping and the corresponding period of vibration is shorter than for the case of damping.

CONCLUSION

It can be concluded:

1. For the strong non-linear differential equation () with the complex function which describes the vibration of a non-linear rotor with gyroscopic effect a closed form solution in the form of the Jacobi elliptic function is obtained.
2. Due to the gyroscopic term the angle position of the rotor center is varying.
3. The position of the rotor center is varying in time according to the elliptic Jacobi function.
4. The adopted elliptic Krylov Bogolubov method is applicable for solving such a differential equation with addition of small functions.
5. The amplitude of vibration, the frequency and the period of vibration depend on the initial conditions and are affected with the additional small function

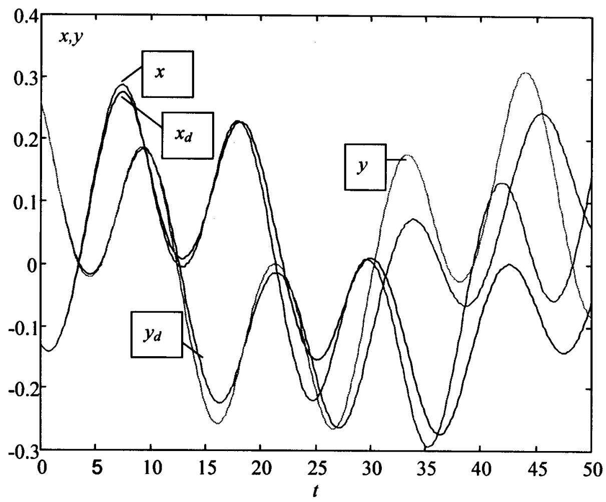

Fig. 2. The time history diagrams of rotor center *x-t* and *y-t* for the case without damping and x_d-*t* and y_d-*t* for the case with damping

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OSCILACIJE ROTORA SA NELINEARNIM SVOJSTVIMA

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U ovom radu su analizirane vibracije strogo nelinearnog rotora pod uticajem giroskopske sile. Kretanje je opisano nelinearnom diferencijalnom jednačinom drugog reda sa kompleksnom funkcijom. Za rešavanje jednačine u radu je razvijena približna analitička metoda koja je bazirana na eliptičkoj Krilov Bogoljubov metodi razradjenoj za sisteme sa jednim stepenom slobode. Nakon odredjivanja tačnog analitičkog rešenja za strogo nelinearnu diferencijalnu jednačinu u obliku Jakobijeve eliptičke funkcije uvedeno je probno rešenje kompletne diferencijalne jednačine istog *oblika kao što je i generalno rešenje. Početna jednačina svede se na četiri diferencijalne jednačine prvog reda. Približno analitičko rešenje ovih jednačina daje rešenje početne diferencijalne jednačine u prvom priblienju. Metoda je primenjena za proučavanje vibracije rotora kod kojeg pored velike nelinearnosti postoje i male nelinearnosti koje su funkcije brzine pomeranja. Analitički dobivena rešenja poredjena su sa numerički dobivenim rešenjima. Razlika je zanemarljivo mala, što svedoči o opravdanosti primene ove metode.*