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AN INVERSE PROBLEM FOR A FIRST ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. *We prove an existence and uniqueness theorem concerning the identification of a nonlinear term in an boundary value problem related to a (nonlinear) first-order ordinary differential equation containing a parameter λ.. We show that the successive iteration method allows to approximate the unknown nonlinear terms.*

Key words: *Existence and uniqueness results. First-order nonlinear ordinary differential equations, Boundary value problems containing parameters.*

In this paper we consider the case of nonlinear first order differential equation containing a parameter λ. The related inverse problem consists in detrmining a nonlinear term f by means of an additional boundary information involving λ. Such a problem reduces to a nonlinear integral equation for f containing its inverse function f^{-1} too.

We begin by considering the following nonlinear Cauchy problem in the large: determine a function y: $[0,1]x[0,\Lambda] \longrightarrow$ R, such that

$$y'(x,\lambda) = \lambda f(y(x,\lambda)) \quad 0 \le x \le 1, 0 \le \lambda \le \Lambda \tag{1}$$

$$y(0,\lambda) = b(\lambda) \quad 0 \le \lambda \le \Lambda \tag{2}$$

where $\Lambda \in (0,+\infty)$, f and b are smooth functions.

Assume that f is everywhere positive and there exists a function $c \in C([0,\Lambda])$, $c(\lambda) \ge b(\lambda)$, $\forall\lambda\in[0.\Lambda]$ such that

$$\int_{b(\lambda)}^{c(\lambda)} f^{-1}(t)dt \ge \lambda \qquad 0 \le \lambda \le \Lambda\,. \tag{3}$$

Under such assumptions it is easy to deduce that the problem admits a unique solutions.

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We can now define what we mean by a solution (y,f) , when $y'(1,\lambda) = \lambda a(\lambda)$ have the inverse problem:

Definition. A pair of functions $(y, f) \in C([0,1]\times[0,\Lambda])\times C(\Re(y))$ satisfying the properties

$$y(\cdot,\lambda)\in C^1([0,1]) \quad \lambda\in[0,\Lambda] \tag{4}$$

$$y(0,0)\le y(x,\lambda)\le y(1,\Lambda) \quad 0\le x\le 1, \quad 0\le\lambda\le\Lambda \tag{5}$$

$$f(b(0)=a(0)) \tag{6}$$

$$f(t)>0 \quad t\in[y(0,0),y(1,\Lambda)] \tag{7}$$

$$f:[y(0,0),y(1,\Lambda)]\to\Re \text{ is strictly increasing;} \tag{8}$$

and equations (1),(2) is called a solution to the inverse problem.

From (5) and (8) it is easy to derive the bounds

$$f(y(0,0))\le f(y(x,\lambda))\le f(y(1,\Lambda)) \qquad 0\le x\le 1, 0\le\lambda\le\Lambda \tag{9}$$

We shall assume that the functions a and b satisfy the following properties:

$$a,b\in C^1([0,\Lambda]), \quad a(\lambda)>0, \quad a'(\lambda)>0, \quad b'(\lambda)>0 \quad 0\le\lambda\le\Lambda \tag{10}$$

than we get

$$\begin{gathered} a(0)=f(b(0))=f(y(0,0))\le f(y(x,\lambda))\le f(y(1,\lambda))=\lambda^{-1}y'(1,\lambda)=a(\lambda)\le a(\Lambda) \\ 0\le x\le 1, \quad 0\le\lambda\le\Lambda \end{gathered} \tag{11}$$

Theorem. Assume that data the a and b satisfy, in addition to properties (10), also the inequality

$$b(\Lambda)-b(0)\le\int_0^\Lambda a(t)dt \tag{12}$$

Then the inverse problem (1),(2) admits a unique a solution (y,f) (cf. definition) such that

$$y\in C^1([0,1]\times[0,\Lambda]), f\in C^1(\Re(y)) \tag{13}$$

We observe that, if (y,f) is solution to the problem (1),(2) according to definition, then (y,f) is a solution to the system

$$\int_{b(\lambda)}^{y(x,\lambda)} f^{-1}(t)dt=\lambda\ x \qquad (x,\lambda)\in[0,1]\times[0,\Lambda] \tag{14}$$

$$a(\lambda)=f(y(1,\lambda)) \tag{15}$$

Introduce then the auxiliary unknown $\Phi(\lambda) = y(1,\lambda)$, $\lambda\in[0,\Lambda]$. Set now $x = 1$ in the equation (14) and differentiate with respect to λ. We derive the following system for the pair (Φ, f)

$$\Phi'(\lambda)f(\Phi(\lambda))^{-1}-b'(\lambda)f(b(\lambda))^{-1}=1 \qquad \lambda\in[0,\Lambda], \qquad a(\lambda)=f(\Phi(\lambda)).$$

This implies

$$\Phi'(\lambda) = [1 + b'(\lambda) f(b(\lambda))^{-1}] a(\lambda) \quad \lambda \in [0, \Lambda] .$$

Moreover, we get $\Phi(0) = b(0)$. Consequently we deduce the representation

$$\Phi(\lambda) = b(0) + \int_0^\lambda [1 + b'(t) f(b(t))^{-1}] a(t) dt \quad \lambda \in [0, \Lambda] .$$

Finally, we derive the following integral equation for f, f^{-1} denoting the inverse function of f:

$$f^{-1}(a(\lambda)) = b(0) + \int_0^\lambda [1 + b'(t) f(b(t))^{-1}] a(t) dt \qquad \lambda \in [0, \Lambda].$$

Proof of the theorem. We prove that the inverse problem (1),(2) admits one solution only. To this purpose we assume that (y_1, f_1), (y_2, f_2) are solutions of (1), (2) and $y_2(1, \Lambda) \le y_1(1, \Lambda)$. Introduce then the functions

$$w(x,\lambda) = y_1(x,\lambda) - y_2(x,\lambda) \qquad x \in [0,1], \lambda \in [0,\Lambda] \tag{16}$$

$$p(x,\lambda) = \int_0^1 f_1'(y_2(x,\lambda) + \theta(y_1(x,\lambda) - y_2(x,\lambda))) d\theta \qquad x \in [0,1], \lambda \in [0,\Lambda] \tag{17}$$

$$q(x,\xi,\lambda) = e^{\lambda \int_\xi^x p(\theta,\lambda) d\theta} \qquad x, \xi \in [0,1], \lambda \in [0,\Lambda] \tag{18}$$

$$z(t) = f_1(t) - f_2(t) \qquad t \in [b(0), y_2(1,\Lambda)] \tag{19}$$

Since $(y_i, f_i)(i = 1,2)$ satisfy the equations (10,(2), then w solves the following

$$w'(x,\lambda) = \lambda p(x,\lambda) w(x,\lambda) + \lambda z(y_2(x,\lambda)) \qquad x \in [0,1], \lambda \in [0,\Lambda] \tag{20}$$

$$w(0,\lambda) = 0 \qquad \lambda \in [0,\Lambda] \tag{21}$$

$$w'(1,\lambda) = 0 \qquad \lambda \in [0,\Lambda] \tag{22}$$

Solving (20)-(22) we get

$$w(x,\lambda) = \lambda \int_0^x q(x,\xi,\lambda) z(y_2(\xi,\lambda)) d\xi \qquad x \in [0,1], \lambda \in [0,\Lambda] \tag{23}$$

Then from (20) and (23) we deduce

$$w'(x,\lambda) = \lambda z(y_2(x,\lambda)) + \lambda^2 p(x,\lambda) \int_0^x q(x,\xi,\lambda) z(y_2(\xi,\lambda)) d\xi \qquad x \in [0,1], \lambda \in [0,\Lambda] \tag{24}$$

Setting $x = 1$ in (24) and using (23), we easily derive the following equation for the single variable function z:

$$z(y_2(1,\lambda)) + \lambda p(1,\lambda) \int_0^1 q(1,\xi,\lambda) z(y_2(\xi,\lambda)) d\xi = 0 \qquad \lambda \in [0,\Lambda] \tag{25}$$

Perform now the change of variable $s = y_2(\xi,\lambda)$. It is easy to check ({[2]) that equation (25) is equivalent to the following

$$z(y_2(1,\lambda)) + p(1,\lambda) \int_{y_2(0,\lambda)}^{y_2(1,\lambda)} q(1, y_2^{-1}(s,\lambda),\lambda)[f_2(s)]^{-1} z(s)ds = 0 \tag{26}$$

$$\lambda \in [0,\Lambda]$$

From equation (26) we immediately deduce that $z = 0$ in the interval $[y_2(0,0), y_2(1,\Lambda)]$. Consequently $f_1 = f_2$ in $[b(0, y_2(1,\Lambda))]$. From (19) we get $y_2(1,\Lambda) = y_1(1,\Lambda)$. Finally, we have

$$\begin{aligned} y_1(x,\lambda) &= y_2(x,\lambda) & x \in [0,1], \lambda \in [0,\Lambda] \\ f_1(t) &= f_2(t) & t \in \Re(y_1) = \Re(y_2). \end{aligned} \tag{27}$$

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JEDAN INVERZNI PROBLEM ZA OBIČNU NELINEARNU DIFERENCIJALNU JEDNAČINU PRVOG REDA

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Mi dokazujemo teoremu o postojanju i jedinstvenosti koja se odnosi na identifikaciju nelinearnog člana u jednom problemu graničnih vrednosti za jednu nelinearnu prvog reda diferencijalnu jednačinu koja sadrži parametar λ. Pokazujemo da metoda sukcesivnih iteracija vodi do aproksimacije nepoznatih nelinearnih članova.