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IMPULSIVE HYBRYD FUZZY DIFFERENTIAL EQUATIONS

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Abstract. *In this paper, we suggest another approach to incorporate impulsive nature into fuzzy differential equations so that one can control the behavior of solutions suitably without replacing fuzzy differential equations by other formulations. We develop the comparison theorems for fuzzy impulsive hybrid systems to establish Lyaponov stability results in terms of two measures differential equations.*

Keywords and Phrases: *Fuzzy Hybrid systems, Impulsive differential Equations*

INTRODUCTION

It is now well known that impulsive differential equations, form a natural description of observed evolution phenomena of several real world problems. As a result, the theory of impulsive differential equations has attracted the attention and its theory is more richer than the theory of differential equations without impulses [1].

The theory of fuzzy differential equations has attracted much attention in recent times [6]. The approach is based on the fuzzification of the differential operator, and therefore suffers from the disadvantage, since the solution $u(t)$ of the corresponding fuzzy differential equation has the property that the $\text{diam}[u(t)]^\alpha$ is nondecreasing as time increases. Consequently this original formulation of fuzzy differential equations does not reflect the rich behavior of solution of corresponding ordinary differential equations without fuzziness. Some alternative formulations of fuzzy initial value problems are now suggested by replacing them by a system of multivalued differential equation as well as by set differential equations [6].

In this paper, we suggest another approach to incorporate impulsive nature into fuzzy differential equations so that one can control the behavior of solutions suitably without replacing fuzzy differential equations by other formulations. Impulsive perturbations can act as controllers and thereby remove the disadvantage created by

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the fuzzification of the differential operator mentioned earlier. Utilizing the existing results in impulsive and fuzzy differential equations, we initiate the study of impulsive fuzzy differential equations and then consider impulsive hybrid fuzzy differential systems.

1. PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of R^n. If $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0 + a], t_0 \geq 0$ and $a > 0$ and denote by $E^n = [u : R^n \to [0,1]$ such that u satisfies (i) to (iv) mentioned below]:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = cl[x \in R^n : u(x) > 0]$ is compact.

For $0 < \alpha \leq 1$ we denote $[u]^\alpha = [x \in R^n : u(x) \geq \alpha]$. Then from (i) to (iv), it follows that the α-level sets $[u]^\alpha \in P_k(R^n)$ for $0 \leq \alpha \leq 1$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(R^n)$. Then we define

$$d[u,v] = \sup_{0 \leq \alpha \leq 1} d_H[[u]^\alpha, [v]^\alpha], \tag{1.1}$$

which defines a metric in E^n and (E^n, d) is a complete metric space. We list the following proprieties of $d[u, v]$:

$$d[u+w, v+w] = d[u,v] \quad \text{and} \quad d[u,v] = d[v,u], \tag{1.2}$$

$$d[\lambda u, \lambda v] = |\lambda| d[u,v], \tag{1.3}$$

$$d[u,v] \leq d[u,w] + d[w,v], \tag{1.4}$$

for all $u, v, w \in E^n$ and $\lambda \in R$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called H-difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

For details on fuzzy differential equations and the calculus described above, see [6].

2. HYBRID FUZZY DIFFERENTIAL EQUATIONS

The problem of stabilizing a continuous plant governed by differential equation through the interaction with a discrete time controller has recently been investigated. This study leads to the consideration of hybrid systems [3, 7]. In this section, we shall extend this approach to fuzzy differential equations.

Consider the hybrid fuzzy differential system

$$u'(t) = f(t, u(t), \lambda_k(z)), \quad u(t_k) = z, \tag{2.1}$$

on $[t_k, t_{k+1}]$ for any fixed $z \in E^n, k = 0, 1, 2, ...$, where $f \in C[R_+ \times E^n \times E^n, E^n]$, and $\lambda_k \in C[E^n, E^n]$. Here we assume that $0 \le t_0 < t_1 < t_2 < ...$ are such that $t_k \to \infty$ as $k \to \infty$ and the existence and uniqueness of solutions of the hybrid system hold on each $[t_k, t_{k+1}]$. To be specific, the system would look like

$$u'(t) = \begin{cases} u_0'(t) = f(t, u_0(t), \lambda_0(u_0)), & u_0(t_0) = u_0, \quad t_0 \le t \le t_1, \\ u_1'(t) = f(t, u_1(t), \lambda_1(u_1)), & u_1(t_1) = u_1, \quad t_1 \le t \le t_2, \\ \quad \cdots & \\ \quad \cdots & \\ u_k'(t) = f(t, u_k(t), \lambda_k(u_k), & u_k(t_k) = u_k, \quad t_k \le t \le t_k + 1, \\ \quad \cdots & \\ \quad \cdots & \end{cases}$$

By the solution of (2.1), we therefore mean the following function

$$u(t) = u(t, t_0, u_0) = \begin{cases} u_0(t), & t_0 < t \le t_1, \\ u_1(t), & t_1 < t \le t_2, \\ \cdots & \\ \cdots & \\ u_k(t), & t_k < t \le t_{k+1}, \\ \cdots & \\ \cdots & \end{cases}$$

We note that the solutions of (2.1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for any fixed $u_k \in E^n$ and $k = 0, 1, 2, ...$

Let $V \in C[E^n, R_+]$. For $t \in (t_k, t_{k+1}], u, z \in E^n$, we define

$$D^+V(u, z) = \limsup_{h \to 0^+} \frac{1}{h}[V(u + hf(t, u, \lambda_k(z))) - V(u)].$$

We can then prove the following comparison theorem in terms of Lyapunov-like function V.

Theorem 2.1 *Asume that*

(i) $V \in C[E^n, R_+], V(u)$ *satisfies* $|V(u) - V(v)| \leq Ld[u, v], L > 0$ *for* $u, v \in E^n$;

(ii) $D^+V(u, z) \leq g(t, V(u), \sigma_k(V(z)), t \in (t_k, t_{k+1}]$, *where* $g \in C[R^3_+, R]$, $\sigma_k \in C[R_+, R_+]$, $u, z \in E^n, k = 0, 1, 2, ...$;

(iii) the maximal solution $r(t) = r(t, t_0, w_0)$ *of the hybrid scalar differential equation.*

$$\left.\begin{aligned} w' = g(t, w(t),\ \sigma_k(w_k)),\ t \in (t_k, t_{k+1}] \\ w(t_k) = w_k, \quad k = 0, 1, 2, ..., \end{aligned}\right\} \tag{2.2}$$

exists on $[t_0, \infty)$.

Then any solution $u(t) = u(t, t_0, u_0)$ *of (2.1) such that* $V(u_0) \leq w_0$ *satisfies the estimate*

$$V(u(t)) \leq r(t), \quad t \geq t_0.$$

Proof: Let $u(t)$ be any solution of (2.1) existing on $([t_0, \infty)$ and set $m(t) = V(u(t))$. Then using (i) and (ii), and proceeding as in the proof of Theorem 4.2.1 in [6], we get the differential inequality

$$D^+m(t) \leq g(t, m(t), \sigma_k(m_k)) \text{ for } t_k < t \leq t_{k+1},$$

where $m_k = V(u(t_k))$. For $t \in [t_0, t_1]$, since $m(t_0) = V(u_0) \leq w_0$, we obtain by Theorem 4.2.1. in [1, 4]

$$V(u_0(t)) \leq r_0(t, t_0, w_0), \quad t_0 \leq t \leq t_1,$$

where $r_0(t) = r_0(t, t_0, w_0)$ is the maximal solution of

$$w_0' = g(t, w_0, \sigma_0(w_0)), \quad w_0(t_0) = w_0 \geq 0, \quad t_0 \leq t \leq t_1,$$

and $u_0(t)$ is the solution of

$$u_0' = f(t, u_0(t), \lambda_0(u_0)), \quad u(t_0) = u_0 \geq 0, \quad t_0 \leq t \leq t_1.$$

Similarly, for $t \in [t_1, t_2]$, it follows that

$$V(u_1(t)) \leq r_1(t, t_1, w_1), \quad t_1 \leq t \leq t_2,$$

where $w_1 = r_0(t_1, t_0, w_0), r_1(t, t_1, w_1)$ is the maximal solution of

$$w_1' = g(t, w_1(t, \sigma_1(w_1)), \quad w_1(t_1) = w_1 \geq 0, \quad t_1 \leq t \leq t_2,$$

and $u_1(t)$ is the solution of

$$u_1' = f(t, u_1(t), \lambda_1(u_1)), \quad u_1(t_1) = u_1, \quad t_1 \leq t \leq t_2.$$

Proceeding similarly, we can obtain

$$V(u_k(t)) \le r_k(t, t_k, w_k), \quad t_k \le t \le t_{k+1},$$

where $u_k(t)$ is the solution of

$$u_k'(t) = f(t, u_k(t), \lambda_k(u_k)), \quad u_k(t_k) = u_k, \quad t_k \le t \le t_{k+1},$$

and $r_k(t, t_k, w_k)$ is the maximal solution of

$$w_k' = g(t, w_k(t), \sigma_k(w_k)), \quad w_k(t_k) = w_k, \quad t_k \le t \le t_{k+1},$$

where $w_k = r_{k-1}(t_k, t_{k-1}, r_{k-2}(t_{k-1}, t_{k-2}, w_{k-1}))$. Thus defining $r(t, t_0, w_0)$as the maximal solution of the comparison hybrid system (3.2) as

$$r(t, t_0, w_0) = \begin{bmatrix} r_0(t, t_0, w_0), & t_0 \le t \le t_1, \\ r_1(t, t_1, w_1), & t_1 \le t \le t_2, \\ \dots & \\ r_k(t, t_k, w_k), & t_k \le t \le t_{k+1}, \\ \dots & \\ \dots & \end{bmatrix}$$

and taking $w_0 = V(u_0)$, we obtain the desired estimate

$$V(u(t)) \le r(t), \quad t \ge t_0.$$

The proof is therefore complete.

3. Impulsive Hybrid Fuzzy Differential System

Consider now the hybrid impulsive fuzzy differential system given by

$$\begin{bmatrix} u' = f(t, u, \lambda(t_k, u_k)), & t \in [t_k, t_{k+1}], \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & t = t_k, \\ u(t_0^+) = u_0, & \end{bmatrix} \tag{3.1}$$

where $f \in C[R_+ \times E^n \times E^n, E^n]$, $I_k : E^n \to E^n$, $\lambda_k \in C[R_+, \times E^n, E^n]$, and $k = 0, 1, 2, \dots$. We assume that $I_0(u_0) = 0$, and the existence of solution of the system

$$\begin{bmatrix} u' = f(t, u, \lambda(t_k, z)), & t \in [t_k, t_{k+1}], \\ u(t_k^+) = z + I_k(z), & t \ne t_k, \\ u(t_0^+) = u_0 & \end{bmatrix} \tag{3.2}$$

on $[t_k, t_{k+1}]$ for any fixed $z \in E^n$ and all $k = 0, 1, 2, \dots$. Note that the solution of (3.2) are piecewise continuous function with points of discontinuity of the first type at $t = t_k$ at which they are assumed to be left continuous.

Let $V : R_+ \times E^n \to R_+^n$. Then V is said to belong to class V_0, if

(i) V is continuous in $(t_k, t_{k+1}] \times E^n$ and for each $u \in E^n, k = 1, 2, \ldots.$ $\lim_{(t,v)\to(t_k^+,u)} V(t,v) = V(t_k^+, u)$ exists;

(ii) V is locally Lipschitzian in u. Then we define, as before,

$$D^+V(t,u,z) \equiv \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u,\lambda_k(t_k,z))) - V(t,u)].$$

We need the following comparison result.

Theorem 3.1 *Assume that*

(i) $V \in C[R^+ \times E^n, R_+], V(t,u)$ *is locally Lipschitzian in* u *that is* $|V(t,u) - V(t,v)| \leq Ld[u,v], L > 0,$ *and* $D^+V(t,u,z) \leq g(t, V(t,u), \sigma_k(t_k,z)),$ $t \in (t_k, t_{k+1}),\ u, z \in E^n,$ *where* $\sigma_k \in C[R_+^2, R],\ g \in C[R_+^3, R]$;

(ii) There exist a $\psi_k \in C[R_+, R_+],\ \psi_k(w)$ *is nondecreasing in* w *and*

$$V(t, u + I_k(u)) \leq \psi_k(V(t,u)),\ k = 1, 2, \ldots, u \in E^n;$$

(iii) the maximal solution $r(t) = r(t, t_0, w_0)$ *of the scalar hybrid impulsive differential equation*

$$\begin{bmatrix} w' = g(t, w, \sigma(t_k, w_k)), & t \in [t_k, t_{k+1}], \\ w(t_k^+) = \psi(w(t_k)), & t = t_k, \\ w(t_0) = w_0 \geq 0, & \end{bmatrix} \tag{3.3}$$

existing on $[t_0, \infty]$. Then any solution $u(t) = u(t, t_0, u_0)$ of (3.1) satisfies

$$V(t, u(t) \leq r(t, t_0, w_0),\ \ t \geq t_0,$$

provided $w_0 \geq V(t_0, u_0)$.

The proof of this comparison Theorem follows on similar lines as in Theorem 3.1 defining $u(t)$ and $r(t)$, piece by piece suitably. We omit the proof to avoid monotony.

Having the foregoing comparison result at our disposal, we can formulate stability criteria of the solutions of (**??**) in terms of two different measures.

We need the following definition before we proceed further.

Definition 3.1 *Let* $\mathcal{K} = [a \in C[R_+, R_+]$: $a(w)$ *is strictly increasing in* w *and* $a(0) = 0$ *and* $\Gamma = [h \in [R_+ \times E^n, R_+]$: *inf* $h(t,u) = 0$ *for* $t = R_+,$ *and* $u \in E^n]$.

Definition 3.2 *Let* $h_0, h \in \Gamma$. *Then we say that* h_0 *is uniformly finer than* h *if there exists a* $\rho > 0$ *and a function* $\phi \in \mathcal{K}$ *such that*

$$h_0(t,u) < \rho \text{implies } h(t,u) \leq \phi(h_0(t,u)).$$

Definition 3.3 *The hybrid setvalued differential equation (3.1) is said to be* *(i)* *(h_0, h)-equistable if, for each $\epsilon > 0, t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that*

$$h_0(t_0, u_0) < \delta \quad \text{implies} \quad h(t, u(t)) < \epsilon, \quad t \geq t_0,$$

where $u(t)$ is any solution of equation (3.1).

Based on the this definition, other stability notions can be formulated in terms of two measures. A few choices of (h_0, h) will demonstrate the generality of the definition of 3.3. Furthermore, the stability notions in terms of (h_0, h)unify a variety of stability concepts found in the literature. It is easy to see that the Definition 3.3 reduces to

(1) the well known stability of the trivial solution of (3.1)if $h(t, u) = h_0(t, u) = d(u, \theta)$, $u \in E^n$;

(2) the stability of an invariant set $\Omega \in E^n$ if $h(t, u) = h_0(t, u) = d_0[u, \Omega] = [infd[u, v] : v \in \Omega]$;

(3) the stability of asymptotically invariant set θ, if $h(t, u) = h_0(t, u) = d(u, \theta) + \sigma(t)$, where $\sigma(t) > 0$ is a decreasing function such that $\sigma(t) \to 0$ as $t \to \infty$;

(4) the stability of conditionally invariant set Ω_1 with respect to Ω_2, where $\Omega_2 \subset \Omega_1 \subset E^n$, if $h(t, u) = d_0[u, \Omega_1]$, $h_0(t, u) = d_0[u, \Omega_2]$;

(5) the stability of the prescribed motion $u_0(t)$ of (3.1) if $h(t, u) = h_0(t, u) = d[u, u_0(t)]$;

(6) the partial stability if $h(t, u_1) = d[u_1, \theta]$, $h_0(t, u = d(u, \theta)$, where u_1 is a compact convex subset of $u \in E^n$;

(7) the orbital stability if $h(t, u) = h_0(t, u) = d[u, B((t_0, u_0))$, where $B((t_0, u_0)) - u_0([t_0, \infty), t_0, , u_0)$ is a closet set in E^n and $u_0(t, t_0, u_0)$ is a prescribed solution of (3.1).

For various definitions and stability results in terms of two measures see [1, 2].

Definition 3.4 *Let $V : R_+ \times E^n \to R_+$, belong to class V_0. Then V is said to be*

(i) *$h-$positive definite if there exists a $\rho > 0$ and a function $b \in K$ such that*

$$b(h(t, u) \leq V(t, u)$$

whenever $h(t, u) < \rho, t \in R_+, u \in E^n$;

(ii) *h_0 decrescent if there exists a $\rho > 0$ and a function $a \in \mathcal{K}$ such that*

$$V(t, u) \leq a(h_0(t, u))$$

whenever $h_0(t, u) < \rho, \quad (t, u) \in R_+ \times E^n$.

We are in a position to prove stability results of (3.1)in terms of two measures.

Theorem 3.2 *Assume that*

(i) $V \in V_0$, V is $h-$positive definite and h_0 decrescent;

(ii) Conditions (i) and (ii)of Theorem 3.1 hold;

(iii) h_0 is finer than h.

Then the stability properties of the trivial solution $w = 0$ of (3.3)imply the corresponding $(h_0, h)-$ stability properties of (3.1)respectively.

Proof: We shall only give the proof of $(h_0, h)-$ stability.
Since V is $h-$ positive definite, there exist a $\lambda > 0$ and $b \in \mathcal{K}$ such that

$$b(h(t,u)) \leq v(t,u), \quad if \quad h(t,u) < \lambda. \tag{3.4}$$

Let $0 < \epsilon < \lambda$ and $t_0 \in T_+$. Suppose that the trivial solution $w = 0$ of 3.3)is stable. Then given $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$satisfying

$$0 < w_0 < \delta_1 \quad \Longrightarrow \quad w(t, t_0, w_0) < b(\epsilon), \quad t \geq t_0 \tag{3.5}$$

where $w(t, t_0, w_0)$ is any solution of 3.3. Choose $w_0 = V(t_0, u_0)$. Since $V(t, u)$ is h_0- decrescent and h_0 is finer than h, there exists a $\lambda_0 > 0$ and a $a \in \mathcal{K}$such that for $h_0(t_0, u_0) < \lambda_0$,

$$h(t_0, u_0) < \lambda \quad and \quad V(t_0, u_0) \leq a(h_0(t_0, u_0)). \tag{3.6}$$

It then follows from (3.4) that if $h_0(t_0, u_0 < \lambda_0$, then

$$b(h(t_0, u_0) \leq V(t_0, u_0) \leq a(h_0(t_0, u_0). \tag{3.7}$$

Choose a $\delta = \delta(t_0, \epsilon)$ such that $\delta \in (0, \lambda_0]$, $a(\delta < \delta_1$ and let $h_0(t_0, u_0) < \delta$. Then (3.7)shows that $h(t_0, u_0) < \epsilon$ since $\delta_1 < b(\epsilon)$. we claim that

$$h(t_0, u_0) < \epsilon \quad wheneverh_0(t_0, u_0) < \delta, \tag{3.8}$$

where $u(t)$ is any solution of (3.1).

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IMPULSIVNE HIBRIDNE FAZI-DIFERENCIJALNE JEDNAČINE

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U ovom radu se predlaže drugi pristup uvodjenja inpulsivne prirode u fazi diferencijalne jednačine tako da one mogu upravljati ponašanjem rešenja prigodno bez prevodjenja fazi-diferencijalnih jednačina u druge formulacije. Mi razvijamo teoreme uporedjenja za fazi impulsivne hibridne sisteme radi postavljanja rezultata Lyapunovljeve stabilnosti u formi dve mere diferencijalnih jednačina.

Ključne reči i rečenice (fraze): *Fazi hibridni sistemi, Impulsivne differencijalne jednačine.*