FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 14, 2003, pp. 843 - 849

ON A PERFECT FLUID SPACE-TIME ADMITTING QUASI CONFORMAL CURVATURE TENSOR

UDC 514.82

Sarbari Guha

Dept. of Physics, St. Xavier's College, 30 Park Street, Kolkata 700 016
E-mail: sarbariguha@rediffmail.com

Abstract. *The notion of the quasi conformal curvature tensor C* of type (1,3) in a Riemannian manifold (M^n,g) (n>3) was introduced by M. C. Chaki and M. L. Ghosh [1] according to whom*

$$C^*(X,Y,Z) = aR(X,Y,Z) + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y]$$

where a and b are constants, R is the Riemann tensor of type (1,3), S is the Ricci tensor of type (0,2), Q is the Ricci tensor of type (1,1) and r is the scalar curvature of the manifold.

In this paper, a four-dimensional perfect fluid space-time with a Lorentz metric of signature (+,+,+,–) and non-zero scalar curvature, admitting a quasi conformal curvature tensor ,has been considered.

It is shown that, if such a fluid space-time with unit timelike velocity vector field obeys Einstein's equation with cosmological constant and its quasi conformal curvature tensor is divergence-free then the fluid is shear-free, irrotational and its energy density is constant over the hypersurface orthogonal to the velocity vector field.

1. INTRODUCTION

The notion of the quasi conformal curvature tensor C* of type (1,3) in a Riemannian manifold (M^n, g) (n>3) was introduced by M. C. Chaki and M. L. Ghosh [1] according to whom

$$C^*(X,Y,Z) = aR(X,Y,Z) + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y] \tag{1.1}$$

Received April 20, 2003

where a and b are constants, R is the Riemann tensor of type (1,3), S is the Ricci tensor of type (0,2), Q is the Ricci tensor of type (1,1) and r is the scalar curvature of the manifold. Defining

$$L(X,Y) = S(X,Y) - \frac{r}{2(n-1)} g(X,Y), \tag{1.2}$$

$$g(NX,Y) = L(X,Y) \tag{1.3}$$

and

$$g(QX,Y) = S(X,Y) \tag{1.4}$$

the following relation is obtained

$$NX = QX - \frac{r}{2(n-1)} X \qquad \text{or} \qquad N = Q - \frac{r}{2(n-1)} I. \tag{1.5}$$

Consequently (1.1) can be expressed as follows:

$$\begin{aligned} C^*(X,Y,Z) = (aR(X,Y,Z) + b[L(Y,Z)X - L(X,Z)Y + g(Y,Z)NX - g(X,Z)NY] \\ - \beta r[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.6}$$

with

$$\beta = \frac{a + (n-2)b}{n(n-1)}. \tag{1.7}$$

In this paper, a four-dimensional perfect fluid space-time with a Lorentz metric of signature (+,+,+,–) and non-zero scalar curvature, admitting a quasi conformal curvature tensor ,has been considered.

It is shown that, if such a fluid space-time with unit timelike velocity vector field obeys Einstein's equation with cosmological constant and its quasi conformal curvature tensor is divergence-free then the fluid is shear-free, irrotational and its energy density is constant over the hypersurface orthogonal to the velocity vector field.

2. PRELIMINARIES

From (1.5) we obtain

$$N = Q - \frac{r}{2(n-1)} I \tag{2.1}$$

Taking the divergence of this equation we have

$$div\, N = div\, Q - \frac{dr}{2(n-1)} \tag{2.2}$$

where "d" denotes the operator of exterior differentiation. But

$$div\, Q = \frac{1}{2} dr. \tag{2.3}$$

Therefore from (2.2) we get

$$div\, N = \frac{(n-2)}{2(n-1)}\, dr. \tag{2.4}$$

Next, differentiating (1.6) covariantly with respect to W, we obtain

$$(\nabla_W C^*)(X,Y,Z) = a(\nabla_W R)(X,Y,Z) + b[(\nabla_W L)(Y,Z)X - (\nabla_W L)(X,Z)Y + g(Y,Z)(\nabla_W N)(X) - g(X,Z)(\nabla_W N)(Y)] - \beta Wr[g(Y,Z)X - g(X,Z)Y]. \tag{2.5}$$

Contracting (2.5) and using (2.4) we get

$$(divC^*)(X,Y,Z) = a(div\, R)(X,Y,Z) + b[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)] + [\frac{(n-2)b}{2(n-1)} - \beta]\, [g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.6}$$

But we know that

$$(div\, R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{2.7}$$

In view of (2.7) we get from (2.6)

$$(div\, C^*)(X,Y,Z) = a[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + b[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)] + [\frac{(n-2)b}{2(n-1)} - \beta]\, [g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.8}$$

Let

$$F(X,Y,Z) = a[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + b[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)] + m[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{2.9}$$

where $m = [\frac{(n-2)b}{2(n-1)} - \beta]$.

In that case

$$(div\, C^*)(X,Y,Z) = 0 \tag{2.10}$$

if and only if

$$F(X,Y,Z) = 0. \tag{2.11}$$

If we write

$$H(X,Y,Z) = [(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)], \tag{2.12}$$

then in view of (1.2) we get

$$(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) = H(X,Y,Z). \tag{2.13}$$

Consequently, we have

$$F(X,Y,Z) = a[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + bH(X,Y,Z) + m[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.14}$$

Let

$$(divR)(X,Y,Z) = 0. \tag{2.15}$$

In that case we shall have

$$F(X,Y,Z) = 0. \tag{2.16}$$

From this it follows that

$$dr(X) = 0. \tag{2.17}$$

Hence, from (1.2) we obtain

$$dL(X,Y) = dS(X,Y). \tag{2.18}$$

We can therefore state the following theorem:

Theorem 1: *A Riemannian manifold (M^n, g) (n>3) will have divergence-free quasi conformal curvature tensor if and only if (div R)(X,Y,Z) = 0.*

3. RESULTS

Let (M^4,g) be a general relativistic perfect fluid space-time with divergence-free quasi conformal curvature tensor. In that case in view of equations (1.2) and (2.13) to (2.17) we shall have

$$(\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z).$$

Hence using (1.4) and (1.5) we get

$$(\nabla_X N)(Y) = (\nabla_Y N)(X). \tag{3.1}$$

Let λ be the cosmological constant, T be the energy-momentum tensor of type (1,1), ρ be the energy density, p be the isotropic pressure and U be the velocity vector field of the fluid, such that $g(U,U) = -1$, that is U is timelike.

Further, let $g(X,U) = A(X)\ \forall X.$

Then the Einstein field equations for the perfect fluid can be expressed as follows [2 (p.336 & 339)]:

$$Q - \frac{r}{2}I + \lambda I = T, \tag{3.2}$$

where we have

$$T = (\rho + p)A \otimes U + pI. \tag{3.3}$$

In other words,

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = (\rho + p)A(X)A(Y) + pg(X,Y). \tag{3.4}$$

Taking a frame field and contracting (3.4) over X and Y we obtain

$$r = \rho - 3p + 4\lambda. \tag{3.5}$$

Hence,

$$X.r = X.\rho - 3(X.p)\ . \tag{3.6}$$

Substituting for Q from (3.2) in (1.5) we get

$$N = T + \frac{r(n-2)}{2(n-1)}I - \lambda I. \tag{3.7}$$

In that case, using (3.1) we obtain

$$(\nabla_X T)(Y) - (\nabla_Y T)(X) = 0 \tag{3.8}$$

since r is constant. From (3.3) we get

$$(\nabla_X T)(Y) = [X.(\rho + p)]A(Y)U + (\rho + p)(\nabla_X A)(Y)U + (\rho + p)(\nabla_X U)A(Y) + (X.p)Y, \quad (3.9)$$

and a similar expression for $(\nabla_Y T)(X)$. Putting $Y = U$ in (3.8) we obtain

$$(\nabla_X T)(U) - (\nabla_U T)(X) = 0. \quad (3.10)$$

Putting $Y = U$ in (3.9) we get in virtue of (3.10)

$$\begin{aligned}(\rho + p)\nabla_X U = -[X.(\rho + p)]U - U(\rho + p)A(X)U + [(X.p)U - (U.p)X] \\ - (\rho + p)[(\nabla_U A)(X)U + (\nabla_U U)A(X)].\end{aligned} \quad (3.11)$$

Using (3.6) and remembering that r is constant we obtain from (3.11)

$$\begin{aligned}(\rho + p)\nabla_X U = -[X.(\rho + p)]U - [U.(\rho + p)]A(X)U + \frac{1}{3}[(X.\rho)U - (U.\rho)X] \\ - (\rho + p)[(\nabla_U A)(X)U + (\nabla_U U)A(X)].\end{aligned} \quad (3.12)$$

Further, we have the energy and force equations [2(p.339)] as follows:

$$g(grad\rho, U) = U.\rho = -(\rho + p)divU \quad \textit{(3.13)}$$

and

$$(\rho + p)(\nabla_U U) = -grad_\perp p = -gradp - [g(gradp, U)U] = -gradp - (U.p)U \quad (3.14)$$

where the spatial pressure gradient $grad_\perp p$ is the component of grad p orthogonal to U. Using (3.14) we can express (3.12) as follows:

$$(\rho + p)(\nabla_X U) = -\frac{2}{3}(X.\rho)U - (U.\rho)A(X)U + (U.p)A(X)U + A(X)gradp - \frac{1}{3}(U.\rho)X. \quad (3.15)$$

Taking the inner product with U we get

$$g[(\rho + p)\nabla_X U, U] = grad\rho + (U.\rho)U. \quad (3.16)$$

Since the left hand side of (3.16) is zero, we obtain

$$grad\rho = -(U.\rho)U. \quad (3.17)$$

It is therefore evident that the velocity U is proportional to a gradient. Hence U is *hypersurface orthogonal* [3]. Using (3.17) we get from (3.15)

$$(\rho + p)\nabla_X U = A(X)[(U.\rho)U + gradp] + \frac{1}{3}(U.\rho)[X + A(X)U]. \quad (3.18)$$

Once again using (3.13) and (3.14) we obtain from (3.18)

$$\nabla_X U = -A(X)\nabla_U U + \frac{1}{3}divU[X + A(X)U]. \quad (3.19)$$

Now, for the vector field U, $\nabla_U U$ is the *acceleration vector* and divU is the *expansion scalar*, both of which may be non-zero [2 (p.340)]. It is also known that [2 (p.95)]

$$(curlU)(X,Y) = g(\nabla_X U, Y) - g(\nabla_Y U, X). \tag{3.20}$$

Let h denote the *projection tensor*, such that $hX = X + A(X)U$ The *vorticity tensor* $\omega(X,Y)$ is the projection of curl of U. From (3.20) we get

$$\omega(X,Y) = g(\nabla_{hX} U, hY) - g(\nabla_{hY} U, hX) = 0 \text{ [by (3.19)]}.$$

Again the *shear tensor* $\sigma(X,Y)$ is given by [4]:

$$\sigma(X,Y) = \frac{1}{2}[g(\nabla_{hX} U, hY) + g(\nabla_{hY} U, hX)] - \frac{1}{3} divUg(hX,Y) = 0 \text{ [by (3.19)]}.$$

From this we can conclude that the space-time under our consideration is both shear-free and irrotational. From (3.17) we obtain $g(grad\rho, X) = g(-U\rho U, X)$ that is

$$X.\rho = -U\rho A(X). \tag{3.21}$$

If X is orthogonal to U, then from (3.21) we shall have

$$X.\rho = 0\ . \tag{3.22}$$

This means that the *energy density* is constant over a spacelike hypersurface orthogonal to the velocity vector U. These results can be stated in the following way:

Theorem 2: *A general relativistic perfect fluid space-time obeying Einstein's equation with cosmological constant and admitting a divergence-free quasi conformal curvature tensor is shear-free, irrotational and its energy density is constant over the spacelike hypersurface orthogonal to the velocity vector field.*

Remarks: The converse of this theorem follows easily from Ray Chaudhuri equation [4] by imposing the condition of shear-free irrotational flow and then using the condition of hypersurface orthogonality.

Acknowledgment. *The author is thankful to Prof. M. C. Chaki, former Sir Asutosh Birth Centenary Professor of Higher Mathematics, University of Calcutta for his guidance and discussions during the work. She is also grateful to Rev. Father P. C. Mathew, Principal, St. Xavier's College, Kolkata for his constant support and encouragement*

REFERENCES

1. M. C. Chaki and M. L. Ghosh, *On quasi conformally flat and quasi conformally conservative Riemannian manifolds*, Anatele Stiint. Ale Universitat. "Al. I. Cuza" Iasi, Tomul XLIII, Mathematica, 1997, 375-381.
2. Barrett O'Neill, Semi-Riemannian Geometry, Academic Press Inc., 1983.
3. D. Kramer, H. Stephani, E. Herlt and M. A. H. MacCallum, Exact solutions of Einstein field equations, Cambridge University Press, 1980, 76-77.
4. K. Ray Chaudhuri, Theoretical Cosmology, Oxford Science Publications, 1979, 80-92.

O PROSTOR-VREMENU, KOJI DOZVOLJAVA KVAZI-KONFORMNI TENZOR KRIVINE, I PREDSTAVLJA IDEALNI FLUID

Sarbari Guha

Pojam kvazi-konformalnog tezora krivina C tipa (1,3) na Riemannian višestrukosti (M*,g) (n>3) su uveli M.C.Chaki i M.L.Ghash [1] u saglasnosti sa:*

$$C^*(X,Y,Z) = aR(X,Y,Z) + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y]$$

gde su gde su a, b konstante, R Reimann-ov tenzor tipa (1,3), S je Ricci-jev tenzor tipa (0,2), Q je Ricci-jev tenzor tipa (1,1), i r je skalar krivine mnogostrukosti.

U ovom radu, četvoro-dimenzionalni prostor-vreme, koji predstavlja idealni fluid, sa Lorentz-ovom metrikom signature (+,+,+, -) i ne nultom skalarnom krivinom, koji dozvoljava kvazikonformni tenzor krivine, je razmotren. Pokazano je daako takav prostor vreme, koji predstavlja fluid, sa jediničnim vremeski sličnim vektorskim poljem brzine zadovoljava Einstein-ovu jednačinu sa kosmološkom konstantom i njegov kvazi-konformni tenzor krivine je bez divergencije, tada je fluid slobodno- smičući, nerotirajući i njegova gustina energije je konstantna nad hiperprostorom ortogonalnim vektorsko polje brzina.