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LETTERS TO EDITOR

A COMMENT ON THE PROBLEM OF ELASTICITY

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Abstract. *It is shown by operational method that the boundary value problem of the theory of elasticity related to stresses, which can be reduced to three strains compatibility equations and to three equilibrium equations, in fact is of sixth order. Hence, one do not have to formulate additional boundary conditions.*

Key words: *theory of elasticity, compatibility equations, operator method.*

1. INTRODUCTION

The problem related to the degree of compatibility conditions of thc theory of elasticity has pure academic interest. However, it still draws attention of many researches, and there is a number of recent publications in this field [1,2]. Some manuals stress out that these conditions are independent [3-5], others say about it's mutual dependence [6-12] (but they often have no considerations concerning which of the compatibility conditions are independent [6,7,9]). At last, some authors do not touch upon this question at all. As concerns paper:, devoted to this problem, some researches propose to keep only three strain compatibility conditions [15-18]. Other authors point out that such way leads to mathematically incorrect results, because a governing system (three equilibrium and three compatibility conditions) is of nine order, whereas the boundary conditions correspond to a six order. Additional boundary conditions are formulated in different ways [17-18]. Finally, some researches consider the elastic problem related to stresses using six compatibility equations (directly [2,11,12] or after some transformations [19, 20]). Then the equilibrium equations are used as additional boundary conditions [11,19-21].

To solve the problem we are going to apply the operator method described in references [22, 23].

2. FORMULATION OF THE PROBLEM

The six compatibility equations related to stresses have the following form [1]:

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$$\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=2\frac{\partial^2\tau_{yz}}{\partial x\partial y},$$
$$\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right)=2\frac{\partial^2\tau_{xy}}{\partial y\partial z},\tag{1}$$
$$\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial z^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial z^2}+\frac{\partial^2 s}{\partial x^2}\right)=2\frac{\partial^2\tau_{xz}}{\partial z\partial x};$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\tau_{zx}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+\frac{\partial\tau_{yz}}{\partial x}\right)=\frac{\partial^2}{\partial y\partial z}\left(\sigma_x-\frac{\nu}{1+\nu}s\right),$$
$$\frac{\partial}{\partial y}\left(\frac{\partial\tau_{xy}}{\partial z}+\frac{\partial\tau_{yz}}{\partial x}+\frac{\partial\tau_{zx}}{\partial y}\right)=\frac{\partial^2}{\partial z\partial x}\left(\sigma_y-\frac{\nu}{1+\nu}s\right),\tag{2}$$
$$\frac{\partial}{\partial z}\left(\frac{\partial\tau_{yz}}{\partial x}+\frac{\partial\tau_{zx}}{\partial y}+\frac{\partial\tau_{xy}}{\partial z}\right)=\frac{\partial^2}{\partial x\partial y}\left(\sigma_x-\frac{\nu}{1+\nu}s\right);$$

where: $s = \sigma_x + \sigma_y + \sigma_z$.

As it has been shown in reference [1], inside of a body, the equations (1), (2) are linearly dependent. This can be demonstrated using three dimensional Fourier transformation in relation to x, y, z.

Observe that the compatibility relations are obtained using some differentiatrons. However, in general this procedure is valid inside of the considered space but not on its boundaries. This observation implies that in general the equation systems (1) and (2) are only equivalent inside of the space. Consider the following example. Let a body be bounded by the planes $z = 0$, h. Inside of the space $0 < z < h$ both equation systems (1) and (2) can be taken as an input one, since they are there equivalent. However, on the boundaries $z = 0$, h the first equation of (1) must be satisfied, i. e.

$$\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)S=2\frac{\partial^2\tau_{yz}}{\partial x\partial y},$$

where: $S = \sigma_x + \sigma_y$.

In other words, it can happen that on the boundaries the equation systems (1) and (2) can be not equivalent, and therefore the boundary surfaces must be considered separately [9].

Let us take as the equations (2) as the input equations, and the following homogeneous equilibrium equations (note, that an account of the mass forces do not change the problem):

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0,$$
$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0,\tag{2}$$
$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0;$$

To wit, each three of independent compatibility relations can be expressed using 17 different methods [3, 18]. Here we focus only on the equations (1) and (3), since the approach used here holds also for other possible variants of equations.

3. RESULTS AND DISCUSSION

Formally, the system of equations (1), (3) has ninth order with regard to the coordinates. It is caused by an artificial increase of the system order introduced by differentiation in the process of their derivation.

However, the partial differential equations (1), (3) have constant coefficients, and one can use the operator method of solution to differential equations with constant coefficients. In accordance with the main idea of this method one can operate with derivatives $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial t$ like with constants and use methods of linear algebra [22, 23].

If one applies the operator method to equations (1) and (3) than the following equation is obtained

$$\mathbf{D}\Phi = XYZ\nabla^2\nabla^2\nabla^2\Phi = 0,$$

where: $\mathbf{D}$ s the determinant of the system (1); $X = \dfrac{\partial}{\partial x}, Y = \dfrac{\partial}{\partial y}, Z = \dfrac{\partial}{\partial z}$;

$\nabla^2 = X^2 + Y^2 + Z^2$; Φ is the potential function.

The stresses are defined by the function Φ using the operator method in the following way:

$$\begin{aligned} &\sigma_x = D_{16}\Phi;\ \sigma_y = D_{26}\Phi;\ \sigma_z = D_{36}\Phi; \\ &\tau_{xy} = D_{46}\Phi;\ \tau_{xz} = D_{56}\Phi;\ \tau_{yz} = D_{66}\Phi, \end{aligned} \tag{5}$$

where: D_{j6} (j = 1 – 6) are the minors of the determinant $\mathbf{D}$, obtained by cancelling the j-th row and 6-th column.

Integrating (4) in relation to x, y, z one gets

$$\nabla^6\Phi = \varphi_1(x, y) + \varphi_2(y, z) + \varphi_3(x, z) \qquad (6),$$

where φ_i are unknown functions.

For an elastic space a condition of the functions $\varphi_i = 0$ holds, since stress derivations decay in infinity. In an arbitrary internal point $A(x_0,y_0,z_0)$ of an elastic body the constants $\varphi_i\,(x_0,y_0,z_0)$ are equal to zero, which can be shown in the following way. Indeed, the compatibility relations must be satisfied in any plane containing the point A. Let us choose three orthogonal planes $x = x_0$, $y = y_0$, $z = z_0$. It is not difficult to check that in those planes the equations related to the potential function F have the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F = 0, \tag{7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)F = 0, \tag{8}$$

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F=0. \tag{9}$$

Observe that a comparison of equations (7)–(9) and equation (5) yields $\varphi_i = 0$ ($i = 1,2,3$).

Therefore, we get the sixth order equation in relation to each of the space variables x, y, z of the form

$$\nabla^2 F = 0. \tag{10}$$

Let us now give a brief comment on the boundary conditions. Knowing the function F which is a solution to the equation (10), one can define the stresse: due to the formula (5). Therefore, the formulated boundary value problem with canonical boundary conditions related to stresses can be solved. However, a boundary value problem related to displacements needs a separate consideration.

REFERENCES

1. Borodachev N. M., *About one approach in the solution of 3D problem of the elasticity in stresses*. International Journal of Applied Mechanics, 31(12), 1995, 38-44.
2. Vasil'yev V. V. and Fedorov L. V., *To the problem of the theory of elasticity, formulated in stresses*. Mechanics of Solids, 31(2), 1996, 82-92.
3. Ostrosablin N. I., *Compatibility equations of small deformations and stress functions*. J. Appl Mech Tech Phys, 38(5), 1997, 774-783.
4. Nowacki W., *Theory of Elasticity*, Mir, Moscow, 1975 (in Russian).
5. Germain P., *Mecanique des Mileux Cintinus Massonerice*, Editeurs, Paris, 1962.
6. Amenzade Yu. A., *Theory of Elasticity*, Mir, Moscow, 1979.
7. Donnel L. H., *Beams, Plates and Shells*, McGraw-Hill, New York, 1976.
8. Christensen R. M., *Mechanics of Composite Materials*, John Willey and Sons, New York, 1979.
9. Hahn H. G., *Elastizita Theorie Grundlagen der Linearen Theorie und Anwendungen auf undimensionale, ebene und zaumliche Probleme*, B. G. Teubner, Stuttgart, 1985.
10. Rabotnov Yu. N., *Mechanics of Deformable Rigid Body*, Nauka, Moscow, 1979 (in Russian).
11. Zubchaninov V. P., *Foundation of the Theory of Elasticity and Plasticity*, Vishaya Shcola, Moscow, 1990 (in Russian).
12. Trefftz E., *Mechanik der elastischen Korper*. Handbuch der Physik, bd. VI, Berlin, 1928.
13. Krutkov Yu. A., *Tensor of the Function of Stresses and General Solutions in the Statical Theory of Elasticity*, AN USSR, Moscow-Leningrad, 1949 (in Russian).
14. Timoshenko S. P. and Goodier J. N., *Theory of Elasticity*, McGraw-Hill, New York, 1954.
15. Terebushko O. I., *Foundations of the Theory of Elasticity and Plasticity*, Nauka, Moscow, 1984 (in Russian).
16. Lihachev V. A. and Fleyshman N. P., *Beltrami-Michell equations of compatibility*, Doklady AN USSR, A(9), 1984, 45-47 (in Russian).
17. Belov P. A., Yelpat'yevskiy A. N. and Lur'ye S. A., *To general solution of the theory of elasticity in stresses*, Strength of Constructions, 4, 1980, 159163 (in Russian).
18. Malyi V. I., *Independent conditions of stress compatibility for elastic isotropic body*, Doklady AN USSR, A(7), 1987, 43-46 (in Russian).
19. Kozak J., *Notes on the field equations with stresses and on the boundary conditions in the linearized theory of elastostatics*, Acta Technica Acad. Sci. Hung., 90(3-4), 1980, 221-245.
20. Poberdya B. Ye., Sheshenin S. V. and Holmatov T., *Problem in Stresses*, Tashkent, Fan, 1988 (in Russian).
21. Poberdya B. E., *Numerical Methods in the Theory of Elasticity and Plasticity*, MGU, Moscow, 1981 (in Russian).
22. Gol'denviezer A. L., *Theory of Elastic Thin Shells*, Pergamon Press, New York, 1961.
23. Levi E. E., *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, Rendiconti del Circolo Matematico di Palermo, 24, 1907, 275-317.