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ON THE RELIABILITY OF THE NEW FINITE ELEMENT HC8/27

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Abstract. *It is now generally recognized that reliable finite element procedure, must fulfill specific mathematical convergence requirements, as consistency and stability conditions, before it is recommended for general use in solid and fluid mechanics. In the present paper it will be shown that three-dimensional finite element HC8/27, based on the primal-mixed formulation in linear elasticity, is consistent and stable. In addition, numerical evaluation of the inf-sup condition for the full three-dimensional primal-mixed scheme in elasticity will be presented. It should be noted that present finite element is stable without use of any stabilization technique.*

Key words: *Finite element method, Reliability*

1. INTRODUCTION

The main motive of the present investigation is the research and developing of the finite element approach for which common problems connected to the primal finite element methods in elasticity will not occur. These problems may be divided in three main groups. The first group of problems is connected to shear locking when low order elements are used. The second group of problems is the presence of spurious or kinematic modes, that is extra zeroes eigenvalues of system matrix, when selectively reduced integration is used. The third group of problems is the occurring of the nearly singular system matrix when the limit of incompressibility is approached. The popular *standard displacement based finite element approach*, as representative of the class of primal approaches, in its *raw* form is not applicable in these situations. In spite of great number of techniques to remedy these bad characteristics, they are all on the account of violation of the consistency and stability issues. For example, very popular general shell element QUAD is not stable in the analysis of banding dominated problems [1,2], regardless of number of local nodes per element.

On the other hand, from the 1960's when term *mixed method* was first used, to describe finite element methods in which two or more variables of interest are treated as fundamental variables, number of reasons have been offer to prefer their use in numerical

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simulation rather than primal ones [1,10]. The mixed finite element schemes are now abundantly used for the analysis of fluid flows, almost incompressible and incompressible materials, plates and shells with plane finite elements.

Nevertheless, the overall stability of mixed approaches is difficult to achieve [9-11]. It is from the reason that in the linear elasticity problems one has to deal with two fundamental variables, that is stresses and displacements. In that case, the first stability condition requires a large displacement approximation space in accordance to stress approximation space. On the other hand, second stability condition requires a large stress approximation space in accordance to displacement. It is clear that first and second stability conditions are in contradiction. Therefore, to avoid instabilities the well balance between the approximation spaces of these two fundamental variables must be accomplished.

The application of primal-mixed approach in elasticity did not encountered much attention until results presented in papers [4-6], where it has been shown that it has many advantages in accordance to displacement based finite element approach. The main reason for that situation results from fact that stability, which governs the convergence and the rate at which approximate solution will converge to the solution of the governing mathematical model, is no obvious as in the case of primal based finite element methods. Fortunately, in the case of the present scheme the first stability conditions is *a priory* satisfied. Consequently, with the careful choice of the stress approximation space over the displacement, the second stability condition will be satisfied [11], also.

Only recently, an extension of the present formulation to the three-dimensional settings was made [7,8], where mathematical model starts from the fully three-dimensional equilibrium equations and linear tri-dimensional strain-displacement and full stress-strain relationships, without any simplifications, correcting terms, tricks or tune-ups. It has been shown that the lowest-order finite element HC8/9 of that scheme is consistent, solvable and robust. Further it has been shown that it satisfies ellipticity condition, known also as ellipticity on the kernel condition – the first stability condition, since the bilinear form connected to the stress space is coercive (see Chapter 4.2). Moreover, it has been shown that it can be used in the analysis of regular model problems of arbitrary geometry, as well as in analysis of compressible and almost incompressible materials. Under regular model problems we consider the model problem subjected to the smooth force and displacement boundary conditions. However, it has been numerically proven that it is not stable, i.e. it does not pass numerical inf-sup test. That results was for expected, since its two-dimensional counterpart does not pass that test also, although it was numerically proven that it is very efficient [6,7].

In the present paper it will be shown that is possible to construct reliable [1] finite element method in linear elasticity that is consistent and automatically satisfies the first and second stability condition, and in addition, at the same time has C^0 continuous stresses.

More clearly, this paper is an answer to the paper of Brezzi *et* al. [9] where several observations concerning mixed finite element schemes are presented. Firstly, "...for linear elasticity problems such a construction (satisfying the first and second stability condition) is yet unachieved and looks rather difficult". Secondly, there is a remark about mixed finite element in elasticity where both variables of interest (stresses and displacements) are continuous "…It is not known if this element is stable". And finally, "…The use of C^0

discretization for the stress filed should be avoided. The main reason for this is the difficulty in the numerical solution of the linear system of equations."

In the present case of finite element HC8/27, a well-balanced continuous stress and displacement interpolation functions are used, assuring that second stability condition is satisfied for model problem with arbitrary model geometry and value of Poisson's ratio.

The present finite element scheme is based on so-called primal-mixed formulation in elasticity that is fully investigated in two-dimensional case [4-7], where all relevant issues as reliability, efficiency and applicability are examined in detail. It was numerically proven that primal-mixed finite element QC4/9 is reliable and more efficient than corresponding finite element Q4 of usually used displacement finite element scheme, in spite of additional three equations per global node connected to the stress components.

The one of main advantage of the present finite element in accordance to displacement-based finite element is calculation of stresses in the processing part of the finite element procedure, so typical procedures as stress recovering and stress smoothing techniques [14] are no needed anymore.

2. Present Formulation

We start from the weak primal-mixed formulation in elasticity [4] which seeks $\{\boldsymbol{u},\boldsymbol{T}\}\in H^1(\Omega)^n \times H^1(\Omega)^{n\times n}_{\text{sym}}$ satisfying $\boldsymbol{u}\big|_{\partial\Omega_u}=\boldsymbol{w}$ and $\boldsymbol{T}\cdot\boldsymbol{n}\big|_{\partial\Omega_t}=\boldsymbol{p}$ such that:

$$\int_\Omega(\boldsymbol{AT}:\boldsymbol{S}-\boldsymbol{S}:\nabla\boldsymbol{u}-\nabla\boldsymbol{v}:\boldsymbol{T})d\Omega=-\int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}d\Omega-\int_{\partial\Omega_t}\boldsymbol{v}\cdot\boldsymbol{p}d\partial\Omega-\sum_{\boldsymbol{e}}\boldsymbol{v}\cdot\boldsymbol{F}\,, \tag{1}$$

for all $\{\boldsymbol{v},\boldsymbol{S}\}\in H^1(\Omega)^n\times H^1(\Omega)^{n\times n}_{\text{sym}}$ satisfying $\boldsymbol{v}\big|_{\partial\Omega_u}=0$ and $\boldsymbol{S}\cdot\boldsymbol{n}\big|_{\partial\Omega_t}=\boldsymbol{0}$.

In this expression $\boldsymbol{u}$ is displacement field, $\boldsymbol{T}$ is the stress field, $\boldsymbol{f}$ is the vector of body forces and $\boldsymbol{p}$ is the vector boundary tractions. $\boldsymbol{F}$ is the concentrated force not considered later in this text. Further, $\boldsymbol{A}=\boldsymbol{K}^{-1}$ is the elastic compliance tensor, while $\boldsymbol{v}$ and $\boldsymbol{S}$ are the displacement and stress weight functions, respectively. Space $H^1(\Omega)^{n\times n}_{sym}$ is the space of all symmetric tensorfields that have square integrable gradient, while space $H^1(\Omega)^n$ is the space of all vectorfields that are square integrable and have square integrable gradients, where n is the number of spatial dimensions of the problem under consideration. It should be noted that trial and test displacement space functions are from the same space $H^1(\Omega)^n$ as in the case of popular displacement based finite element approach. Further, the stress boundary conditions are introduced as essential, although not necessary from the point of view of underlying variational principle.

In the case of the present approach, the space Π of unknown functions decomposes as the product of two spaces $\Pi = T\times U$, where space T is the space of the stress functions and U is the space of displacement space functions. So, the present weak formulation (1) can be written in the form:

$$\text{Find } \boldsymbol{u}\in U \text{ and } \boldsymbol{T}\in T \text{ such that}$$
$$B((\boldsymbol{S},\boldsymbol{u}),(\boldsymbol{T},\boldsymbol{v}))=F(v) \tag{2}$$
$$\text{for all } \boldsymbol{v}\in V \text{ and } \boldsymbol{S}\in S\,,$$

where B has the special form:

$$B((\boldsymbol{S},\boldsymbol{u}),(\boldsymbol{T},\boldsymbol{v})) = a(\boldsymbol{S},\boldsymbol{T}) + b(\boldsymbol{S},\boldsymbol{u}) + b(\boldsymbol{T},\mathbf{v}) \,. \tag{3}$$

Let's explain the basic properties of the form a in Eq. (3). For the present linear elasticity problem $\boldsymbol{T}, \boldsymbol{S} \in H^1(\Omega)^{n\times n}_{sym}$, forma a has the physical sense of deformation energy which is positive definite:

$$a(\boldsymbol{T},\boldsymbol{S}) = \int_\Omega \boldsymbol{AT} : \boldsymbol{S} d\Omega \geq \alpha \|\boldsymbol{T}\|^2 \,,\ \alpha \geq 0 \,. \tag{6}$$

Consequently, the form $a : T \times T \to R$ is bilinear *coercive* symmetric quadratic form, resulting with the fact that present scheme satisfies the first Brezzi condition, also known as ellipticity condition (see Chapter 4.2).

Further, form $b : T \times U \to R$ is nonsymmetrical bilinear form, while the form $F : U \times U \to R$ is the linear form [15], where the space R is the space of real numbers.

After discretization of the starting problem (1) by finite element method, it has been shown in [4] that present scheme can be written as the system of linear equations of order $n = n_u + n_t$, where n_u is the number of displacement degrees of freedom, while n_u is the number of stress degrees of freedom:

$$\begin{bmatrix} \mathbf{A_{vv}} & -\mathbf{D_{vv}} \\ -\mathbf{D_{vv}}^{\mathbf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t_v} \\ \mathbf{u_v} \end{bmatrix} = \begin{bmatrix} -\mathbf{A_{vp}} & \mathbf{D_{vp}} \\ \mathbf{D_{pv}}^{\mathbf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t_p} \\ \mathbf{u_p} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{f_p} + \mathbf{p_p} \end{bmatrix} . \tag{4}$$

In this expression, unknown (variable) and known (prescribed) values of the stresses and displacements, denoted by the indices v and p respectively, are decomposed. The nodal stresses t^{Lst} and displacements u_{Kq} components are consecutively ordered in the column matrices $\mathbf{t}$ and $\mathbf{u}$ respectively. The members of the matrices $\mathbf{A}$ and $\mathbf{D}$, of the column matrices $\mathbf{f}$ and $\mathbf{p}$ (discretized body and surface forces) in (4), are respectively:

$$A_{\Lambda uv\Gamma st} = \sum_e \int_{\Omega_i} \Omega^N_\Lambda S_N g^a_{(\Lambda)u} g^b_{(\Lambda)v} A_{abcd}\, g^c_{(\Gamma)s} g^d_{(\Gamma)t} T_L \Omega^L_\Gamma d\Omega \,; \tag{5}$$

$$D^{\Gamma q}_{\Lambda uv} = \sum_e \int_{\Omega_i} \Omega^N_\Lambda S_N U^K_a \Omega^\Gamma_K g^a_{(\Lambda)u}\ g^{(\Gamma)q}_{(\Lambda)v} d\Omega \,, \tag{6}$$

$$f^{\Lambda q} = \sum_e \int_{\Omega_i} g^{(\Lambda)q}_a \Omega^\Lambda_M V^M f^a d\Omega, \quad p^{\Lambda q} = \sum_e \int_{\partial\Omega_{it}} g^{(\Lambda)q}_a \Omega^\Lambda_M V^M p^a d\partial\Omega \tag{7}$$

In these expressions, Ω^N_Λ is the incidence matrix that maps global nodes into the local nodes of an element, S_N, T_L are the stress approximation functions, while V^M,U^K are the displacement approximation functions. The term A_{abcd} is the fourth order elastic compliance tensor in its covariant form. Further, in (4) $g^a_{(\Lambda)u}$ and $g^{(\Gamma)q}_{(\Lambda)v}$ are Euclidian's shifters. In addition, the Einstein convention on the summation of repeated indices is used. By the letters N and M the local stress nodes are denoted, ranging in the present case from {8,27} in each finite element. Further, letter K stands for 8 displacement node per each element, in the present case.

From the expression (4) it can be seen that present system matrices is symmetric and indefinite. Further, it has positive, zero and negative eigenvalues due to the fact that the saddle point problem is discretized. It is different from displacement based finite element method where solution is sought as an extreme point, where resulting system matrix will have only zero and positive eigenvalues. As it is already mentioned, in the present case discretization of stresses and displacements has to be made in the compatible way in order to avoid instabilities.

3. PRESENT FINITE ELEMENT HC8/27.

In the recent years, considerable attention has been devoted to the development of reliable finite elements. In the present case the reliability of the finite element HC8/27 shown in Fig.1., is examined in detail. The basic properties of the lower order finite elements HC8/8 and HC8/9, of the present formulation, can be found in [8]. In the present notation, letter H stands for element hexahedral geometry, while letter C indicates continuous interpolation of displacement and stress fields. These letters are followed by number of displacement and stress local nodes per element, respectively. Circles depict displacement nodes, while stress nodes are represented by tetrahedrons. The present finite element HC8/27 has 3 degrees of freedom per displacement and 6 degrees of freedom per each stress node.

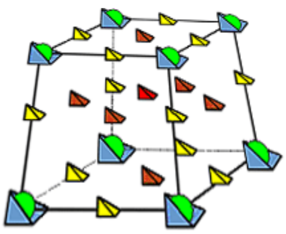

Fig. 1. The finite element HC8/27.

Both unknown fields are in corner nodes approximated by the tri-linear interpolation functions $P_1 - P_8$ given by Eq. (7).

In addition, stress field is enriched by $P_9 - P_{27}$ quadratic hierarchic shape functions connected to the additional 19 hierarchic nodes, from which twelve are connected to midside nodes at the element edges, next six are connected to midside nodes of element sides and one to, so-called, *bubble* node placed at the element center.

$$\begin{aligned}
&P_1(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1-\xi^1)(1-\xi^2)(1-\xi^3);\ P_2(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1+\xi^1)(1-\xi^2)(1-\xi^3)\\
&P_3(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1+\xi^1)(1+\xi^2)(1-\xi^3);\ P_4(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1-\xi^1)(1+\xi^2)(1-\xi^3)\\
&P_5(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1-\xi^1)(1-\xi^2)(1+\xi^3);\ P_6(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1+\xi^1)(1-\xi^2)(1+\xi^3)\\
&P_7(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1+\xi^1)(1+\xi^2)(1+\xi^3);\ P_8(\xi^1,\xi^2,\xi^3)=\frac{1}{8}(1-\xi^1)(1+\xi^2)(1+\xi^3)
\end{aligned} \tag{8}$$

$$P_9=\frac{1}{4}(1-(\xi^1)^2)(1-\xi^2)(1-\xi^3) \qquad P_{13}=\frac{1}{4}(1-\xi^1)(1-\xi^2)(1-(\xi^3)^2)$$
$$P_{10}=\frac{1}{4}(1+\xi^1)(1-(\xi^2)^2)(1-\xi^3) \qquad P_{14}=\frac{1}{4}(1+\xi^1)(1-\xi^2)(1-(\xi^3)^2)$$
$$P_{11}=\frac{1}{4}(1-(\xi^1)^2)(1+\xi^2)(1-\xi^3) \qquad P_{15}=\frac{1}{4}(1+\xi^1)(1+\xi^2)(1-(\xi^3)^2)$$
$$P_{12}=\frac{1}{4}(1-\xi^1)(1-(\xi^2)^2)(1-\xi^3) \qquad P_{16}=\frac{1}{4}(1-\xi^1)(1+\xi^2)(1-(\xi^3)^2)$$
$$P_{17}=\frac{1}{4}(1-(\xi^1)^2)(1-\xi^2)(1+\xi^3) \qquad P_{21}=\frac{1}{2}(1-(\xi^1)^2)(1-(\xi^2)^2)(1-\xi^3)$$
$$P_{18}=\frac{1}{4}(1+\xi^1)(1-(\xi^2)^2)(1+\xi^3) \qquad P_{22}=\frac{1}{2}(1-(\xi^1)^2)(1-(\xi^2)^2)(1+\xi^3)$$
$$P_{19}=\frac{1}{4}(1-(\xi^1)^2)(1+\xi^2)(1+\xi^3) \qquad P_{23}=\frac{1}{2}(1-\xi^1)(1-(\xi^2)^2)(1-(\xi^3)^2)$$
$$P_{20}=\frac{1}{4}(1-\xi^1)(1-(\xi^2)^2)(1+\xi^3) \qquad P_{24}=\frac{1}{2}(1+\xi^1)(1-(\xi^2)^2)(1-(\xi^3)^2) \tag{9}$$

$$P_{25}=\frac{1}{2}(1-(\xi^1)^2)(1-\xi^2)(1-(\xi^3)^2)$$
$$P_{26}=\frac{1}{2}(1-(\xi^1)^2)(1+\xi^2)(1-(\xi^3)^2) \tag{10}$$
$$P_{27}=(1-(\xi^1)^2)(1-(\xi^2)^2)(1-(\xi^3)^2)$$

The numerical integration is performed by 3×3×3 numerical Gaussian integration. It should be noted that, besides additional degrees of freedom connected to the stress global nodes, in the two-dimensional case it was shown that this formualtion has better aspect of the time needed for prescribed accuracy than displacement finite element method [6,7].

4. THE MATHEMATICAL CONVERGENCE REQUIREMENTS

As the finite element mesh is refined, the solution of that discrete problem should approach to the analytical solution of the mathematical model, i.e. to converge. The convergence requirements for shape functions of isoparametric element can be grouped into three categories, that is: completeness, compatibility and stability. Consequently, we may say that consistency and stability imply convergence.

Completeness criterion requires that elements must have enough approximation power to capture the analytical solution in the limit of a mesh refinement process. Therefore, the approximation functions must be of certain polynomial order that ensures that all integrals in the corresponding weak formulation are finite. Further, *compatibility* requirement demands that shape functions provide displacement continuity between elements. On that manner, it will be provided that no artificial material gaps will appear during the deformation of finite element mesh. As the mesh is refined, such gaps could multiply and may absorb or release spurious energy.

Completeness and compatibility are two aspects of the so-called *consistency* condition between the discrete and mathematical models. A finite element model that passes both completeness and continuity requirements is called *consistent.*

Further, if the method is *stable* the non-physical zero-energy modes (kinematic modes) in finite element model problem will be prevented. The kinematic modes are name for extra zeroes eingenvalues of the corresponding finite element system matrix. The finite element is stable if it satisfies two necessary conditions i.e., the first condition represented by the *ellipticity on the kernel* condition and second condition represented by the *inf-sup* condition [1].

It should be noted that satisfaction of the completeness criterion is necessary for the convergence, while violating others two criteria not necessary means that solution will not converge. However, if the method is not stable, at least what could happen is that approximate solution will not converge to the analytical solution at the same rate as the best approximation error [10]. That is, in the best scenario the approximate solution will slowly converge to the analytical solution of the mathematical model. The worse scenario will be the getting away from analytical solution, as the mesh is refined, as shown in numerical example from Chapter 6. With that in connection, interesting reader my notice that reasons from illusory high accuracy of the rough meshes in the displacement base finite element approach, are investigated in detail in [14]. Finally, in the worst case the system matrix will be singular, thus without solution of the given model problem at all.

4.1 *Consistency condition*

In the present case, the *completeness* requirement is satisfied as local approximation finite element functions posses all polynomial terms of degree $k = 3$, needed for the present three-dimensional case. Further, both finite approximation subspaces of stress and displacements are from space H^1 over the domain of the model problem. Consequently, it is provided that these spaces are continuous over the interelement a boundary, resulting that compatibility requirement is satisfied, also.

4.2 *First stability condition*

The *ellipticity on the kernel* condition [9] is given by:

$$a(\boldsymbol{z},\boldsymbol{z}) \leq \alpha_h \parallel \boldsymbol{z} \parallel \text{ for all } \boldsymbol{z} \in Z_h \,,\; Z_h = \{\boldsymbol{z} \in S_h \big|\; b(\boldsymbol{z},\boldsymbol{v}) = 0 \text{ far all } \boldsymbol{v} \in V_h\} \qquad (11)$$

Spaces S_h and V_h are finite dimensional subspaces of test stress spaces $\boldsymbol{S}$ and displacement spaces $\boldsymbol{v}$, respectively.

In the present case, the test and trial stress local functions are from spaces $S_h \subset (H^1)^{n \times n}$. Therefore, the corresponding bilinear form a in (3) is quadratic. In addition, as it is already mentioned in Chapter 2, in the physical sense it represents the deformation energy, which is for linear elasticity problems always positive definite. Consequently, it is symmetric and bounded, also. From all this properties of the present bilinear form a, we can conclude that first stability condition is automatically satisfied.

4.3 *Second stability condition*

The second condition for stability is satisfied if for the meshes of increasing density, value γ_h, following from LBB (Ladyzhenskaya, Babuška, Brezzi) condition and defined in e.g. [12], p.76, Eq.(3.22), remains bounded above zero:

$$\gamma \le \gamma_h = \inf_{\boldsymbol{v} \in V_h} \sup_{\boldsymbol{S} \in S_h} \frac{b(\boldsymbol{S}, \boldsymbol{v})}{\| \boldsymbol{S} \| \| \boldsymbol{v} \|} \tag{12}$$

For the present choice of approximation spaces V_h and S_h , the upper condition can be read as:

$$\gamma \le \gamma_h = \inf_{\boldsymbol{v}_h \in H^{1(n)}} \sup_{\boldsymbol{S}_h \in H^{1(n \times n)}} \frac{b(\boldsymbol{S}_h, \boldsymbol{v}_h)}{\| \boldsymbol{S}_h \| \| \boldsymbol{v}_h \|}, \tag{13}$$

where in the present case:

$$b(\boldsymbol{T}_h, \boldsymbol{u}_h) = \sum_e \int_{\Omega_i} \boldsymbol{T}_h : \nabla \boldsymbol{u}_h d\Omega_e, \| \boldsymbol{T}_h \|^2 = \sum_e \int_{\Omega_i} \boldsymbol{T}_h : \boldsymbol{T}_h d\Omega, \ \| \boldsymbol{u}_h \|^2 = \sum_e \int_{\Omega_i} \nabla u_h^{\mathbf{T}} : \nabla \boldsymbol{u}_h d\Omega. \tag{14}$$

In addition, condition (12) ensures solvability and optimality of the finite element solution [1]. It is interesting to note that any loading does not enter the test [1].

It should be noted that discrete LBB condition in the present case is equivalently stated [11] that for each $\boldsymbol{v} \in V_h$ there is an i such that:

$$\nabla \boldsymbol{v}_h \in (S_h)_i \ . \tag{15}$$

Because verification of condition like (12) involves an infinite number of meshes, a numerical inf-sup test should be performed for a sequence of several meshes of increasing refinement [1]. Consequently, in the present case, numerical *inf-sup* test in matrix notations is stated as the generalized eigenvalue problem given by:

$$\mathbf{D}_h^{\mathrm{T}} \mathbf{A}_h^{-1} \mathbf{D}_h \, \boldsymbol{x} = \lambda \, \mathbf{K}_h \, \boldsymbol{x} \, , \tag{16}$$

where **D** and **A** are matrix entries in (4), matrix **K** is the stiffness matrix from the relating displacement finite element method. The square root of the smallest eigenvalue of the above problem $\sqrt{\lambda_{\min}}$ is equal to the inf-sup value γ_h in Eq.(12). The test involves [2] the determination of γ_h for several meshes with increasing refinement with the mesh density indicator $1/N$, via calculation of $\lambda_{\min}$.

If the inf-sup values, for chosen sequence of finite element meshes, do not show decrease toward zero, meaning that the $\lambda_{\min}$ values stabilize at some positive level, then it can be considered that inf-sup test is passed. It should be noted that decreasing of the inf-sup values on log-log diagram would be seen as curve with moderate or excessive slope. This approach is already used in [7,8] for the testing of stability of lower order finite element HC8/9 of the present scheme.

5. INF-SUP TEST NUMERICAL RESULTS

In this section the results of the numerical inf-sup test for the present finite element HC8/27 with respect to the finite element HC8/9 for which the stresses approximation space is less fine [10,13], are reported. The results are presented in the log-log diagram, where on the horizontal axe the mesh factor N given as $1/N$ is presented. Mesh factor in the present case represents the uniform subdivision of the starting one-element model, along its axes. On the diagram's vertical axe the smallest eigenvalues of the corresponding generalized eigenvalues problem (16) are given. In addition, all element matrices in (16) are evaluated using full numerical integration.

The simple unit square block, shown in Figure 2, loaded by uni-axial force is analyzed. The largest circles in Figure 2 depict suppressed displacement degree of freedom in x-direction, while smaller and smallest circles depict suppressed displacements in y and z directions, respectively. Therefore, only some displacement degrees of freedom are constrained to allow the present tension test in x-direction. The model problem is gradually refined using meshes with mesh density indicator $N = 1,2,3,4$.

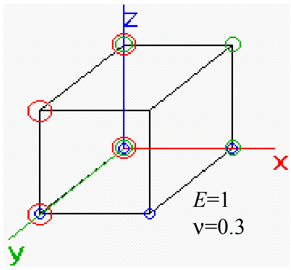

Fig. 2. The unit brick model problem.

The first model problem has only displacement degrees prescribed as essential. The maximal eigenvalues of generalized eigenvalue problem (16) are equal to unity, for all considered finite element meshes. Minimal eigenvalues are shown in the Figure 3. Although, in the present case only the sequence of three finite element meshes are considered, the instability of finite element HC8/9 is obvious, while stability of finite element HC8/27 is numerically proven.

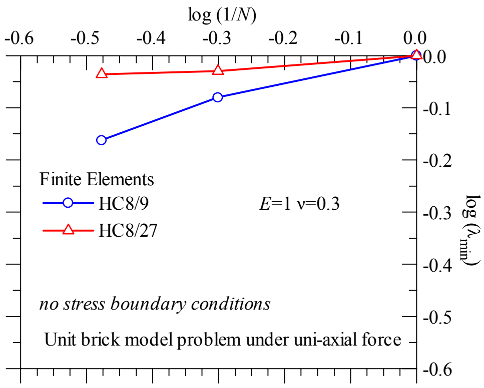

Fig. 3. Inf-sup results.

In the next test, the stress boundary conditions are introduced as essential, which means that on the free physical boundaries of the test model problem normal and shear stress components are set to zero. Consequently, the number of stress degrees of freedom is reduced, resulting with smaller system matrix and therefore, smaller matrices **A** and **D** in (16). It is interesting to note that introduction of essential stress boundary condition always improve the results. From the obtained inf-sup results shown in the Table 1 and Figure 4, it can be concluded that finite element HC8/27 is obviously stable.

Table 1. Finite element HC8/27 Inf-sup test results.

Finite element HC8/27 The minimal eigenvalues of the problem (16)				
N	1	2	3	4
λ_{min}	0.52521005	0.45950700	0.47725788	0.46886696

In the Figure 5, the inf-sup test results for the present finite element and different gradually increased values of Poisson's ratio toward incompressibility, are shown. We may see that this element is stable even for the almost incompressibility case. Consequently, it can be said that finite element HC8/27 is robust.

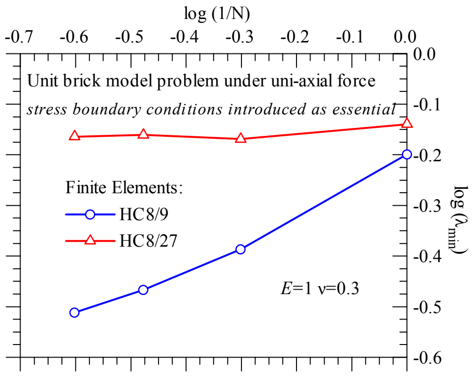

Fig. 4. Inf-sup results. Stress boundary condition introduced as essential.

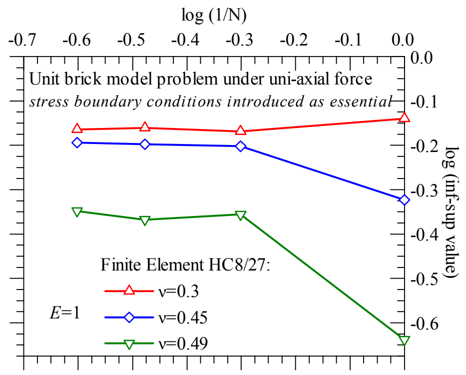

Fig. 5. Inf-sup results. Increasing values of Poisson ratio.

It is interesting to see that if the L^2 norms for evaluation of the test (16), for which any material characteristics do not enter the test, as suggested in [1], the similar eigenvalues were obtained, see Figure 6. It should be noted that in the present paper L^2 norms were obtained using $E = 1$ and $\nu = 0$ for calculating the coefficients of matrix $\mathbf{A}$ and $\mathbf{K}$ [6].

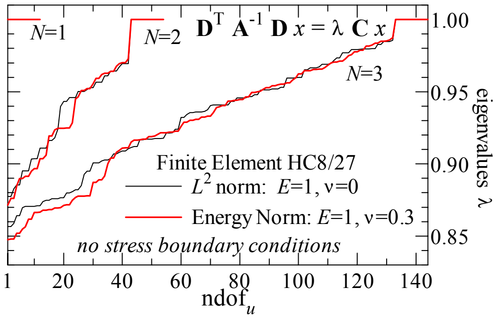

Fig. 6. Eigenvalues of problem (16) when L^2 and Energy norm are introduced.

It should be noted that when stress approximation spaces are such that resulting finite elements are not stable, like in the case of element HC8/9, the occurring of the kinematic modes (zero eigenvalues of the corresponding generalized eigenvalue problem (16)) in the discrete inf-sup test can be prevented by not employing stress boundary conditions at all, or applying these only on the part of the boundary where surface forces are introduced. In addition, the numerical inf-sup test results for the plate bending analysis and hyperbolic paraboloid show that present finite element is stable also, which is not presented here.

CONCLUSION

In the present paper the reliability, that is consistency and stability, of the new primal-mixed three-dimensional finite element HC8/27, so-called Taylor-Hood finite element in elasticity, where displacements and stresses shape functions are *a priori* continuous fundamental variables, is analyzed in detail. It has been shown that present finite element method is consistent, which means that is satisfies completeness and compatibility criteria. In addition it has been shown that it satisfies *a priori* ellipticity on the kernel conditions, widely known as first stability conditions. Further, it has been shown that with the careful choice of stress approximation finite element sub-spaces, the second stability condition, known as inf-sup test, is satisfied also, providing that approximate solution will converge to the analytical solution at the same rate as best approximation error. The numerical experiments performed on standard benchmark examples not presented here show the superior behavior of the present method, as well.

The present finite element, as a first mixed finite element in elasticity with such a spectrum of good properties, can be recommended for general use in 3d elasticity. Consequently, the usual big library of case-dependent finite elements, such as families of one-dimensional, plane and solid elements, could be replaced with the present finite element. Naturally, since the present approach involves much more degrees of freedom than primal approaches, special attention must be paid on the development on the most effective solution strategy, which is subject of the future investigation.

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O POUZDANOSTI KONAČNOG ELEMENTA HC8/27

Dubravka Mijuca

U radu je pokazano da je konačni element HC8/27 primalno-mešovite formulacije u punoj trodimenzionoj elastičnosti pouzdan, što podrazumenva da je konzistentan i stabilan. Kao takav, može da se preporuči kao univerzalan konačni element element u mehanici čvrstog tela za analizu problema proizvoljne geometrije, uslova oslanjanja, zadatih opterećenja i vrednosti koeficijenta smicanja ν. *Na taj način se uobičajene biblioteke elemenata, bazirane na familijama jednodimenzionih, ravnih i trodimenzionih elemenata, zamenjuju samo jednim trodimenzionim elementom, kome korisnik može da bira broj lokalnih čvorova po elementu za aproksmaciju napona, a u zavisnosti od toga da li je polazni fizički problem problem regularan ili ne.*