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RELATIVE MOTION OF SYSTEMS IN A PARAMETRIC FORMULATION OF MECHANICS

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Abstract. *A parametric formulation of mechanics, formulated by the author himself, is based on the separation of the double role of time by the aid of a family of varied paths, so that the time as independent variable remains unchanged, while it as a parameter is transformed into a new parameter, which depends on a chosen path and is taken as an additional generalized coordinate.*
In this paper this parametric formulation of mechanics has been extended to the relative motion of arbitrary rheonomic systems. In this way, the corresponding Lagrangian and Hamiltonian equations, as well as the energy change law for such systems have been formulated and analised, and the obtained results are illustrated by a simple example.

1. INTRODUCTION

Recently, in the study of the rheonomic systems, V. Vujičić [1-3] gave a modification of the analytical mechanics of such systems, with the aim to include the influence of nonstationary constraints to the laws of motion. Supposing that these constraints always can be written in the form $F[\vec{r}_\nu, f(t)] = 0$, he introduced this function $f(t)$ as an additional generalized coordinate $q^0 = f(t)$, and on this basis formulated an extended system of the Lagrangian and Hamiltonian equations, with the additional equations corresponding to q^0. As a consequence of these equations, the energy conservation law for such systems is obtained in the form $\mathcal{E} = T + U + P = \text{const}$, where P, so-called rheonomic potential, arose from the nonstationary constraints.

A different approach to this problem was given by the author himself (Đ. Mušicki [4-5]) in the form of a parametric formulation of mechanics. It is based on the separation of the double role of time (independent variable and a parameter), using a family of varied paths, and on the transition to a new parameter, which depends on the chosen parth, and which is taken as an additional generalized coordinate. In this way, the main general principles of mechanics and the energy relations were formulated, and the obtained results are in accordance with the corresponding ones of Vujičić, because of the formal

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similarity of the roles of time and the introduced parameter, but with other approach and different interpretation.

All these results refer to the absolute motion of the rheonomic systems, but it is known that the Lagrangian and Hamiltonian formalism are also applicable to the relative motion of these systems. In this case, the Lagrangian equations can be used either with respect to an absolute frame of reference, or directly with respect to a moving nonintertial frame of reference, when all the quantities will be relative. This is demonstrated, for example, in the famous textbooks on the theoretical physics by Landau and Lifschitz [6] and on the theoretical mechanics by P. Appell [7], where it is shown how this method can be applied to obtain the fundamental equation of the relative motion in the vector form, and to study the relative motion in both ways through several examples. The extension of this application of the Lagrange's method to a system of particles in the relative motion was given by L. Pars [8], who formulated the Lagrangian of such systems in the general form as a function of the relative generalized coordinates and the relative velocities, but without analysing the corresponding Lagrangian equations.

This analysis was carried out by A. Lur'je [9], who studied these Lagrangian equations in details, and formulated the corresponding Hamiltonian equations on the basis of the associated canonical transformations. In this analysis two types of these Hamiltonian equations are introduced, according to whether the terms which represent the transport and Coriolis's force were pointed out explicitly or not. Recently, M. Lukačević and O. Jeremić [10] obtained the Hamiltonian equations for the scleronomic systems directly, starting from the Pars's Lagrangian of the system of particles in its relative motion, and eliminating certain formalities in the Lur'je's work. Namely, instead of his formal definitions of two types of generalized momenta, these authors defined the relative generalized momenta in the usual way, as $P_\alpha = \partial\Lambda / \partial\dot{q}^\alpha$, where Λ is the Lagrangian of a system in its relative motion. In such a way, starting from the Hamilton's principle for the relative motion as a constrained variational problem, they obtained the corresponding Hamiltonian equations, and explained them better from the point of view of the canonical transformations, after finding the corresponding generating function.

In this paper we shall extend this parametric formulation of mechanics to the relative motion of the rheonomic systems of particles, and study the corresponding Lagrangian and Hamiltonian equations, as well as the general energy change law for such systems.

2. A PARAMETRIC FORMULATION OF MECHANICS

Let us consider the motion of a mechanical system of N particles with respect to a moving, noninertial frame of reference $AX'Y'Z'$, whose position at each instant t is determined by the given time functions of the pole velocity $\vec{\upsilon}_A(t)$ and of the angular velocity $\omega(t)$. Suppose that this motion is limited by k nonstationary holonomic constraints

$$f_\mu[\vec{r}_\nu^{\,'}, \varphi(t)] = 0\,, \quad (\mu = 1,2,...,k; \nu = 1,2,...,N) \tag{2.1}$$

where we assumed that time in such constraints always appears through certain function $\varphi(t)$, and determine the position of this system with respect to the considered noninertial frame of reference by a set of generalized coordinates $q^i (i = 1,2,...,n)$, where $n = 3N - k$. In the case of such mechanical systems the time has a double role: on one hand it is the independent variable, as in the mechanics in general, and on the other hand, in the equations of nonstationary constraints it has the character of a parameter. The principal idea of this parametric formulation of mechanics is based on the separation of this double role of time with the aid of a family of varied paths, what remains valid for the considered relative motion as well. In order to avoid any possible confusion, we shall denote time as a parameter by t_p, while time as independent variable - evolution time without any index.

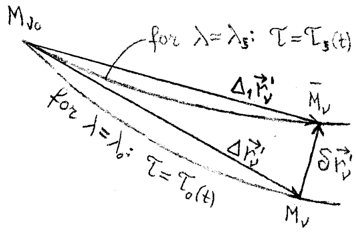


Fig. 1.

Let us imagine a family of varied paths of the particles, starting from their positions $M_{\nu 0}(\nu = 1,2,...,N)$ in the instant $t = t_0$ (Fig. 1) and defined in the usual way, i.e. very close to the actual path and in accordance with the constraints. These varied paths can be determined by the equations of motion in the vector form, or in the generalized coordinates:

$$\vec{r}_\nu' = \vec{r}_\nu'(t,\lambda) \quad (\nu = 1,2,...,N) \Leftrightarrow q^i = q^i(t,\lambda), \quad (i = 1,2,...,n) \tag{2.2}$$

where λ is a variable parameter. For each particle M_ν let us present two possible displacements: $\Delta\vec{r}_\nu$ along the actual path and $\Delta_1\vec{r}_\nu$ along a varied path, as vector sums of the corresponding elementary displacements in some finite time interval (t_0, $t_0+\Delta t$). Their differences represent the corresponding virtual displacements

$$\delta\vec{r}_\nu' \overset{def}{=} \Delta_1\vec{r}_\nu' - \Delta\vec{r}_\nu' = \vec{r}_\nu'(t_0+\Delta t,\lambda_\xi) - \vec{r}_\nu'(t_0+\Delta t,\lambda_0), \tag{2.3}$$

which according to the definition of the varied paths must be very small quantities, even for an arbitrary large time interval Δt.

If we wish to present so defined varied paths in generalized coordinates, the position of the system in the instant $t = t_0$ will be determined by a set $q_0^i(i = 1,2,...,n)$, and in the instant $t = t_0 + \Delta t$ on the actual path by a set q^i, and on the varied path by $\bar{q}^i$ ($i = 1,2,...,n$). In this case, the evolution of the system along the actual and along the varied path will be presented by a set of diagrams, presented on the Fig. 2, in which the corresponding curves $q^i = q^i(t,\lambda_0)$ and $q^i = q^i(t,\lambda_\xi)$ are close to each other. Therefore, the corresponding variations δq^i at each instant must be very small quantities, what shows certain similiraty with the varied paths in the Hamilton's principle.

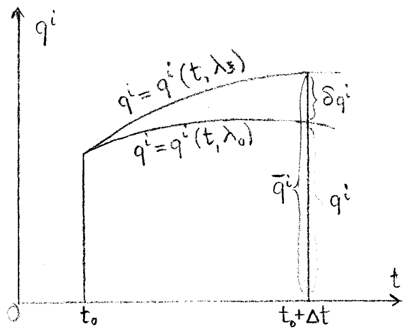


Fig. 2.

Instead of time as a parameter, let us introduce a new parameter τ, which depends on the chosen paths by the following relation, solvable for t_p

$$\tau = \tau(t_p,\lambda) \Rightarrow t_p = t_p(\tau,\lambda), \tag{2.4}$$

and retain the evolution time t as independent variable. Then, for any value of this parameter $\lambda = \lambda_\xi$, there exists some function of time $\tau = \tau(t_p, \lambda_\xi) \equiv \tau_\xi(t_p)$, associated with the corresponding varied path. Therefore, the values of this parameter which corespond to the positions of the particle on the actual part (M_ν) and on the varied parth ($\overline{M}_\nu$) at the same instant $t = t_0 + \Delta t$ are mutually different, namely $\tau + \Delta\tau$ and $\tau + \Delta_1\tau$. In order to adjust this parameter to the constraints, let us choose this function $\tau(t_p, \lambda)$ so that for $\lambda = \lambda_0$, i.e. for the actual path, it coincides with the function $\varphi(t_p)$, which appears in the equations of constraints (2.1), and let us take so defined quantity τ as an additional generalized coordinate

$$q^0 = \tau(t_p, \lambda_0) = \varphi(t_p) \tag{2.5}$$

In all the relations where time has the role of a parameter, one can replace time by so introduced parameter, which retains the same value along any fixed path, actual or a varied path. So, for $\lambda = \lambda_0$, i.e. for the actual path, the constraints (2.1) can be expressed in the form

$$f_\mu[\vec{r}_\nu{}', \varphi(t_p)] = f_\mu[\vec{r}_\nu{}', \tau] = 0. \tag{2.6}$$

Similarly, by introducing an extended set of generalized coordinates, the Lagrangian $L(\vec{r}_\nu{}', \vec{\upsilon}_\nu{}', t)$ will be transformed into

$$L = L[\vec{r}_\nu{}'(q^\alpha), \frac{\partial \vec{r}_\nu{}'}{\partial q^\alpha}\dot{q}^\alpha, t(q^0, \lambda_0)] \equiv L^*(q^\alpha, \dot{q}^\alpha), \tag{2.7}$$

where the summation over the repeated indices is understood.

In this parametric formulation of mechanics, the total work of the ideal reaction forces along arbitrary virtual diplacements can be found in the following way. Starting from the definition of the ideal reaction forces for the holonomic systems, and from the condition for the virtual displacements

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu{}'} \cdot \delta\vec{r}_\nu{}' + \frac{\partial f_\mu}{\partial \tau} \cdot \delta\tau = 0,$$

which is a consequence of (2.6), one obtains

$$\vec{R}_\nu^{id} \cdot \delta\vec{r}_\nu{}' = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu{}'} \cdot \delta\vec{r}_\nu{}' = R_0 \delta\tau, \tag{2.8}$$

where R_0 is given by

$$R_0 = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau} = \vec{R}_\nu^{id} \cdot \frac{\partial \vec{r}_\nu{}'}{\partial q^0}. \tag{2.9}$$

This quantity R_0, which arises from the nonstationary constraints and only for the rheonomic systems is different from zero, represents the generalized reaction force corresponding to the additional generalized coordinate q^0. The property that this work $\vec{R}_\nu^{id} \cdot \delta\vec{r}_\nu{}'$ for the rheonomic systems is different from zero is an essential characteristic of this parametric formulation of mechanics, in contrast to its usual formulation, where $\vec{R}_\nu^{id} \cdot \delta\vec{r}_\nu{}' = 0$.

By using this quantity R_0, the influence of nonstationary constraints can be included in the general principles of mechanics and the corresponding differential equations of motion. So, starting from the fundamental equation of dynamics in respect to an absolute frame of reference, and bearing in mind (2.8), one can obtain the d'Alembert-Lagrange's principle in this parametric formulation as [4]

$$(\vec{F}_\nu + \vec{R}_\nu^* - m_\nu \vec{a}_\nu)\cdot \delta\vec{r}_\nu = -R_0 \delta\tau , \tag{2.10}$$

where $\vec{R}_\nu^*$ is the nonideal reaction force, and $\vec{a}_\nu$ the acceleration for the ν-th particle. This principle can be also expressed in the generalized coordinates, putting $\delta\vec{r}_\nu = (\partial\vec{r}_\nu / \partial q^\alpha)\delta q^\alpha$, and decomposing the generalized forces into the potential and nonpotential ones $Q_\alpha = -\partial U / \partial q^\alpha + Q_\alpha^*$. In this way, by grouping the similar terms, it can be presented in the form

$$\left(Q_\alpha^* - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^\alpha} + \frac{\partial \mathscr{L}}{\partial \dot{q}^\alpha} \right)\delta q^\alpha = 0 , \tag{2.11}$$

where $\mathscr{L}$ and $\tilde{Q}_\alpha^*$ are given by

$$\mathscr{L}(q^\alpha, \dot{q}^\alpha) = L - P = T - (U + P), \quad \tilde{Q}_\alpha^* = Q_\alpha^* + R_\alpha^* \tag{2.12}$$

Here T and U are the kinetic and the potential energy of the system, Q_α^* and R_α^* the generalized nonpotential active and nonideal reaction forces respectively, and P is defined by

$$P \overset{def}{=} -\int R_0 dq^0 \quad \Leftrightarrow \quad R_0 = -\frac{dP}{dq^0} . \tag{2.13}$$

Since all the variations δq^α are independent, from (2.11) it follows immediately that all the expressions between parenthesis must be equal to zero, and therefore

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathscr{L}}{\partial \dot{q}^\alpha} = Q_\alpha^* \quad (\alpha = 0,1,2,...,n) \tag{2.14}$$

These are the corresponding Lagrangian equations in this parametric formulation of mechanics, the number of which is $n + 1$, and so introduced Lagrangian $\mathscr{L}$ through the term P comprises the influence of nonstationary constraints. The quantity P, defined by (2.13) was introduced by Vujičić [1] in his modification of the mechanics of rheonomic systems, and named the rheonomic potential. It can be obtained by previous finding R_0, either immediately from its definition (2.9) after determining the multipliers λ_μ in the usual way, or from the Lagrangian equations (2.14), by finding the solutions $q^i = q^i(t)$ $(i = 1,2,...,n)$ of the first n of them and inserting those ones into the last Lagrangian equation. In this way, one obtains this quantity in the form $R_0 = F(q^i,t)$, and putting here these solutions for q^i, with the aid of (2.4) it becomes certain function only of q^0, then P can be found by integration (2.13), also as some function of q^0.

3. Lagrangian of a System in Its Relative Motion

Let us examine how one can apply the Lagrangian equations (2.14) in this parametric formulation of mechanics directly to the relative motion of a mechanical system of particles with respect to a moving, noninertial frame of reference. First, we must effectuate the transformation of the coordinates from an absolute frame of reference to this noninertial one, which represents a punctual transformation. Since the form of the Lagrange's equations remains invariant in any transformation of this kind, these Lagrangian equations can be utilized directly in the same form in the considered relative motion of the system. Then, all the quantities and their changes must be treated as the relative ones, i.e. in a way as they are seen by an observer in this noninertial frame of reference, as in the usual formulation of mechanics, which will be denoted here by ', for example $\vec{\upsilon}_\nu$.

In this aim, the corresponding Lagrangian of this system must be expressed as a function of the relative coordinates and relative velocities of the particles, as well as of the given time functions $\vec{\upsilon}_A(t)$ and $\vec{\omega}(t)$, which determine the position of this noninertial frame of reference in each instant. In order to obtain so formulated Lagrangian, let us start from this Lagrangian in an inertial system of reference, and substitute the absolute velocities $\vec{\upsilon}_\nu$ of the particles by the sum of the corresponding relative and transport velocities

$$\vec{\upsilon}_\nu = \vec{\upsilon}_\nu^{rel} + \vec{\upsilon}_\nu^{tr} = \vec{\upsilon}_\nu' + \vec{\upsilon}_A + \vec{\omega}\times\vec{r}_\nu' \tag{3.1}$$

Then, this Lagrangian in our parametric formulation, according to (2.12) will be

$$\begin{aligned}\mathscr{L}'(\vec{r}_\nu', \vec{\upsilon}_\nu') &= \frac{1}{2}m_\nu\vec{\upsilon}_\nu^2 - (U'+P) = \frac{1}{2}m_\nu\vec{\upsilon}_A^2(t) + \frac{1}{2}m_\nu\vec{\upsilon}_\nu'^2 + \frac{1}{2}m_\nu(\vec{\omega}\times\vec{r}_\nu')^2 + \\ &+ m_\nu\vec{\upsilon}_A(t)\cdot\vec{\upsilon}_\nu' + m_\nu\vec{\upsilon}_A(t)\cdot(\vec{\omega}\times\vec{r}_\nu') + m_\nu\vec{\upsilon}_\nu'(t)\cdot(\vec{\omega}\times\vec{r}_\nu') - U[\vec{r}_0(t)+\vec{r}_\nu'] - P\end{aligned} \tag{3.2}$$

The sum of the fourth and the fifth term, on the basis of a general relation between the absolute and the relative derivative of any vector function $\vec{A}(t)$

$$\frac{d\vec{A}}{dt} = \frac{d_r\vec{A}}{dt} + \vec{\omega}\times\vec{A}\,, \tag{3.3}$$

where the symbol d_r/dt denotes the relative derivative, can be transformed into

$$m_\nu\vec{\upsilon}_A(t)\cdot\vec{\upsilon}_\nu' + m_\nu\vec{\upsilon}_A(t)\cdot(\vec{\omega}\times\vec{r}_\nu') = \frac{d}{dt}[m_\nu\vec{\upsilon}_A(t)\cdot\vec{r}_\nu'] - m_\nu.\vec{r}_\nu'\cdot\frac{d\,\vec{\upsilon}_A}{dt}$$

Here all the terms which can be expressed as the total time derivative of any function can be omitted, since all the Lagrangians which differ by a total time derivative are equivalent. In this way, after certain identical transformations, utilizing the known properties of the vector and mixed vector products, this Lagrangian obtains the form

$$\begin{aligned}\mathscr{L}'(\vec{r}_\nu', \vec{\upsilon}_\nu') &= \frac{1}{2}m_\nu\vec{\upsilon}_\nu'^2 + \frac{1}{2}m_\nu\omega^2\vec{r}_\nu'^2 - \frac{1}{2}m_\nu(\vec{\omega}\cdot\vec{r}_\nu')^2 - \\ &- m_\nu\vec{r}_\nu'\cdot\vec{a}_A + m_\nu\vec{r}_\nu'\cdot(\vec{\upsilon}_\nu'\times\vec{\omega}) - (U'_{(\vec{r}_\nu',t)} + P),\end{aligned} \tag{3.4}$$

where $\vec{a}_A$ is the acceleration of the pole A.

In order to point out the physical meaning of some terms in this expression, let us decompose it in the following way

$$\mathscr{L}'(\vec{r}_\nu^{\,'},\vec{\upsilon}_\nu^{\,'}) = T' + \vec{\omega}\cdot\vec{L}_A^{'} - (\Pi + P) \tag{3.5}$$

where T', $\vec{L}_A^{'}$ and Π are given by

$$\begin{gathered} T' = \frac{1}{2} m_\nu \vec{\upsilon}_\nu^{\,'2}, \quad \vec{L}_A^{'} = \vec{r}_\nu^{\,'} \times m_\nu \vec{\upsilon}_\nu^{\,'} \\ \Pi = U'(\vec{r}_\nu^{\,'},t) + m\vec{r}_c^{\,'}\cdot\vec{a}_A - \frac{1}{2} J_A^{'}\omega^2, \quad J_A^{'} = m_\nu(\vec{\omega}_0 \times \vec{r}_\nu^{\,'})^2 \end{gathered} \tag{3.6}$$

and the rheonomic potential P, specific for this parametric formulation of mechanics, is defined by (2.13). So formulated T' is the relative kinetic energy, $\vec{L}_A^{'}$ the relative angular momentum of the considered system, and J'_A is the relative moment of inertia of this system in its rotation around the axis through the pole A, all these quantities computed with respect to the noninertial frame of reference in the instant t. The second and the third term in Π have the character of certain potential energy, being dependent only on the variable $\vec{r}_\nu^{\,'}$, the first of them arising from the translation, and the second from the rotation of this frame of reference, so-called centrifugal potential energy.

This Lagrangian of a system in its relative motion differs from the corresponding one in the usual formulation of mechanics, obtained by L. Pars [8], only by the specific term P for this parametric formulation of mechanics, which expresses the influence of the nonstationary constraints to this problem. Our proof represents a generalization of the corresponding one, presented in the vector form by Landau and Lifschitz [6] to a system of particles, in accordance with the cited Pars's result, extending it to our parametric formulation of mechanics.

4. LAGRANGIAN EQUATIONS FOR THE RELATIVE MOTION

Now, let us express this Lagrangian in the generalized coordinates q^α (α=0,1,2,...,n), which determine the position of this system with respect to the noninertial frame of reference, in the sense of this parametric formulation of mechanics. In this aim, putting $\vec{\upsilon}_\nu^{\,'} = (\partial\vec{r}_\nu^{\,'}/\partial q^\alpha)\dot{q}^\alpha$, the first term in (3.5), i.e. the relative kinetic energy of the system can be presented as

$$T' = \frac{1}{2} m_\nu \vec{\upsilon}_\nu^{\,'\,2} = \frac{1}{2} A_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta, \quad (\alpha,\beta = 0,1,2,...n) \tag{4.1}$$

where the corresponding metric tensor in this noninertial frame of reference is given by

$$A_{\alpha\beta} = m_\nu \frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha}\cdot\frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\beta} \tag{4.2}$$

In a similar way, the second term in (3.5) becomes a linear function of $\dot{q}^\alpha$

$$\vec{\omega}\cdot\vec{L}_A = B_\alpha \dot{q}^\alpha, \quad B_\alpha = \vec{\omega}\cdot\left(m_\nu \vec{r}_\nu^{\,'} \times \frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha}\right), \tag{4.3}$$

and the terms Π and P are certain functions only of the generalized coordinates q^α

$$\Pi = \Pi(q^\alpha) \quad (\alpha = 0,1,2,...,n), \quad P = P(q^0) \tag{4.4}$$

Therefore, this Lagrangian in the generalized coordinates will be presented as a quadratic function of the variables $\dot{q}^\alpha$

$$\mathscr{L}'(q^\alpha, \dot{q}^\alpha) = \frac{1}{2} A_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta + B_\alpha \dot{q}^\alpha - \Pi(q^\alpha) - P(q^0) \tag{4.5}$$

Since the corresponding Lagrangian equations (2.14) in our application are treated directly with respect to the noninertial frame of reference, they must be used in the form

$$\frac{d_r}{dt}\frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha} - \frac{\partial \mathscr{L}'}{\partial q^\alpha} = \widetilde{Q}_\alpha^* \quad (\alpha = 0,1,2,...,n) \tag{4.6}$$

These equations, according to their form, coincide with the ones obtained by M. Lukačević and O. Jeremić [10], but in this case the number of these equations and the domain of their validity are greater than in the usual formulation, and they have quite different meaning. Our results refer to the more general, rheonomic systems with arbitrary nonstationary constraints, and are given in our parametric formulation of mechanics, with an additional Lagrangian equation, what will be of special interest for the corresponding energy relations.

If we want to point out explicitly the terms specific for the relative motion, i.e. the inertial forces which represent the so-called transport and Coriolis's ones, let us write the Lagrangian (4.5) as

$$\mathscr{L}'(q^\alpha, \dot{q}^\alpha) = L' + \mathscr{L}_1' - P, \tag{4.7}$$

where

$$L' = T' - U', \quad Z_1' = \vec{\omega}\cdot\vec{L}_A - (m\vec{r}_c^{\,'}\cdot\vec{a}_A - \frac{1}{2}J_A'\omega^2) \tag{4.8}$$

All the necessary partial derivatives of so decomposed Lagrangian can be found bearing in mind that $\mathscr{L}_1'$ depends on q^α and $\dot{q}^\alpha$ through all the variables $\vec{r}_\nu^{\,'}$ and $\vec{\upsilon}_\nu^{\,'}$ ($\nu = 1,2,...,N$) respectively. In this way, after finding the corresponding partial derivatives on the basis of (3.4), by inserting the obtained expressions into the Lagrangian equations (4.6), and grouping the similar terms, these equations can be presented in the form

$$\frac{d_r}{dt}\frac{\partial L'}{\partial \dot{q}^\alpha} - \frac{\partial L'}{\partial q^\alpha} = \widetilde{Q}_\alpha^* + Q_\alpha^{tr} + Q_\alpha^{cor} + \delta_\alpha^0 R_0, \quad (\alpha = 0,1,2,...,n) \tag{4.9}$$

where Q_α^{tr} and Q_α^{cor} are given by

$$\begin{aligned} Q_\alpha^{tr} &= \vec{F}_\nu^{tr}\cdot\frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha} = -m_\nu[\vec{a}_A + \dot{\vec{\omega}}\times\vec{r}_\nu^{\,'} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_\nu^{\,'})]\cdot\frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha} \\ Q_\alpha^{cor} &= \vec{F}_\nu^{cor}\cdot\frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha} = -2m_\nu(\vec{\omega}\times\vec{\upsilon}_\nu^{\,'})\cdot\frac{\partial \vec{r}_\nu^{\,'}}{\partial q^\alpha} \end{aligned} \tag{4.10}$$

These quantities Q_α^{tr} and Q_α^{cor} represent the generalized forces which correspond to the resultants of all the transport and Coriolis's forces, acting on the particles of the system in this noninertial frame of reference, while the quantity R_0 is a characteristic of this parametric formulation of mechanics.

The obtained results are in accordance with the ones formulated by A. Lur'je [9], p. 433-436), but given here in a different form and with annother approach and interpretation of these results, according to our parametric formulation of mechanics.

5. HAMILTONIAN EQUATIONS FOR THE RELATIVE MOTION

The transition from the Lagrangian to the corresponding Hamiltonian formalism for the considered relative motion of a system of particles in this parametric formulation can be carried out in a similar way as in the usual formulation. Namely, let us introduce the generalized momenta by

$$p_\alpha \stackrel{def}{=} \frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha}, \quad (\alpha = 0,1,2,...,n) \tag{5.1}$$

the number of wich is $n + 1$, where $\mathscr{L}'$ is the corresponding Lagrangian given by (4.5). Then, the generalized momenta will have the form

$$p_\alpha = A_{\alpha\beta}\dot{q}^\beta + B_\alpha, \quad (\alpha,\beta = 0,1,2,...,n) \tag{5.2}$$

from which, after multiplying these equations by the conjugate metric tensor $A^{\alpha\gamma}$, summing over repeated index, and applying the property $A_{\alpha\beta}\, A^{\alpha\gamma} = \delta_\beta^\gamma$, it follows

$$\dot{q}^\gamma = A^{\alpha\gamma}(p_\alpha - B_\alpha) \quad (\alpha,\beta = 0,1,2,...,n) \tag{5.3}$$

Since the matrix $A_{\alpha\beta}$ for the regular mechanical systems is nonsingular, the matrix $A^{\alpha\gamma}$ always exists, and therefore the system of equations (5.2) can always be solved for the variables $\dot{q}^\beta$, what allows us the transition to the Hamiltonian formalism.

The corresponding canonical, Hamiltonian equations can be obtained as usually, for example starting from the variation of the Lagrangian, applying the Lagrangian equations (4.6), and passing from the variables $\dot{q}^\alpha$ to the generalized momenta, what can be written as

$$\delta(p_\alpha \dot{q}^\alpha - \mathscr{L}') = (\widetilde{Q}_\alpha^* - \dot{p}_\alpha)\delta q^\alpha + \dot{q}^\alpha \delta p_\alpha \tag{5.4}$$

Therefore, the expression in the parenthesis can be considered as a function of the variables q^α and p_α, from where by comparison with the variation of this expression one obtains

$$\frac{d_r p_\alpha}{dt} = -\frac{\partial H'}{\partial p_\alpha} + \widetilde{Q}_\alpha^*, \quad \frac{d_r q^\alpha}{dt} = -\frac{\partial H'}{\partial p_\alpha}, \quad (\alpha = 0,1,2,...,n) \tag{5.5}$$

where the Hamiltonian is given by

$$H'(q^\alpha, p_\alpha) = p_\alpha \dot{q}^\alpha - \mathscr{L}'(q^\alpha, \dot{q}^\alpha) \tag{5.6}$$

There are the corresponding Hamiltonian equations in this parametric formulation of mechanics, with $2(n + 1)$ independent ones. This Hamiltonian can be formed by eliminating the generalized velocities by means of (5.3), which is for the regular systems always possible.

In this case we can also emphasize the terms characteristic for the relative motion, similarly as in the Lagrangian equations. Therefore, we must introduce the generalized momenta associated to $L' = T' - U'$

$$p_\alpha^{(r)} \overset{def}{=} \frac{\partial L'}{\partial \dot{q}^\alpha} = \frac{\partial}{\partial \dot{q}^\alpha}(T' - U'), \quad (\alpha = 0,1,2,...,n) \tag{5.7}$$

and decompose the Hamiltonian into three parts

$$\mathscr{H}'(q^\alpha, p_\alpha) = H' + \mathscr{H}_1' + P, \tag{5.8}$$

where

$$H' = p_\alpha^{(r)}\dot{q}^\alpha - L', \quad \mathscr{H}_1' = \frac{\partial \mathscr{L}_1'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathscr{L}_1' \tag{5.9}$$

Then, after finding the corresponding partial derivatives, and inserting them into the Hamiltonian equations (5.5), by grouping the similar terms these equations can be presented in the form

$$\begin{aligned} \frac{d_r p_\alpha^{(r)}}{dt} &= -\frac{\partial H'}{\partial q^\alpha} + \tilde{Q}_\alpha^* + Q_\alpha^{tr} + Q_\alpha^{cor} + \delta_\alpha^0 R_0 \\ \frac{d_r q^\alpha}{dt} &= -\frac{\partial H'}{\partial p_\alpha}, \quad (\alpha = 0,1,2,...,n) \end{aligned} \tag{5.10}$$

where Q_α^{tr} and Q_α^{cor} are given by (4.10).

The physical meaning of these Hamiltonians can be perceived in a similar way as for the usual Hamiltonian. Namely, on the basis of (4.5) and (5.2) we have

$$\mathscr{H}'(q^\alpha, p_\alpha) = (A_{\alpha\beta}\dot{q}^\beta + B_\alpha)\dot{q}^\alpha - \left(\frac{1}{2}A_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + B_\alpha\dot{q}^\alpha - \Pi - P\right)$$

and, after canceling the linear terms with respect to $\dot{q}^\alpha$, one obtains

$$\mathscr{H}'(q^\alpha, p_\alpha) = E'^{ext} = \frac{1}{2}m_\nu\vec{\upsilon}_\nu'^2 + \Pi + P. \tag{5.11}$$

Thus, the Hamiltonian of a system of particles in its relative motion with respect to a noninertial frame of reference represents the generalized relative mechanical energy of this system, extended by the rheonomic potential P, which arises from the nonstationary constraints. However, the meaning of the Hamiltonian H' defined by (5.9), is simpler, since here all the terms characteristic for the relative motion cancel out, so that this Hamiltonian represents the relative total energy $E' = T' + U'$ of the system, extended also by the rheonomic potential P.

The obtained results and equations have the same form as the ones presented by A. Luje [9], and in the cited paper by M. Lukačević and O. Jeremić [10] in the usual formulation of mechanics. But, here these equations represent an extended system of Hamiltonian equations, with $2(n + 1)$ of them, and they have quite different meaning in this parametric formulation of mechanics, including the influence of the nonstationary constraints, which is absent in the usual formulation of mechanics.

6. ENERGY CHANGE LAW FOR THE RELATIVE MOTION

In order to examine the corresponding energy relations in the relative motion of a system of particles with respect to a noninertial frame of reference, let us first present the energy change law in the usual formulation of mechanics, in the vector form and in the generalized coordinates.

a) In this aim, let us start with the fundamental equation of the relative motion for each particle of the system

$$m_\nu \frac{d_r \vec{\upsilon}'_\nu}{dt} = \vec{F}_\nu + \vec{R}^{id}_\nu + \vec{R}^*_\nu - m_\nu \vec{a}^{tr}_\nu - m_\nu \vec{a}^{cor}_\nu, \quad (\nu = 1,2,...,N) \tag{6.1}$$

in which the reaction forces are decomposed into the ideal and the nonideal ones, while the transport and Coriolis's accelerations are given by

$$\vec{a}^{tr}_\nu = \vec{a}_A + \dot{\vec{\omega}} \times \vec{r}'_\nu + \vec{\omega} \times (\vec{\omega} \times \vec{r}'_\nu), \quad \vec{a}^{cor}_\nu = 2\vec{\omega} \times \vec{\upsilon}'_\nu \tag{6.2}$$

If we multiply these equations by $d_r \vec{r}'_\nu = \vec{\upsilon}'_\nu dt$, and sum over the repeated index, one obtains the corresponding kinetic energy law

$$d_r T' = d_r \left(\frac{1}{2} m_\nu \upsilon'^2_\nu \right) = \vec{F}_\nu \cdot d_r \vec{r}'_\nu + \vec{R}^{id}_\nu \cdot d_r \vec{r}'_\nu + \vec{R}^*_\nu \cdot d_r \vec{r}'_\nu - m_\nu \vec{a}^{tr}_\nu \cdot d_r \vec{r}'_\nu - m_\nu \vec{a}^{cor}_\nu \cdot d_r \vec{r}'_\nu \,. \tag{6.3}$$

The second term represents the elementary total work of the ideal reaction forces along the possible displacements $d_r \vec{r}'_\nu$, which is of special interest for this analysis of the energy relations, and can be found in the following way. By applying the condition for these displacements

$$\frac{\partial f_\mu}{\partial \vec{r}'_\nu} \cdot d_r \vec{r}'_\nu + \frac{\partial f_\mu}{\partial t} dt = 0 \,,$$

which is a consequence of (2.1), this work will be equal to

$$\vec{R}^{id}_\nu \cdot d_r \vec{r}'_\nu = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}'_\nu} \cdot d_r \vec{r}'_\nu = -\lambda_\mu \frac{\partial f_\mu}{\partial t} dt \,, \tag{6.4}$$

which is different from zero only for the systems with nonstationary constraints. The other terms can be presented in a more suitable form, decomposing the active forces into the potential and the nonpotential ones, and computing the elementary works of all these forces, utilizing the known rules in the vector calculus, for example

$$-m_\nu \vec{\omega} \times (\vec{\omega} \times \vec{r}'_\nu) \cdot d_r \vec{r}'_\nu = -m_\nu [\vec{\omega}(\vec{\omega} \cdot \vec{r}'_\nu) - \vec{r}'_\nu (\vec{\omega} \cdot \vec{\omega})] \cdot d_r \vec{r}'_\nu =$$
$$= m_\nu (\vec{\omega} \times \vec{r}'_\nu) \cdot (\vec{\omega} \times d_r \vec{r}'_\nu) = \frac{1}{2} m_\nu d_r (\vec{\omega} \times \vec{r}'_\nu)^2$$

In this way, the kinetic energy law can be transfomed to

$$\frac{d_r E'}{dt} = \frac{d_r}{dt}(T' + \Pi) = \frac{\partial U'}{\partial t} - \lambda_\mu \frac{\partial f_\mu}{\partial t} + (\vec{F}^*_\nu + \vec{R}^*_\nu) \cdot \vec{\upsilon}'_\nu - m_\nu (\dot{\vec{\omega}} \times \vec{r}'_\nu) \cdot \vec{\upsilon}'_\nu \,, \tag{6.5}$$

where Π is given by (3.6).

This is the general energy change law for the relative motion in the usual and vector formulation, where the second term on the right-hand side expresses the influence of the nonstationary constraints. It $\partial U'/\partial t = 0$, $\vec{F}_\nu^* \cdot \vec{\upsilon}_\nu' = 0$, $\vec{R}_\nu^* \cdot \vec{\upsilon}_\nu' = 0$ and $\vec{\omega} = const$, from here it follows that

$$T' + U' + \int \lambda_\mu \frac{\partial f_\mu}{\partial t} dt = const , \tag{6.6}$$

what can be interpreted as a kind of the energy conservation law.This energy change law represents some generalization of the similar, but usually simpler law for the relative motion (see f.e. [11], p. 186-202) to the systems with arbitrary nonstationary constraints. This integral form of the energy conservation law, opposite to the usual formulation, contrains a term arising from these constraints.

b) Let us start from the Lagrangian equations for the relative motion in the usual formulation [10]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \widetilde{Q}_i^*, \quad (i = 1,2,...,n) \tag{6.7}$$

where

$$L(q^i, \dot{q}^i, t) = \frac{1}{2} m_\nu \vec{\upsilon}_\nu'^2 + \vec{\omega} \cdot \vec{L}_A - (U + m\vec{r}_c' \cdot \vec{a}_A - \frac{1}{2} J_A \omega^2) , \tag{6.8}$$

multiply it by $dq^i = \dot{q}^i dt$, and sum over the repeated index. In this way, after transforming the first term, one obtains

$$d\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i = d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i = \frac{\partial L}{\partial q^i} dq^i + \widetilde{Q}_i^* dq^i$$

and bearing in mind that

$$dL = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i + \frac{\partial L}{\partial t} dt ,$$

the previous relation can be written as

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = -\frac{\partial L}{\partial t} + \widetilde{Q}_i^* \dot{q}^i , \tag{6.9}$$

Since the relative kinetic energy of a system of particles, according to its dependence on $\dot{q}^i$, in the general case has three terms: $T' = T_2' + T_1' + T_0'$, the expression in parenthesis on the basis of (6.8) and Euler's theorem is equal to

$$E' = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = T_2' - T_0' + \Pi . \tag{6.10}$$

This form of the energy change law differs from the previous one (6.5), here the influence of the nonstationary constraints is absent, and therefore the corresponding conservation laws are also different. Under the condition that all the terms on the right-hand side are equal to zero, in the first case we have $E' = T' + \Pi = \text{const}$, and in the second one $\mathcal{E}' = T_2' - T_0' + U = const$ (analogous to the Jacobi's energy integral).

c) However, if we start from the extended system of Lagrangian equations in this parametric formulation of mechanics, another energy chage law will be obtained, as it is shown by V. Vujičić [2] in his modification of the mechanics of rheonomic systems, but with a quite different interpretation. In our case, let us start from the corresponding Lagrangian equations (4.6) for the relative motion, multiply them by $dq^\alpha = \dot{q}^\alpha dt$, and sum over the repeated index, what can be written as

$$d\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right) - \frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha}d\dot{q}^\alpha - \frac{\partial \mathscr{L}'}{\partial q^\alpha}q^\alpha = \tilde{Q}^*_\alpha dq^\alpha , \tag{6.11}$$

where the Lagrangian $\mathscr{L}$ is given by (4.5). Its total differential in this case will be

$$d\mathscr{L}' = \frac{\partial \mathscr{L}'}{\partial q^\alpha}dq^\alpha + \frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha}d\dot{q}^\alpha ,$$

here the explicit dependence of the Lagrangian on time is included in the first term through the variable q^0, so that by grouping the similar terms, this relation obtains the form

$$\frac{d_r \mathcal{E}'^{ext}}{dt} = \frac{d_r}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathscr{L}\right) = \tilde{Q}^*_\alpha \dot{q}^\alpha , \tag{6.12}$$

Having in mind the physical meaning of the Hamiltonian (5.11), it can be presented explicitly as

$$\frac{d_r \mathcal{E}'^{ext}}{dt} = \frac{d_r}{dt}(T' + \Pi + P) = \tilde{Q}^*_\alpha \dot{q}^\alpha , \tag{6.13}$$

where, according to (3.6)

$$\Pi = U' + m\vec{r}_c^{\,'} \cdot \vec{a}_A - \frac{1}{2}J_A^{'}\omega^2 , \tag{6.14}$$

This is the corresponding general energy change law for the relative motion of a system of particles with respect to a noninertial frame of reference in this parametric formulation of mechanics. From here it is evident that if $\tilde{Q}^*_\alpha \dot{q}^\alpha = 0$, i.e. if the effect of all the generalized nonpotential active and nonideal reaction forces is equal to zero, the energy conservation law is valid in the form

$$E'^{ext} = T' + \Pi + P = \frac{1}{2}m_\nu \vec{\upsilon}_\nu^{\,'2} + U' + m\vec{r}_c^{\,'} \cdot \vec{a}_A - \frac{1}{2}J_A^{'}\omega^2 + P = const , \tag{6.15}$$

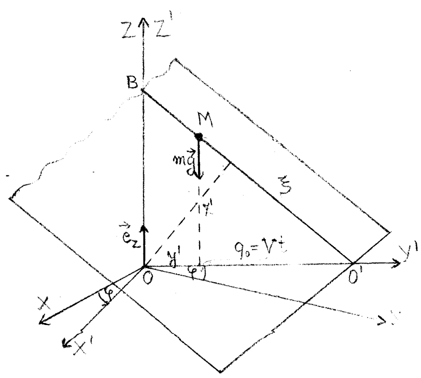
This energy change law and the corresponding conservation law differ from the ones in the previous case. However, in essence they are equivalent to the corresponding laws in the usual vector formulation (6.5) and (6.6), since the last term in the second relation according to (2.9) and (2.13) is equal to the rheonomic potential P for $q^0 = t$

$$P_{(q^0=t)} = -\int R_0 dt = \int \lambda_\mu \frac{\partial f_\mu}{\partial t}dt . \tag{6.16}$$

In this way, as in the case of the absolute motion, from the point of view of the energy relations this parametric formulation of mechanics is equivalent to the asemble of the usual Lagrangian formalism and the law for the work of the ideal reaction forces, which is not contained in this usual formulation.

7. An Example

Let us illustrate the obtained results by the relative motion of a particle with respect to a noninertial frame of reference *AX'Y'Z'*, which rotates with a constant angular velocity $\vec{\omega} = \overrightarrow{d\varphi}/dt$ around the *Z*-axis of a immobile frame of reference *OXYZ* with the same origin of coordinates (Fig. 3).



(Fig.3)

Let this particle move in the constant Earth's gravitational field along an inclined smooth line *O'B*, attached to this noninertial frame of reference, and displacing uniformly with the velocity *V* along the *Y*-axis. If we denote by α the angle between the direction of the motion of this particle and the *Y*-axis, then in each instant *t* must be

$$\mathrm{tg}\,\alpha = \frac{z'}{Vt - y'}$$

and therefore, this motion is restricted by a nonstationary constraint

$$f_1(y', z', t) \equiv (Vt - y')\sin\alpha - z'\cos\alpha = 0, \qquad f_2(x') \equiv x' = 0. \tag{7.1}$$

The motion of this particle has one degree of freedom, and take for the generalized coordinate $q_1 = \xi$, which determines the position of the particle at each instant t with respect to this noninertial frame of reference *AX'Y'Z'*. Let us choose for a parameter, i.e. an additional generalized coordinate $q_0 = Vt$, presented on the figure, then the relations between these generalized and rectangular coordinates are

$$x' = 0, \quad y' = q_0 - \xi\cos\alpha, \quad z' = \xi\sin\alpha \tag{7.2}$$

In this case the relative angular momentum $\vec{L}_A = \vec{r}' \times m\vec{\upsilon}'$ is normal to the plane *OO'B*, and thus also normal to the $\vec{\omega}$, so that $\vec{\omega}\cdot\vec{L}_A' = 0$, and since the pole *A* which coincides with the origin of coordinates is immobile, its acceleration is $\vec{a}_A = 0$. Therefore, the Lagrangian of this particle in the considered relative motion, according to (3.5) and (3.6) is reduced to

$$\mathscr{L}'(\vec{r}', \vec{\upsilon}') = T' - (\Pi + P) = T' - (U' - \frac{1}{2}J_A\omega^2 + P) \tag{7.3}$$

The relative kinetic and potential energy of this particle with respect to this noninertial frame of reference, expressed in the choosen generalized coordinates are

$$\begin{gathered} T' = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) = \frac{1}{2}m(\dot{\xi}^2 + \dot{q}_0^2 - 2\dot{\xi}\dot{q}_0\cos\alpha) \\ U' = mgz' = mg\xi\sin\alpha, \end{gathered} \tag{7.4}$$

and so-called centrifugal potential energy, containing the relative moment of inertia of this particle in its rotation about Z-axis, according to (7.2) is

$$\frac{1}{2}J'_A\omega^2 = \frac{1}{2}m(x'^2+y'^2)\omega^2 = \frac{1}{2}m(q_0-\xi\cos\alpha)^2\omega^2 \tag{7.5}$$

In this manner, the previous expression for the Lagrangian will be presented as

$$\mathscr{L}'(q^\alpha,\dot{q}^\alpha) = \frac{1}{2}m(\dot{\xi}^2+\dot{q}_0^2-2\dot{\xi}\dot{q}_0\cos\alpha) - mg\xi\sin\alpha + \frac{1}{2}m(q_0-\xi\cos\alpha)^2\omega^2 - P \tag{7.6}$$

where the rheonomic potential P can be found from the Lagrangian equations.

In our case, the corresponding extended system of Lagrangian equations (4.6) has only two equations, and since $Q^*_\alpha = 0$ and $R^*_\alpha = 0$ $(\alpha = 0,1)$, we have

$$\frac{d}{dt}\frac{\partial\mathscr{L}'}{\partial\dot{\xi}} - \frac{\partial\mathscr{L}'}{\partial\xi} = 0, \quad \frac{d}{dt}\frac{\partial\mathscr{L}'}{\partial\dot{q}_0} - \frac{\partial\mathscr{L}'}{\partial q_0} = 0 \tag{7.7}$$

or in explicit form, since according to (2.13) $dP/dq_0 = -R_0$

$$\begin{aligned} &\frac{d}{dt}(m\dot{\xi} - m\dot{q}_0\cos\alpha) + mg\sin\alpha + m(q_0-\xi\cos\alpha)\cos\alpha\omega^2 = 0 \\ &\frac{d}{dt}(m\dot{q}_0 - m\dot{\xi}\cos\alpha) - m(q_0-\xi\cos\alpha)\omega^2 - R_0 = 0. \end{aligned} \tag{7.8}$$

In order to find R_0, let us eliminate $d(m\dot{\xi})/dt$ from these equations, having in mind that $\dot{q}_0 = V = const$. The first of these ones gives

$$\frac{d}{dt}(m\dot{\xi}) = -mg\sin\alpha - m\cos\alpha(q_0-\xi\cos\alpha)\omega^2$$

and after inserting this expression into the second equation, we get

$$\begin{aligned} R_0 &= -\cos\alpha\frac{d}{dt}(m\dot{\xi}) - m(q_0-\xi\cos\alpha)\omega^2 = \\ &= mg\sin\alpha\cos\alpha - m(q_0-\xi\cos\alpha)\sin^2\alpha\,\omega^2. \end{aligned} \tag{7.9}$$

But, in aim to obtain R_0 as a function only of q_0, one must find the solution $\xi = \xi(t)$ of the first equation (7.8) and insert it into this expression for R_0. This equation explicitly has the form

$$\frac{d^2\xi}{dt^2} - \cos^2\alpha\;\omega^2\xi = -g\sin\alpha - Vt\cos\alpha\;\omega^2$$

or, more concisely

$$\frac{d^2\xi}{dt^2} - k^2\xi = -at + b, \tag{7.10}$$

where

$$k^2 = \cos^2\alpha\;\omega^2, \quad a = -V\cos\alpha\;\omega^2, \quad b = -g\sin\alpha \tag{7.11}$$

The general solution of this differential equation is equal to the sum of such solution of the corresponding homogeneous equation and one particular solution of this differential equation

$$\xi = C_1 sh(kt + C_2) + \xi_p\,,$$

where C_1 and C_2 are the constants of integration, determined by the initial conditions. If we seek one particular solution in the form $\xi_p = ct + d$, these constants can be found from the condition that this expression must satisfy the differential equation (7.10), and so this general solution obtains the form

$$\xi = C_1 \mathrm{sh}(kt + C_2) - \frac{a}{k^2}t - \frac{b}{k^2}. \tag{7.12}$$

By inserting it into (7.9), this quantity R_0 becomes a function of time

$$R_0 = m\sin^2\alpha\cos\alpha\;\omega^2\left[C_1\mathrm{sh}(kt + C_2) - \frac{a}{k^2}t - \frac{b}{k^2}\right] +$$
$$+ mg\sin\alpha\cos\alpha - m\sin^2\alpha\,\omega^2 Vt,$$

and since $q_0 = Vt$, this expression can be presented also in the form of a function of q_0

$$R_0 = A + Bq_0 + C_1^{'}\mathrm{sh}\left(\frac{k}{V}q_0 + C_2\right), \tag{7.13}$$

where because of the conciseness the corresponding constants are introduced. Then, the rheonomic potential P can be obtained, according to (2.13), by integration with respect to q^0

$$P = -\int R_0 dq_0 = -Aq_0 - \frac{1}{2}Bq_0^2 - C'\mathrm{ch}\left(\frac{k}{v}q_0 + C_2\right), \tag{7.14}$$

where $C' = C_1^{'}V/k$. This result represents a generalization of the corresponding one obtained by Vujičić [2], expanded here to the relative motion of a particle. From here one can see that in the case of the relative motion the rheonomic potential can depend also on the quantities characteristic for such motion, as the angular velocity $\vec{\omega}$ of the noninertial frame of reference.

In the considered case all the conditions for the validity of the energy conservation law are satisfied, so that according to (6.15) we have

$$\mathcal{E}^{'ext} = T' + \Pi + P = \frac{1}{2}m\vec{\upsilon}'^2 + U' - \frac{1}{2}J_A^{'}\omega^2 + P = const \tag{7.15}$$

If we substitute these quantities by the obtained expressions, this energy conservation law gets the following explicit form

$$\begin{aligned}\mathcal{E}^{'ext} &= \frac{1}{2}m(\dot{\xi}^2 + \dot{q}_0^2 - 2\dot{\xi}\dot{q}_0\cos\alpha) + mg\xi\sin\alpha - \frac{1}{2}m(q_0 - \xi\cos\alpha)^2\omega^2 - \\ &- Aq_0 - \frac{1}{2}Bq_0^2 - C'\mathrm{ch}\left(\frac{k}{V}q_0 + C_2\right) = const\end{aligned} \tag{7.16}$$

This energy integral in the parametric formulation of mechanics differs from the corresponding one for the absolute motion in the usual formulation by the centrifugal potential energy and by the rheonomic potential, the first being specific for the relative motion, and the second one for this parametric formulation.

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RELATIVNO KRETANJE SISTEMA U PARAMETARSKOJ FORMULACIJI MEHANIKE

Đorđe Mušicki

Parametarska formulacija mehanike, formulisana od samog autora, zasniva se na razdvajanju dvostruke uloge vremena za reonomne sisteme (nezavisno promenljiva i parametar) pomoću izvesne familije variranih putanja. Pri tome vreme kao nezavisna promenljiva ostaje nepromenjeno, dok se umesto vremena kao parametra uvodi novi parametar, koji zavisi od izabrane putanje iz ove familije i uzima se kao dopunska generalisana koordinata.

U ovom radu ova parametarska formulacija mehanike proširena je na relativno kretanje proizvoljnih reonomnih sistema. Na taj način, formulisani su i analizirani odgovarajući prošireni sistemi Lagrange-ovih i Hamilton-ovih jednačina, kao i opšti zakon promene energije za ovakve sisteme, a dobijeni rezultati su ilustrovani jednim prostim primerom.