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# A SIMPLIFIED MODEL FOR THE DYNAMIC ANALYSIS OF CABLE - STAYED BRIDGES

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**George T. Michaltsos**

National Technical University of Athens

**Abstract**. *This paper deals with a model, which simplifies the finding of the natural eigenfrequencies of a cable-stayed bridge, with dense distribution of cables. Then, it is possible to study the vertical forced vibrations of the bridge. Finally, an example is given, which, clearly, shows the simplicity of the method.*

## 1. INTRODUCTION

Many researchers have, in the last 100 years, studied the dynamic response of railway bridges and, later, of highway bridges under the influence of moving loads. Extensive references to the literature on this subject can be found in Frýba's book [1].

The cable-stayed bridges are a particular form of bridges, which have been of great interest in recent years, particularly because of their special shape and, also, because they are an alternative solution to suspension bridges for long spans [2].

For the dynamic behaviour of cable-stayed bridges particular attention is given to the free vibrations, aerodynamic stability (or instability) and to the seismic analysis. There are many studies which have obtained serious results concerning the dynamic response of several types of cable-stayed bridges, or of service loads [3]. A numerical analysis of the dynamic response of cable-stayed bridges has been developed, taking into account the vibration behaviour of the stay cables and studying the influence of the coupled deck-cable motions [4].

We must refer to the recent studies of Nazmy and Abdel-Ghaffar [5], Chatterjee and others [6], dealing with the lateral and torsional dynamic behaviour, Bruno and Colotti [7], proposing a fan-shaped bridge scheme as an analytical model and studying the eigenfrequencies.

Finally, we must mention the works of Achkrive-Preumont [8], who deals with the active vibration control of C-S bridges, Khalil [9] who studies some special characteristics of C-S bridges and attempts to solve the associated problems, Bosdogianni-Olivari [10]

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who study the oscillations of a bridge under combined effect of wind and rain and, finally, Virlogeux [11] who studies the rapid progress and the improvements of C-S bridges in the last years.

For the dynamic study of cable-stayed bridges, one must take into account the influence of the dynamic deformations of the pylons and, also, the serious influence of the axial forces of the deck, caused by the cable-tensions, which excite the bridge in a simultaneous axial dynamic movement.

The corresponding dynamic moments, which rise because of the above axial dynamic forces, can be, as a first approximation, neglected.

The present work investigates the dynamic behaviour of cable-stayed bridges. Taking into account the interaction between cables and deck we try to find an analytical expression for the load $q(x)$, which expresses the effect of the cables on the bridge deck. Afterwards we are able to form a linear system to find the tensions of the cables (for the case of rare distribution of the cables) or the differential equation giving the deck deformations (for the case of dense distribution of the cables). Then, we can proceed to the first goal, which is the finding of the eigenfrequencies of the bridge, using a simplified version of Galerkin's method. The definition of shape modes is the second goal. Then one can find, easily, the eigenfrequencies, shape modes etc.

Finally he dynamic analysis of a cable-stayed bridge subjected to moving loads is studied. and an illustrated example is presented.

## 2. ANALYSIS

For the following analysis, the model of a cable-stayed bridge scheme, like the one of Figure 1, is adopted. The optimal values of the ratios $\ell_1/(H-h)$ and $\ell_2/2(H-h)$ are usually assumed as: $\ell_1/(H-h)$=5/3, and $\ell_2/2(H-h)$=5.

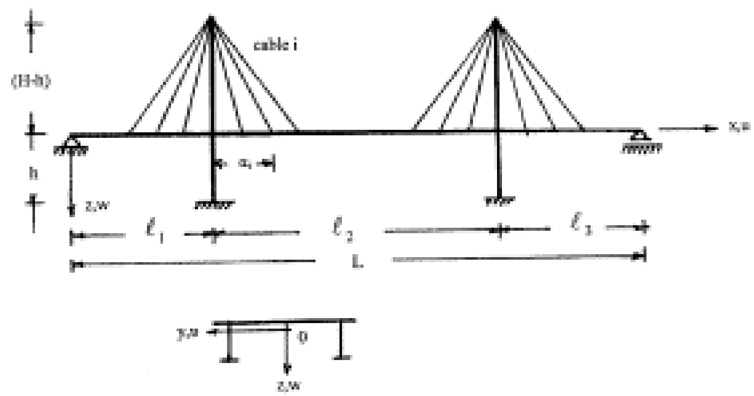


Fig. 1. Structural scheme of a cable stayed-bridge.

We accept the following assumptions:

a. The deck of the bridge, during its free vibration, is simply supported by the pylon without any other connection. So, the deck of the bridge can be characterized as a three- span continuous beam.

b. It is assumed that, under the dead load, there is no configuration of the deck or of the pylons, but the cables are stressed by the axial forces $P_{ig}$ , which can be calculated through a static analysis.
c. For the case of the dynamic behaviour, we will have: $P_i = P_{ig} + P_{id}$, where: $P_{ig} = constant$ (as before in b) is the force, which is given by the static analysis, $P_{id} = P_i\, f(t)$ is a time depended and so changed dynamic portion of the force $P_i$ . Because it is valid that $p < g$, it will be $p < g \;\; P_{id} < P_{ig}$ and then $P_{id}$ can be positive or negative, up to the value of $P_{ig}$ : $|P_{id}| \leq P_{ig}$ .
d. As it is well-known, the behaviour of the cables upon an axial force is non-linear, due to its own weight action. For a dynamic analysis we consider that the initial tension $\sigma_g$ of the cables corresponds to the starting equilibrium configuration under the own load g and so, we can adopt the tangent modulus of elasticity and not that which is given by Dischinger's formula: $\widetilde{E} = E/(1+\dfrac{E\gamma^2\ell_0^2}{12\cdot\sigma_0^3})$ .
e. We ignore the influence of the bridge deck surface roughness, because its contribution on the vibration of the bridge is negligible for bridges with long span, like the cable-stayed bridges.
f. A 2D analysis is considered while the influence of axial forces either of the pylon or of the deck is neglected.

### 2.1 Deformation of the system bridge-pylon

The relative deformation of the top of the pylon of Figure 2, to the point of the support of the deck on the pylon, is:

$$u = u_p - u_d = \frac{P_i H^3}{6E_p I_p}\left[2-3\left(\frac{h}{H}\right)^2+\left(\frac{h}{H}\right)^3\right] \tag{1}$$

where: $P_i$ the projection of the axial force of the i cable on the horizontal axis
$E_p$ the modulus of elasticity of the material of the pylon and
$J_p$ the moment of inertia of the cross-section of the pylon.

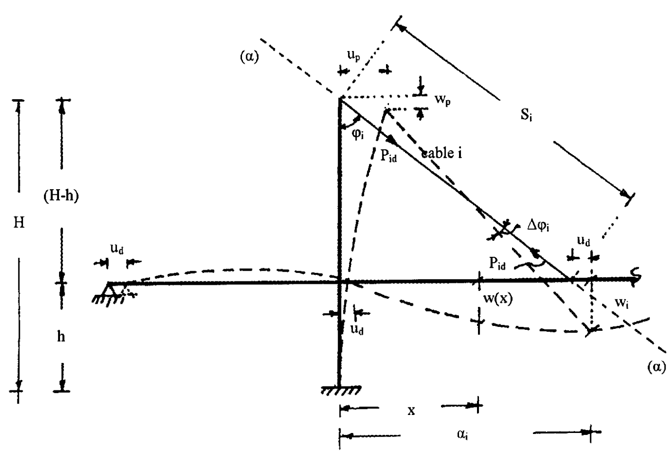


Fig. 2. Deformation of the girder. Cables at the right of the pylon.

**2.2 Relations between $P_{id}$, $w_i$, $u$**

The force which acts at the top of the pylon, because of a dynamic loading and causes the deformation $u$, is: $\sum_{i=1}^{\rho} P_{id} \cdot \sin\varphi_i$, where ρ is the number of the cables, which are connected at the top of the pylon. Then we have:

$$\delta u = u_p - u_d = \frac{H^3}{6E_p I_p}\left[2-3\left(\frac{h}{H}\right)^2+\left(\frac{h}{H}\right)^3\right]\sum_{i=1}^{\rho} P_{id}\cdot\sin\varphi_i \tag{2}$$

If $s_i$ is the length of the cable i under the static loads and $\Delta s_i$ its deformation because of the dynamic loading, it will be:

$$\Delta s_i = \frac{s_i \cdot P_{id}}{E_s \cdot A_i} \tag{3}$$

where $E_s$ is the tangent modulus of elasticity of the material of the cable and $A_i$ the area of its cross-section.

From the geometry of figure 2, we have (projecting on axis a-a):

$$u_p \cdot \sin\varphi_i + w_p \cdot \cos\varphi_i + (s_i + \Delta s_i)\cos\Delta\varphi_i = s_i + u_d \cdot \sin\varphi_i + w_i \cdot \cos\varphi_i$$

or neglecting $w_p$ (as a very small quantity) and putting $\cos\Delta\varphi_i = 1$, we get:

$$(u_p - u_d)\sin\varphi_i + \Delta s_i = w_i \cos\varphi_i \quad \text{or}$$

$$\left\{\frac{H}{6E_p I_p}\left[2-3\left(\frac{h}{H}\right)^2+\left(\frac{h}{H}\right)^3\right]\cdot\sum_{i=1}^{\rho} P_{id}\cdot\sin\varphi_i\right\}\cdot\sin\varphi_i + \frac{s_i P_{id}}{E_s A_i} = w_i\cos\varphi_i \quad \text{for } i=1 \text{ to } \rho$$

and finally:

$$\begin{aligned} &\sin\varphi_i \sum_{i=1}^{\rho}(P_{id}\sin\varphi_i) + \frac{s_i \cdot P_{id}}{B\cdot E_s \cdot A_i} = \frac{\cos\varphi_i}{B}\cdot w_i \\ &\text{where: } B = \frac{H^3}{6E_p I_p}\cdot\left[2-3\left(\frac{h}{H}\right)^2+\left(\frac{h}{H}\right)^3\right] \text{ and } i=1 \text{ to } \rho \end{aligned} \tag{4}$$

From equation (4), we obtain the following system:

$$\begin{aligned} &(\alpha_1 + \sin\varphi_1)\cdot P_{1d} + \sin\varphi_2 P_{2d} + \ldots\ldots\ldots + \sin\varphi_\rho P_{\rho d} = \beta_1 \cdot w_1 \\ &\sin\varphi_1 \cdot P_{1d} + (\alpha_2 + \sin\varphi_2)P_{2d} + \ldots\ldots\ldots + \sin\varphi_\rho P_{\rho d} = \beta_2 \cdot w_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\sin\varphi_1 \cdot P_{1d} + \sin\varphi_2 P_{2d} + \ldots\ldots\ldots + (\alpha_\rho + \sin\varphi_\rho) P_{\rho d} = \beta_\rho \cdot w_\rho \\ &\textit{where}: \quad \alpha_i = \frac{s_i}{B\cdot E_s\cdot A_i\cdot\sin\varphi_i}, \quad \beta_i = \frac{1}{B\cdot\tan\varphi_i} \end{aligned} \tag{5}$$

Solving the above system (5), we can find:

$$\left.\begin{aligned}
&P_{id} = \gamma_{i1} w_1 + \gamma_{i2} w_2 + \cdots\cdots + \gamma_{i\rho} w_\rho\ , \quad \textit{with} \quad i = 1\ \textit{to}\ \rho \quad \textit{and}: \\
&\gamma_{ij} = \frac{\begin{vmatrix}
(a_1+\sin\varphi_1)\cdots\sin\varphi_{j-1} & \sin\varphi_{j+1}\cdots\cdots\cdots\ \sin\varphi_\rho \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\sin\varphi_1\cdots\cdots\cdots(a_{j-1}+\sin\varphi_{j-1}) & \sin\varphi_{j+1}\cdots\cdots\cdots\ \sin\varphi_\rho \\
\sin\varphi_1\cdots\cdots\cdots\ \sin\varphi_{j-1} & (a_{j+1}+\sin\varphi_{j+1})\cdots\sin\varphi_\rho \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\sin\varphi_1\cdots\cdots\cdots\ \sin\varphi_{j-1} & \sin\varphi_{j+1}\cdots\cdots\cdots(a_\rho+\sin\varphi_\rho)
\end{vmatrix}}{\begin{vmatrix}
(a_1+\sin\varphi_1) & \sin\varphi_2\cdots\cdots\cdots\cdots\ \sin\varphi_\rho \\
\sin\varphi_1 & (a_2+\sin\varphi_2)\cdots\cdots\sin\varphi_\rho \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\sin\varphi_1 & \sin\varphi_2\cdots\cdots\cdots\cdots(a_\rho+\sin\varphi_\rho)
\end{vmatrix}}\cdot\beta_j
\end{aligned}\right\} \tag{6}$$

**2.3. The free vibration of a bridge with rare distribution of cables**

The equation of the vertical motion of the deck bridge is:

$$EIyw''''(x,t) + c_b \cdot \dot{w}(x,t) + m\ddot{w}(x,t) + \sum_{i=1}^{\rho} P_{id} \cdot \cos\varphi_i \cdot \delta(x-\alpha_i) = 0 \tag{7}$$

where: $\delta(x-\alpha_i)$ the Dirac's Delta function, while the moments caused by the tensions of the cables are neglected as very small.

Then, because of equation (6a) and of the relation, $w_i = w(\alpha_i)$ equation (7) becomes:

$$EI_y w''''(x,t) + c_b \cdot \dot{w}(x,t) + m\ddot{w}(x,t) + \sum_{i=1}^{\rho} [\gamma_{i1} w(\alpha_1) + \ldots + \gamma_{i\rho} w(\alpha_\rho)] \cdot \cos\varphi_i \cdot \delta(x-\alpha_i) = 0 \tag{7a}$$

We search a solution of separate variables, under the form:

$$w(x,t) = X(x) \cdot T(t) \tag{8}$$

Because of equation (8), equation (7a) becomes:

$$EI_y X'''' \cdot T + c_b X \cdot \dot{T} + \left[\sum_{i=1}^{\rho} [\gamma_{i1} X(\alpha_1) + \ldots + \gamma_{i\rho} X(\alpha_\rho)] \cdot \cos\varphi_i \cdot \delta(x-\alpha_i)\right] \cdot T + m \cdot X \cdot \ddot{T} = 0$$

or

$$\frac{X'''' + \dfrac{1}{EI_y}\left[\sum_{i=1}^{\rho} [\gamma_{i1} \cdot X(\alpha_1) + \ldots + \gamma_{i\rho} \cdot X(\alpha_\rho)] \cdot \cos\varphi_i \cdot \delta(x-\alpha_i)\right]}{\dfrac{m}{EI_y} X} = -\frac{\ddot{T} + \dfrac{c_b}{m}\dot{T}}{T} = \omega^2$$

and finally:

$$\left.\begin{aligned}&X'''' + \frac{1}{EI_y}\left[\sum_{i=1}^{\rho}[\gamma_{i1}X(\alpha_i)+...+\gamma_{i\rho}X(\alpha_\rho)]\cdot\cos\varphi_i\cdot\delta(x-\alpha_i)\right]-\lambda\cdot X=0\\ &\text{where:}\quad \lambda=\frac{m\omega^2}{EI_y}\end{aligned}\right\}\tag{9}$$

In order to apply Galerkin's method, we put:

$$X(x)=c_1\Psi_1(x)+c_2\Psi_2(x)+...+c_n\Psi_n(x)\tag{10}$$

where: $c_i$ unknown coefficients, which will be determined and $\Psi_i(x)$ are arbitrary chosen functions of $x$, which satisfy the boundary conditions. As such functions, we choose the shape functions of the corresponding continuous beam (which has the same characteristics with the bridge but without cables), given by the appendix.

Introducing equation (10) into (9), multiplying the outcome successively by $\Psi_1$, $\Psi_2$, … $\Psi_n$ and integrating the results from 0 to $L$, we obtain the following homogeneous, linear system without second member of n equations, with unknowns $c_1$, $c_2$, …$c_n$.

$$\left.\begin{aligned}&\int_0^L(c_1\Psi_1''''+...+c_n\Psi_n'''')\Psi_\sigma dx+\\ &+\frac{1}{EI_y}\{[\gamma_{11}\{c_1\Psi_1(\alpha_1)+...+c_n\Psi_n(\alpha_1)\}+...+\gamma_{1\rho}\{c_1\Psi_1(\alpha_\rho)+...+c_n\Psi_n(\alpha_\rho)\}]\cdot\cos\varphi_1\cdot\Psi_\sigma(\alpha_1)+\\ &\quad+[\gamma_{21}\{c_1\Psi_1(\alpha_1)+...+c_n\Psi_n(\alpha_1)\}+...+\gamma_{2\rho}\{c_1\Psi_1(\alpha_\rho)+...+c_n\Psi_n(\alpha_\rho)\}]\cdot\cos\varphi_2\cdot\Psi_\sigma(\alpha_2)+\\ &\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad+[\gamma_{\rho1}\{c_1\Psi_1(\alpha_1)+...+c_n\Psi_n(\alpha_1)\}+...+\gamma_{\rho\rho}\{c_1\Psi_1(\alpha_\rho)+...+c_n\Psi_n(\alpha_\rho)\}]\cdot\cos\varphi_\rho\cdot\Psi_\sigma(\alpha_\rho)-\\ &\quad-\lambda\int_0^L(c_1\Psi_1+...+c_n\Psi_n)\Psi_\sigma dx=0 \qquad\qquad (\text{for }\ \sigma=1\ \text{ to }\ n)\end{aligned}\right\}\tag{11}$$

The above equations (11) constitute a linear homogeneous system of $c_1$, $c_2$, ….$c_n$, without second member. In order to have a non trivial solution, the determinant of the coefficients of $c_1$, $c_2$,….,$c_n$, must be equal to zero:

$$|\Delta_{ij}|=0\tag{12}$$

From the above equation (12) one can find the eigenfrequencies of the bridge.

### 2.4 The free vibration of a bridge with dense distribution of cables

Let us consider that the cables are placed very densely, at a distance $\delta << (\alpha_2-\alpha_1)$ (see Figure 3). So, we can consider a distributed load $q(x)$, extended from $\alpha_1$ to $\alpha_2$ which at $x_I$ will be:

$$q(x_i)=\frac{1}{\delta}P_{id}\cdot\cos\varphi_i\tag{13}$$

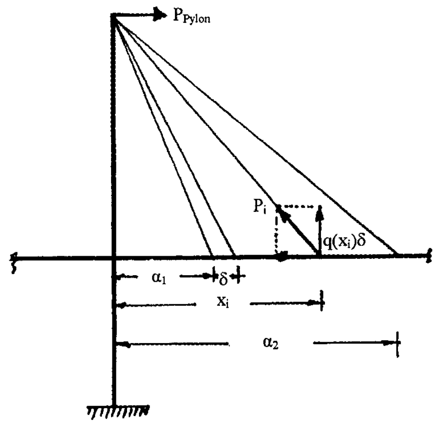


Fig. 3. The load $q(x)$, which expresses the effect of the cables on the bridge deck.

It is evident that it is valid:

$$P_{pylon} = \int_{\alpha_1}^{\alpha_2} q(x) \cdot \tan\varphi \cdot dx \tag{14}$$

And equation (2) becomes:

$$\begin{aligned}(\delta u) = u_p - u_d &= \frac{H^3}{6E_pI_p} \cdot \left[2 - 3\left(\frac{h}{H}\right)^2 + \left(\frac{h}{H}\right)^3\right] \cdot \int_{\alpha_1}^{\alpha_2} q(x) \cdot \tan\varphi \cdot dx = \\ &= \frac{H^3}{6E_pI_p(H-h)} \cdot \left[2 - 3\left(\frac{h}{H}\right)^2 + \left(\frac{h}{H}\right)^3\right] \cdot \int_{\alpha_1}^{\alpha_2} x \cdot q(x) \cdot dx\end{aligned} \tag{15}$$

For the cable $i$, is valid that:

$$\delta u \cdot \sin\varphi_i + \frac{s_i \cdot P_{id}}{E_S \cdot A_i} = w_i \cdot \cos\varphi_i \tag{16}$$

and equation (16), because of (15), becomes:

$$\left\{\frac{H^3}{6E_pI_p(H-h)} \cdot \left[2 - 3\left(\frac{h}{H}\right)^2 + \left(\frac{h}{H}\right)^3\right] \cdot \int_{a_1}^{a_2} x \cdot q(x)\, dx\right\} \cdot \sin\varphi + \frac{s_i}{E_S A(x)} \cdot \frac{\delta q(x_i)}{\cos\varphi_i} = w_i \cdot \cos\varphi_i \tag{17}$$

where:

$$A(x) = \frac{A_i}{\delta} \tag{18}$$

the, conventional, distributed area of cross-section of the cables, which is a function of $x$.

*2.4.1 First case A(x)=constant.*

We symbolize:

$$\left.\begin{array}{r}\dfrac{H^3}{6EJ_p(H-h)}\left[2-3\left(\dfrac{h}{H}\right)^2+\left(\dfrac{h}{H}\right)^3\right]=A\\[2ex]\dfrac{1}{E\,A(x)}=B\end{array}\right\}\tag{19}$$

On the other hand we have: $s_i\,\dfrac{x_i}{\sin\varphi}$. Then, equation (17) becomes:

$$A\cdot\sin\varphi\cdot\int_{\alpha_1}^{\alpha_2}xq(x)dx+B\cdot\frac{xq(x)}{\sin\varphi\cdot\cos\varphi}=w(x)\cos\varphi\ \text{ or}$$

$A\cdot\sin^2\varphi\cdot\cos\varphi\int_{\alpha_1}^{\alpha_2}xq(x)dx+Bxq(x)=w(x)\cdot\sin\varphi\cdot\cos^2\varphi$ or from the geometry of figure 2:

$$A\cdot\frac{(H-h)x^2}{[x^2+(H-h)^2]^{3/2}}\cdot\int_{\alpha_1}^{\alpha_2}xq(x)+Bxq(x)=w(x)\cdot\frac{(H-h)^2x}{[x^2+(H-h)^2]^{3/2}}\tag{20}$$

After integration from $\alpha_1$ to $\alpha_2$ we get:

$$A\left(\int_{\alpha_1}^{\alpha_2}\frac{(H-h)\,x^2dx}{[x^2+(H-h)^2]^{3/2}}\right)\left(\int_{\alpha_1}^{\alpha_2}xq(x)\,dx\right)+B\int_{\alpha_1}^{\alpha_2}xq(x)\,dx=(H-h)^2\int_{\alpha_1}^{\alpha_2}\frac{xw(x)\,dx}{[x^2+(H-h)^2]^{3/2}}\tag{21}$$

We can find:

$$I_c=\int_{\alpha_1}^{\alpha_2}\frac{(H-h)\,x^2dx}{[x^2+(H-h)^2]^{3/2}}=$$

$$=(H-h)\left\{\ell_n\frac{\alpha_2+\sqrt{\alpha_2^2+(H-h)^2}}{\alpha_1+\sqrt{\alpha_1^2+(H-h)^2}}-\frac{\alpha_2\sqrt{\alpha_1^2+(H-h)^2}-\alpha_1\sqrt{\alpha_2^2+(H-h)^2}}{\sqrt{[\alpha_1^2+(H-h)^2]\cdot[\alpha_2^2+(H-h)^2]}}\right\}$$

After some manipulations we reach:

$$(AI_c+B)\int_{\alpha_1}^{\alpha_2}xq(x)dx=(H-h)^2\int_{\alpha_1}^{\alpha_2}\frac{xw(x)dx}{[x^2+(H-h)^2]^{3/2}}\tag{22}$$

And finally from equation (20), we can find:

$$q(x)=\frac{(H-h)^2}{B\cdot[x^2+(H-h)^2]^{3/2}}\cdot\left[w(x)-\frac{A(H-h)\cdot x}{AI_c+B}\cdot\int_{\alpha_1}^{\alpha_2}\frac{x\cdot w(x)dx}{[x^2+(H-h)^2]^{3/2}}\right]\tag{23}$$

*2.4.2. Second case*, $A(x)$ = *variable.*

It is usual to suppose [12] that the area of the cross-sections of the cables changes under the formula:

$$A(x) = \frac{g}{\sigma_g \cdot \cos\varphi} \tag{24}$$

where:

$g$ : the uniform distributed deck own load

$\sigma_g$ : the initial tension of the stays curtain, due to the above g . It is : $\sigma_g = \sigma_\alpha \dfrac{g}{g+p}$

where $\sigma_a$ the allowable stress of cables and p the design live load.

Then we symbolize:

$$\left.\begin{aligned} \frac{H^3}{6EJ_p(H-h)}\left[2-3\left(\frac{h}{H}\right)^2+\left(\frac{h}{H}\right)^3\right] &= A \\ \frac{\sigma_g}{E\cdot g} &= B \end{aligned}\right\} \tag{25}$$

and because of $s_i = \dfrac{x_i}{\sin\varphi_i}$ , equation (17) becomes:

$$A\cdot\sin\varphi\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx + B\cdot\frac{x\cdot q(x)}{\sin\varphi} = w(x)\cdot\cos\varphi \qquad \text{or}$$

$$A\cdot\sin^2\varphi\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx + B\cdot x\cdot q(x) = w(x)\cdot\sin\varphi\cdot\cos\varphi$$

or from the geometry of Figure 2:

$$A\cdot\frac{x^2}{x^2+(H+h)^2}\cdot\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx + B\cdot x\cdot q(x) = w(x)\cdot\frac{(H-h)\cdot x}{x^2+(H-h)^2} \tag{26}$$

After integration from $\alpha_1$ to $\alpha_2$ , we get:

$$A\cdot\left(\int_{\alpha_2}^{\alpha_1}\frac{x^2\cdot dx}{x^2+(H-h)^2}\right)\cdot\left(\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx\right) + B\cdot\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx = (H-h)\cdot\int_{\alpha_2}^{\alpha_1}\frac{x\cdot w(x)\cdot dx}{x^2+(H-h)^2} \tag{27}$$

We can find:

$$I_V = \int_{\alpha_2}^{\alpha_1}\frac{x^2\cdot dx}{x^2+(H-h)^2} = (\alpha_2-\alpha_1)-(H-h)\left(arc\cdot\tan\frac{\alpha_2}{H-h} - arc\cdot\tan\frac{\alpha_1}{H-h}\right)$$

After some manipulations we reach:

$$A\cdot I_V\cdot\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx + B\cdot\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx = (H-h)\cdot\int_{\alpha_1}^{\alpha_2}\frac{x\cdot w(x)\cdot dx}{x^2+(H-h)^2} \qquad \text{or}$$

$$\int_{\alpha_1}^{\alpha_2} x\cdot q(x)\cdot dx = \frac{H-h}{AI_V+B}\cdot\int_{\alpha_1}^{\alpha_2}\frac{x\cdot w(x)\cdot dx}{x^2+(H-h)^2} \tag{28}$$

And, finally, from equation (26), we can find:

$$q(x)=\frac{(H-h)}{B\cdot[x^2+(H-h)^2]}\cdot\left\{w(x)-\frac{Ax}{AI_V+B}\cdot\int_{\alpha_1}^{\alpha_2}\frac{x\cdot w(x)\cdot dx}{x^2+(H-h)^2}\right\} \tag{29}$$

Or, under a general form, we can write:

$$\left.\begin{array}{l}
q(x,t)=F_1(x)\cdot[w(x,t)-F_2(x)\cdot\int_{\alpha_1}^{\alpha_2}F_3(x)\cdot w(x,t)\cdot dx] \\
\text{Where: for A = const.} \\
F_1(x)=\dfrac{(H-h)^2}{B\cdot[x^2+(H-h)^2]^{3/2}}\ ,\ F_2(x)=\dfrac{A(H-h)}{AI_C+B}\cdot x\ ,\ F_3(x)=\dfrac{x}{[x^2+(H-h)^2]^{3/2}} \\
\text{and for } A=\text{variable} \\
F_1(x)=\dfrac{H-h}{B\cdot[x^2+(H-h)^2]}\ ,\ F_2(x)=\dfrac{Ax}{A\,I_V+B}\ ,\ F_3(x)=\dfrac{x}{x^2+(H-h)^2}
\end{array}\right\} \tag{30}$$

For the case of Figure 4, namely for the case of a bridge deck that is at the left of the pylon, we have:

$$\left.\begin{array}{l}
\text{for A = const.} \\
F_1(x)=\dfrac{(H-h)^2}{B\cdot[(\ell-x)^2+(H-h)^2]^{3/2}}\ ,F_2(x)=\dfrac{A(H-h)}{AI_C+B}\cdot(\ell-x)\ ,F_3(x)=\dfrac{\ell-x}{[(\ell-x)^2+(H-h)^2]^{3/2}} \\
\text{for } A=\dfrac{g}{\sigma_g\cdot\cos\varphi}\ , \\
F_1(x)=\dfrac{H-h}{B\cdot[(\ell-x)^2+(H-h)^2]}\ ,\ F_2(x)=\dfrac{A(\ell-x)}{A\,I_V+B}\ ,\ F_3(x)=\dfrac{\ell-x}{(\ell-x)^2+(H-h)^2}
\end{array}\right\} \tag{30a}$$

We can now find the eigenfrequencies of the bridge.
The equation of the vertical motion of the deck bridge is:

$$EI_y w''''(x,t)+c_b\dot{w}(x,t)+m\ddot{w}(x_1t)=-q(x,t) \tag{31}$$

with $q(x,t)$ given by equation (30).
We search for a solution of separate variables, under the form:

$$w(x,t)=X(x)\cdot T(t) \tag{32}$$

Because of equations (32) and (30), equation (31) becomes:

$$EI_yX''''\cdot T+c_b\cdot X\dot{T}+mX\ddot{T}=-F_1\left[X-F_2\int_{\alpha_1}^{\alpha_2}F_3\cdot X\cdot dx\right]\cdot T\quad\text{or}$$

$$\frac{X'''' + \dfrac{1}{EI_y} F_1 \left[ X - F_2 \int_{\alpha_1}^{\alpha_2} F_3 \cdot X \cdot dx \right]}{\dfrac{m}{EI_y} \cdot X} = -\frac{\ddot{T} + \dfrac{c_b}{m} \dot{T}}{T} = \omega^2 \text{ or}$$

$$\left. \begin{aligned} & X'''' + \frac{1}{EI_y} F_1 \left[ X - F_2 \int_{\alpha_1}^{\alpha_2} F_3 \cdot X \cdot dx \right] - \lambda X = 0 \\ & \ddot{T} + \frac{c_b}{m} \cdot \dot{T} + \omega^2 T = 0 \\ & \text{where : } \quad \lambda = \frac{m\omega^2}{EI_y} \end{aligned} \right\} \tag{33}$$

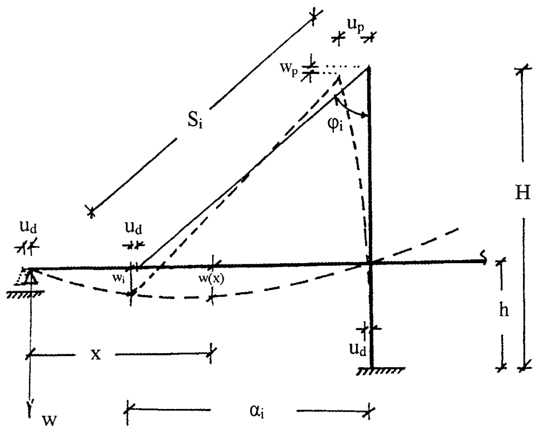


Fig. 4. Deformation of the girder. Cables at the left of the pylon.

In order to apply Galerkin's method, we set (as in §2.3) :

$$X(x) = c_1\Psi_1(x) + c_2\Psi_2(x) + ..... + c_n\Psi_n(x) \tag{34}$$

with the notes and restrictions of §2.3.

Introducing (34) into (33), multiplying the outcome successively by $\Psi_1$ , $\Psi_2$ ,….,$\Psi_n$ and integrating the results from 0 to L, we obtain the following homogeneous, linear system, without second member of n equations, with unknowns $c_1$ ,$c_2$ ,…..,$c_n$ .

$$c_1(A_{i1} - \lambda B_{i1}) + c_2(A_{i2} - \lambda B_{i2}) + ....... + c_n(A_{in} - \lambda B_{in}) = 0, \quad (i = 1,2,....., n) \tag{35}$$

where:

$$\left. \begin{aligned} A_{ij} &= \int_0^L [\Psi_j'''' + \frac{1}{EI_y} \cdot F_1(x) \cdot \Psi_j - \frac{1}{EI_y} \cdot F_1(x) \cdot F_2(x) \cdot (\int_{\alpha_1}^{\alpha_2} F_3(x)\Psi_j \cdot dx)] \cdot \Psi_i \cdot dx \\ B_{ij} &= \int_0^L \Psi_i \cdot \Psi_j \cdot dx \end{aligned} \right\} \tag{36}$$

In order for the above system to have non-trivial solutions, the determinant of its coefficients must be zero:

$$|\Gamma_{ij}| = 0\,,\ \text{with}\ i,j = 1,2,....,n\ \text{and}\ \Gamma_{ij} = A_{ij} - \lambda \cdot B_{ij} \tag{37}$$

From equations (37), we determine the values of λ and from equation (33b), the spectrum of the flexural eigenfrequencies $\omega_i$ .

From the first $(n-1)$ equations of system (35), we can finally find:

$$\left.\begin{array}{l} \dfrac{c_j}{c_1} = \dfrac{\begin{vmatrix} \Gamma_{12}\ldots\ldots\ldots\ldots\Gamma_{11}\ldots\ldots\ldots\ldots\Gamma_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Gamma_{(n-1)2}\ldots\ldots\ldots\Gamma_{(n-1)1}\ldots\ldots\ldots\Gamma_{(n-1)n} \end{vmatrix}}{|\Gamma_{ij}|} \quad \text{with}: i = 1,...,(n-1), \quad j = 2,..,(n) \\ \text{and therefore}: \quad X_n(x) = c_1 \displaystyle\sum_{j=2}^{n} \left( \Psi_1 + \frac{c_j}{c_1} \Psi_j \right) \end{array}\right\} \tag{38}$$

From equation (33b), we can find the time function of the free vibration:

$$\left.\begin{array}{lll} T(t) = e^{-\beta t}(A_1 \sin \overset{*}{\omega} t + B_1 \cos \overset{*}{\omega} t) & for & \dfrac{c_b}{2 \cdot m} < \omega \\ T(t) = e^{-\beta t}(A_2 + B_2 \cdot t) & for & \dfrac{c_b}{2 \cdot m} = \omega \\ T(t) = e^{-\beta t}(A_3 \sinh \overset{*}{\omega} t + \cosh \overset{*}{\omega} t) & for & \dfrac{c_b}{2 \cdot m} > \omega \end{array}\right\} \tag{38a}$$

where: $\beta = \dfrac{c_b}{2m}$ is the, so called, "damping quantity" and $\overset{*}{\omega} = \sqrt{\omega^2 - \beta^2}$ .

The two last cases **are impossible** to take place for usual cable-stayed bridges. That is possible to happen only **when** $I_y \to 0$ **and** $I_p$ **has very small values** (like some primitive bridges in jungle where girder strength is practically nonexistent). Nevertheless, it is remarkable that through appropriate values of $A_3$ and $B_3$, (which are depended on initial conditions), the third case conducts to a flutter instability.

### *2.5. The forced vibrations*

The equation of motion for the flexural forced vibration, is given by:

$$EI_y w''''(x,t) + c_b \cdot \dot{w}(x,t) + m\ddot{w}(x,t) + q(x,t) = p(x,t) \tag{39}$$

where $q(x,t)$ is given by equ.(30) while p(x,t) is the external dynamic load.

We seek a solution under the form:

$$w(x,t) = \sum_n X_n(x) \cdot P_n(x) \tag{40}$$

where $P_n(t)$, unknown functions of the time, which will be determined and $X_n(x)$ are functions of $x$, arbitrarily chosen, which satisfy the boundary conditions.

As such functions, we choose the shape functions of the corresponding continuous beam (which continuous beam has the same characteristics with the bridge but without cables), given by the appendix.

Then equation (39) becomes:

$$EI_y \sum_n X_n'''' \cdot P_n + c_b \sum_n X_n \dot{P}_n + m \sum_n X_n \ddot{P}_n + F_1 [\sum_n X_n P_n - F_2 \int_{\alpha_1}^{\alpha_2} F_3 \sum_n X_n P_n] = p(x,t) \quad (41)$$

The Shape functions $X_n$, satisfy the equation of motion of the freely vibrating corresponding continuous beam:

$$EI_y X_n'''' - m\omega_n^2 X_n = 0$$

And equation (41) becomes:

$$m \sum_n \omega_n^2 X_n P_n + c_b \sum_n X_n \dot{P}_n + m \sum_n X_n \ddot{P}_n + F_1 [\sum_n X_n P_n - F_2 \int_{\alpha_1}^{\alpha_2} F_3 \sum_n X_n P_n] = p(x,t) \quad (42)$$

After multiplication by $X_i$ and integration of the outcome from 0 to $L$, we get:

$$\begin{aligned}
& m(\omega_1 P_1 \int_0^L X_1 X_i dx + \omega_2 P_2 \int_0^L X_2 X_i dx + \cdots + \omega_n P_n \int_0^L X_n X_i dx) + \\
& + c_b (\dot{P}_1 \int_0^L X_1 X_i dx + \dot{P}_2 \int_0^L X_2 X_i dx + \cdots + \dot{P}_n \int_0^L X_n X_i dx) + \\
& + m (\ddot{P}_1 \int_0^L X_1 X_i dx + \ddot{P}_2 \int_0^L X_2 X_i dx + \cdots + \ddot{P}_n \int_0^L X_n X_i dx) + \\
& + (P_1 \int_0^L F_1 X_1 X_i dx + P_2 \int_0^L F_1 X_2 X_i dx + \cdots + P_n \int_0^L F_1 X_n X_i dx) - \\
& - [P_1 \cdot (\int_{\alpha_1}^{\alpha_2} F_3 X_1 dx)(\int_0^L F_1 F_2 X_i dx) + P_2 (\int_{\alpha_1}^{\alpha_2} F_3 X_2 dx)(\int_0^L F_1 F_2 X_i dx) + \cdots + P_n (\int_{\alpha_1}^{\alpha_2} F_3 X_n dx)(\int_0^L F_1 F_2 X_i dx)] = \\
& = \bar{f}(t) \int_0^L \overline{p}(x) X_i \, dx
\end{aligned}$$

(because, we considered that $p(x,t) = \overline{p}(x) \cdot \bar{f}(t)$ ).

Or finally

$$\left.\begin{aligned}
& \sum_{i=1}^n D_{ij} P_i + c_b \sum_{i=1}^n G_{ij} \dot{P}_i + m \sum_{i=1}^n G_{ij} \ddot{P}_i = R_j \\
& for \quad j = 1 \ to \ n \\
\text{and :} \quad & D_{ij} = m\omega_i \int_0^L X_j X_i \, dx + \int_0^L F_1 X_j X_i \, dx - (\int_{\alpha_1}^{\alpha_2} F_3 X_j dx)(\int_0^L F_1 F_2 X_i dx) \\
& G_{ij} = \int_0^L X_j X_i \, dx \\
& R_j = \int_0^L \overline{p}(x) X_j dx
\end{aligned}\right\} \quad (43)$$

In order to solve the above differential system, we use the Carlson-Laplace transformation with the following initial conditions:

$$w(x,0) = \dot{w}(x,0) = 0 \tag{45}$$

$$\left.\begin{array}{ll} \text{We put:} & L\,P_i(t) = g_i(p) \\ \text{and therefore:} & L\,\dot{P}_i(t) = p\,g_i(p) - pP_i(0) = p\,g_i(p) \\ & L\,\ddot{P}_i(t) = p^2 g_i(p) - p^2 P_i(0) - p\dot{P}_i(0) = p^2 g_i(p) \\ & L\,\bar{f}(t) = F(p) \end{array}\right\} \tag{46}$$

Then, equation (43), get the form:

$$(D_{1j} + c_b G_{1j}\,p + mG_{1j}\,p^2)\,g_1(p) + (D_{2j} + c_b G_{2j}\,p + m\,G_{2j}\,p^2)\,g_2(p) + \cdots\cdots$$
$$\cdots + (D_{nj} + c_b\,G_{nj}\,p + m\,G_{nj}\,p^2)\,g_n(p) = R_j F(p) \tag{47}$$
$$with \quad j = 1 \ to \ n$$

The usual forms of functions $F(p)$, are rational functions of $p$.

Then , solving the system (47), $g_i(p)$ takes the following form:

$$g_i(p) = \frac{N_i(p)}{M_i(p)} \qquad with \quad i = 1 \ to \ n \tag{48}$$

where $N_i$, $M_i$ are polynomials with respect to p with $M_i(p)$ of higher order than $N_i(p)$.

Heaviside's rule can thus be applied, leading finally to equation:

$$P_i(t) = L^{-1} g_i(p) = L^{-1}\frac{N_i(p)}{M_i(p)} = \frac{N_i(0)}{M_i(0)} + \sum_{k=1}^{\ell} \frac{N_i(\rho_k)\,e^{\rho_k t}}{\rho_k\,M_i'(\rho_k)} \tag{49}$$

## 3. NUMERICAL RESULTS AND DISCUSSION

In this section a numerical investigation based on the equations obtained in the previous paragraphs has been developed. The above equations connect the deformations of the deck to the load $q(x)$ (that expresses the effect of the tensions of the cables). The individual and coupling effects of the mechanical and geometrical parameters are discussed in detail.

The mathematical model discussed herein is related to a real three-span bridge with a fan-shaped system of cables along the girder (like in Fig. 5), middle span $\ell_2 = 400$ m, equal site spans $\ell_1 = \ell_3 = 120$ m, weight per unit length $g = \sim 8000$ kN/m, and moment of inertia of the deck $I_d = \sim 0.21$ m$^4$. The above data are combined with the design loads $p/g = 0.5$ and 1.0 (this ratio is connected to the cross-section area of the cables), moments of inertia of the pylon $I_p / I_d = 0.5$, 1.0, 1.5, 2.0, 2.5, total height of pylons $L/H = 4$, 5, 6, 7, and distance of the deck under-surface from the surface of the earth $h/H = 0.50$, 0.25, and 0.125.

The three eigenfrequencies $\omega_1$, $\omega_2$ and $\omega_3$ are found for the above cases. Numerical results, concerning the above research, are given in Table A.

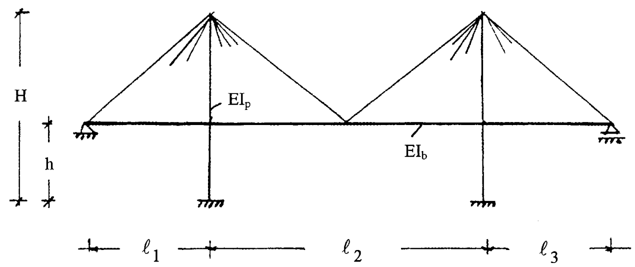


Fig. 5. Cable stayed-bridge for the illustrated example.

From the plots of Fig. 6, 7, 8 we can see, respectively, the three eigenfrequencies of the free vibrations as a function of the parameter h/H for the cases $I_p / I_b = 0.5$, 1.5, 2.5. The plots of Fig. 9, 10, 11 show the three eigenfrequencies as a function of parameter $I_p / I_b$, for the cases h/H = 0.5, 0.25, and 0.125.

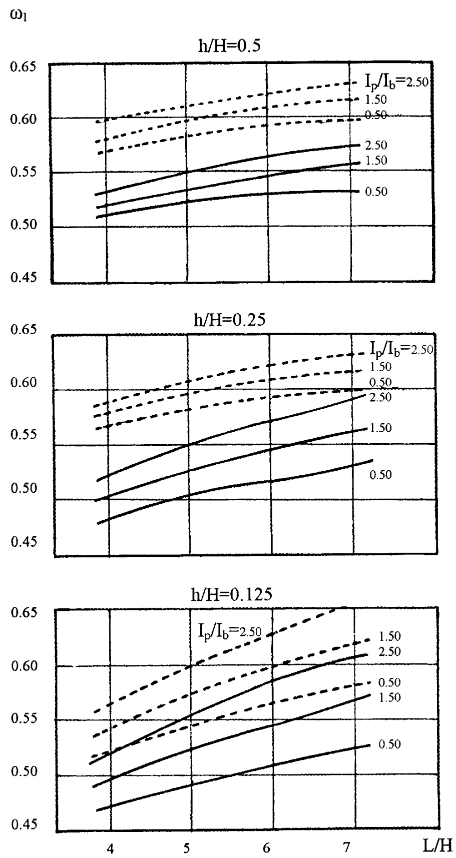


Fig. 6. Relationship between the first eigenfrequency $\omega_1$ and the ratio $L/H$ for various values of $I_p/I_b$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

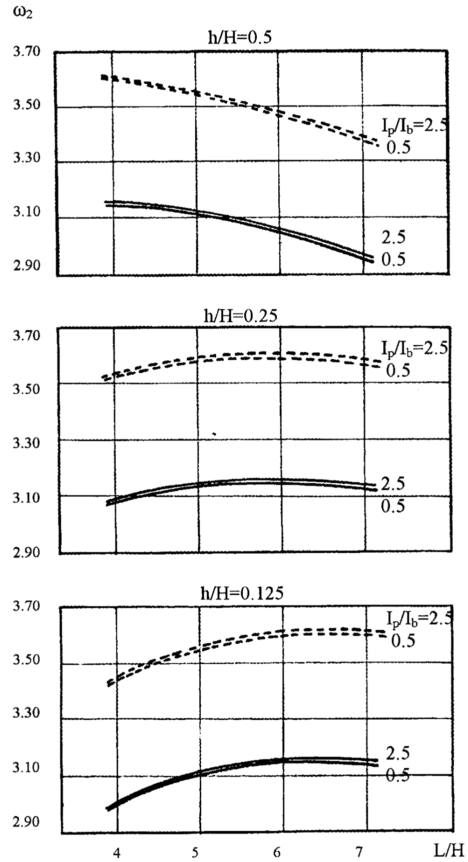


Fig. 7. Relationship between the first eigenfrequency $\omega_2$ and the ratio $L/H$ for various values of $I_p/I_b$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

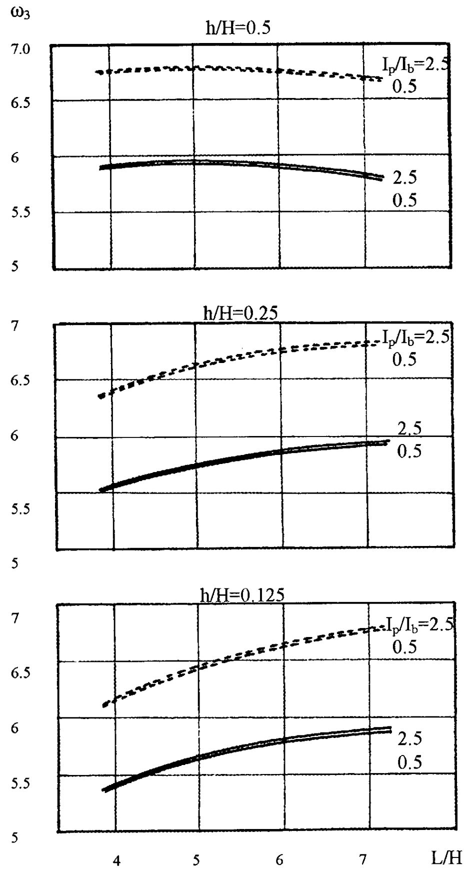


Fig. 8. Relationship between the first eigenfrequency $\omega_3$ and the ratio $L/H$ for various values of $I_p/I_b$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

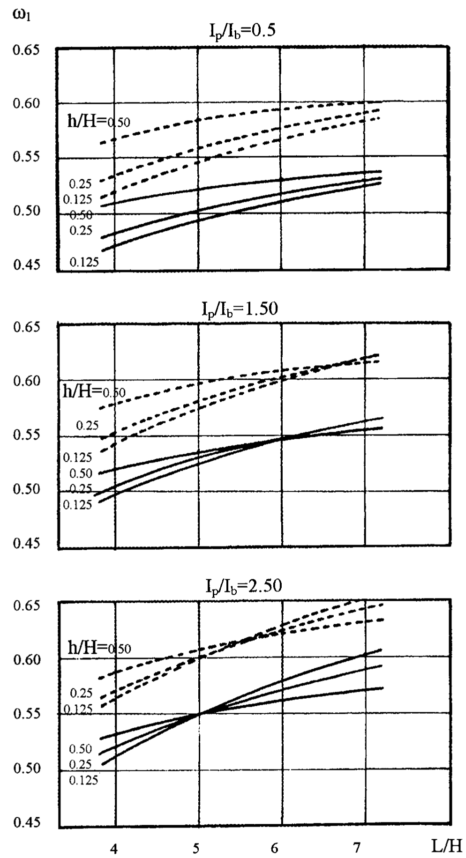


Fig. 9. Relationship between the first eigenfrequency $\omega_1$ and the ratio $L/H$ for various values of $h/H$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

The first significant remark is that if the characteristics of the deck remain unchanged, while the other parameters (like the pylon height, pylon moment of inertia, cross-section area of the cables, distance of the deck under-surface from the surface of the earth) change, then we can affect the eigenfrequencies only a little.

The maximum effect of the above secondary parameters (like the pylon height, pylon moment of inertia, cross-section area of the cables, distance of the deck under-surface from the surface of the earth) is for the first eigenfrequency ~15.5%, for the second ~6.8% and for the third one ~9.5%.

Figures 6 to 11 show clearly those influences.

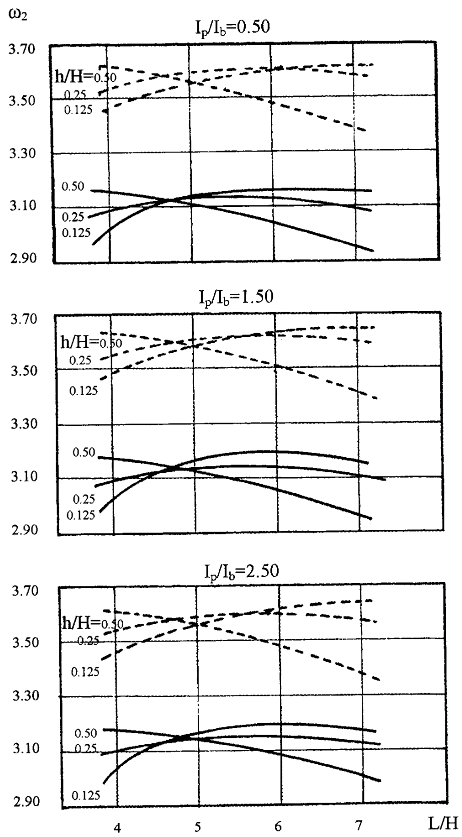


Fig. 10. Relationship between the first eigenfrequency $\omega_2$ and the ratio $L/H$ for various values of $h/H$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

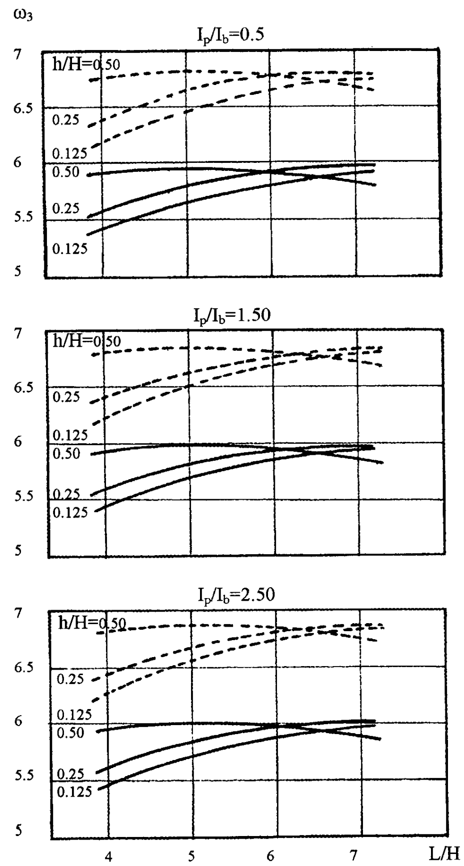


Fig. 11. Relationship between the first eigenfrequency $\omega_3$ and the ratio $L/H$ for various values of $h/H$.
___ $p/g = 0.50$ - - - - $p/g = 1.00$

## 4. CONCLUSIONS

On the basis of the model chosen, we may draw the following conclusions:

1. The function $q(x)$, giving the load that expresses the effect of the cables on the bridge deck, is determined.
2. A simplified model, based on the above found load $q(x)$ is used for a quick dynamic test of a bridge.
3. The basic parameters, which affect the eigenfrequencies of a bridge, are those of the deck of the bridge.
4. The parameters connected to the pylons and the cables of a bridge affect less the eigenfrequencies and their effect comes up to about 12%.

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## APPENDIX

### 1. Two-span continuous beam

Eigenfrequencies equation:

$$\begin{aligned}&\cosh\lambda\ell_2\cdot\sin\lambda\ell_1\cdot\sin\lambda\ell_2\cdot\sinh\lambda\ell_1+\cosh\lambda\ell_1\cdot\sin\lambda\ell_1\cdot\sin\lambda\ell_2\cdot\sinh\lambda\ell_2\\&-\cos\lambda\ell_2\cdot\sinh\lambda\ell_1\cdot\sinh\lambda\ell_1\cdot\sinh\lambda\ell_2-\cos\lambda\ell_1\cdot\sinh\lambda\ell_2\cdot\sinh\lambda\ell_1\cdot\sinh\lambda\ell_2=0\end{aligned}$$

Shape-functions equation:

$$\Psi_n(x_1)=\frac{1}{\sin\lambda_n\ell_1}\sin\lambda_n x_1-\frac{1}{\sinh\lambda_n\ell_1}\sinh\lambda_n x_1 \qquad \mathit{for}\ 0\le x_1\le\ell_1$$

$$\Psi_n(x_2)=-\cot\lambda_n\ell_2\sin\lambda_n x_2+\cos\lambda_n x_2+\coth\lambda_n\ell_2\sinh\lambda_n x_2-\cosh\lambda_n x_2 \quad \mathit{for}\ 0\le x_2\le\ell_2$$

where: $\lambda_n=\left(\dfrac{m\,\omega_n^2}{E\,I}\right)^{0.25}$

### 2. Three-span continuous beam

Eigenfrequencies equation:

$$\begin{aligned}
&\cosh\lambda_n\ell_2\cdot\cosh\lambda_n\ell_3\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1+\\
&\cosh\lambda_n\ell_1\cdot\cosh\lambda_n\ell_3\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_2-\\
&\cos\lambda_n\ell_2\cdot\cosh\lambda_n\ell_3\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_2-\\
&\cos\lambda_n\ell_1\cdot\cosh\lambda_n\ell_3\cdot\sin\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_2+\\
&\cosh\lambda_n\ell_1\cdot\cosh\lambda_n\ell_2\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_3-\\
&\cos\lambda_n\ell_3\cdot\cosh\lambda_n\ell_2\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_2\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3+\\
&\cos^2\lambda_n\ell_2\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3-\\
&2\cdot\cos\lambda_n\ell_2\cdot\cosh\lambda_n\ell_2\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3+\\
&\cosh^2\lambda_n\ell_2\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3-\\
&\cos\lambda_n\ell_1\cdot\cosh\lambda_n\ell_2\cdot\sin\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3+\\
&\sin\lambda_n\ell_1\cdot\sin^2\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_3-\\
&\cos\lambda_n\ell_3\cdot\cosh\lambda_n\ell_1\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_2\cdot\sinh\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3-\\
&\cos\lambda_n\ell_2\cdot\cosh\lambda_n\ell_1\cdot\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3+\\
&\cos\lambda_n\ell_2\cdot\cos\lambda_n\ell_3\cdot\sin\lambda_n\ell_1\cdot\sinh\lambda_n\ell_{12}\cdot\sinh\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3+\\
&\cos\lambda_n\ell_1\cdot\cos\lambda_n\ell_3\cdot\sin\lambda_n\ell_2\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3+\\
&\cos\lambda_n\ell_1\cdot\cos\lambda_n\ell_2\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3-\\
&\sin\lambda_n\ell_1\cdot\sin\lambda_n\ell_3\cdot\sinh\lambda_n\ell_1\cdot\sinh^2\lambda_n\ell_2\cdot\sinh\lambda_n\ell_3=0
\end{aligned}$$

Shape-functions equation:

$$\Psi_n(x_1)=\frac{1}{\sin\lambda_n\ell_1}\sin\lambda_n x_1-\frac{1}{\sinh\lambda_n\ell_1}\sinh\lambda_n x_1 \qquad \textit{for } 0\le x_1\le\ell_1$$

$$\begin{aligned}
\Psi_n(x_2)=&\left(-\cot\lambda_n\ell_2+\frac{C}{\sin\lambda_n\ell_2}\right)\sin\lambda_n x_2+\cos\lambda_n x_2\\
&+\left(\coth\lambda_n\ell_2-\frac{C}{\sinh\lambda_n\ell_2}\right)\sinh\lambda_n x_2-\cosh\lambda_n x_2 \qquad \textit{for } 0\le x_2\le\ell_2
\end{aligned}$$

$$\begin{aligned}
\Psi_n(x_3)=&-C\cdot\cot\lambda_n\ell_3\cdot\sin\lambda_n\ell_3+C\cdot\cos\lambda_n x_3\\
&+C\cdot\coth\lambda_n\ell_3\cdot\sinh\lambda_n x_3-C\cdot\cosh\lambda_n x_3 \qquad \textit{for } 0\le x_3\le\ell_3
\end{aligned}$$

where: $\lambda_n=\left(\dfrac{m\,\omega_n^2}{E\,I}\right)^{0.25}$

and $C=\dfrac{\sin\lambda_n\ell_2-\sinh\lambda_n\ell_2}{\sin\lambda_n\ell_2\cdot\sinh\lambda_n\ell_2(\coth\lambda_n\ell_2+\coth\lambda_n\ell_3-\cot\lambda_n\ell_2-\cot\lambda_n\ell_3)}$

Table A

| | p/g | 0.50 | | | | | | | | | 1.00 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | h/H | 0.500 | | | 0.250 | | | 0.125 | | | 0.500 | | | 0.250 | | | 0.125 | | |
| L/H | $I_p/I_b$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
| 4 | 0.5 | 0.5107 | 3.1469 | 5.9193 | 0.4833 | 3.0844 | 5.5946 | 0.4722 | 3.0148 | 5.4034 | 0.5690 | 3.6085 | 6.7882 | 0.5349 | 3.5353 | 6.4106 | 0.5206 | 3.4541 | 6.1881 |
| | 1.0 | 0.5163 | 3.1481 | 5.9198 | 0.4934 | 3.0862 | 5.5952 | 0.4853 | 3.0117 | 5.4042 | 0.5741 | 3.6095 | 6.7886 | 0.5440 | 3.5369 | 6.4111 | 0.5324 | 3.4561 | 6.1888 |
| | 1.5 | 0.5219 | 3.1493 | 5.9202 | 0.5033 | 3.0880 | 5.5959 | 0.4980 | 3.0193 | 5.4050 | 0.5791 | 3.6106 | 6.7890 | 0.5529 | 3.5385 | 6.4117 | 0.5440 | 3.4581 | 6.1895 |
| | 2.0 | 0.5227 | 3.1506 | 5.9207 | 0.5129 | 3.0899 | 5.5965 | 0.5104 | 3.0216 | 5.4058 | 0.5841 | 3.6117 | 6.7894 | 0.5618 | 3.5402 | 6.4122 | 0.5554 | 3.4601 | 6.1901 |
| | 2.5 | 0.5329 | 3.1518 | 5.9212 | 0.5224 | 3.0917 | 5.5972 | 0.5224 | 3.0239 | 5.4066 | 0.5891 | 3.6127 | 6.7898 | 0.5704 | 3.5417 | 6.4128 | 0.5665 | 3.4621 | 6.1908 |
| 5 | 0.5 | 0.5243 | 3.1057 | 5.9659 | 0.5042 | 3.1423 | 5.8096 | 0.4951 | 3.1090 | 5.6699 | 0.5850 | 3.5606 | 6.8422 | 0.5593 | 3.6027 | 6.6606 | 0.5474 | 3.5637 | 6.4981 |
| | 1.0 | 0.5312 | 3.1075 | 5.9666 | 0.5162 | 3.1447 | 5.8530 | 0.5107 | 3.1120 | 5.6710 | 0.5911 | 3.5620 | 6.8428 | 0.5702 | 3.6048 | 6.6613 | 0.5615 | 3.5663 | 6.4991 |
| | 1.5 | 0.5379 | 3.1091 | 5.9672 | 0.5280 | 3.1472 | 5.8114 | 0.5258 | 3.1149 | 5.6721 | 0.5972 | 3.5635 | 6.8434 | 0.5808 | 3.6069 | 6.6621 | 0.5752 | 3.5689 | 6.4999 |
| | 2.0 | 0.5446 | 3.1107 | 5.9679 | 0.5395 | 3.1496 | 5.8123 | 0.5405 | 3.1179 | 5.6731 | 0.6033 | 3.5649 | 6.8439 | 0.5913 | 3.6090 | 6.6629 | 0.5888 | 3.5715 | 6.9009 |
| | 2.5 | 0.5512 | 3.1124 | 5.9686 | 0.5507 | 3.1520 | 5.8131 | 0.5546 | 3.1308 | 5.6742 | 0.6093 | 3.5664 | 6.8445 | 0.6016 | 3.6111 | 6.6637 | 0.6017 | 3.5740 | 6.5018 |
| 6 | 0.5 | 0.5317 | 3.0424 | 5.9356 | 0.5184 | 3.1495 | 5.9145 | 0.5111 | 3.1455 | 5.8220 | 0.5936 | 3.4869 | 6.8069 | 0.5757 | 3.6109 | 6.7824 | 0.5663 | 3.6061 | 6.6748 |
| | 1.0 | 0.5398 | 3.0445 | 5.9365 | 0.5322 | 3.1526 | 5.9157 | 0.5293 | 3.1492 | 5.8234 | 0.6007 | 3.4887 | 6.8077 | 0.5882 | 3.6136 | 6.7834 | 0.5824 | 3.6093 | 6.6760 |
| | 1.5 | 0.5476 | 3.0467 | 5.9374 | 0.5457 | 3.1556 | 5.9168 | 0.5465 | 3.1529 | 5.8247 | 0.6074 | 3.4906 | 6.8085 | 0.6004 | 3.6163 | 6.7844 | 0.5981 | 3.6125 | 6.6772 |
| | 2.0 | 0.5554 | 3.0488 | 5.9383 | 0.5588 | 3.1586 | 5.9179 | 0.5631 | 3.1565 | 5.8260 | 0.6148 | 3.4925 | 6.8092 | 0.6124 | 3.6189 | 6.7854 | 0.6133 | 3.6157 | 6.6783 |
| | 2.5 | 0.5630 | 3.0509 | 5.9392 | 0.5716 | 3.1617 | 5.9191 | 0.5792 | 3.1602 | 5.8273 | 0.6247 | 3.4943 | 6.8100 | 0.6241 | 3.6216 | 6.7864 | 0.6281 | 3.6189 | 6.6795 |
| 7 | 0.5 | 0.5364 | 2.9566 | 5.8524 | 0.5306 | 3.1256 | 5.9633 | 0.5264 | 3.1501 | 5.9217 | 0.5988 | 3.3869 | 6.7000 | 0.5897 | 3.5831 | 8.8389 | 0.5832 | 3.6112 | 6.7906 |
| | 1.0 | 0.5458 | 2.9594 | 5.8536 | 0.5465 | 3.1295 | 5.9649 | 0.5468 | 3.1547 | 5.9234 | 0.6072 | 3.3893 | 6.7100 | 0.6041 | 3.5868 | 6.8403 | 0.6017 | 3.6153 | 6.7021 |
| | 1.5 | 0.5550 | 2.9621 | 5.8548 | 0.5620 | 3.1333 | 5.9663 | 0.5664 | 3.1593 | 5.9251 | 0.6155 | 3.3918 | 6.7121 | 0.6181 | 3.5898 | 6.8416 | 0.6196 | 3.6193 | 6.7936 |
| | 2.0 | 0.5640 | 2.9651 | 5.8560 | 0.5771 | 3.1372 | 5.9678 | 0.5853 | 3.1639 | 5.9269 | 0.6236 | 3.3942 | 6.7132 | 0.6318 | 3.5931 | 6.8429 | 0.6370 | 3.6233 | 6.7951 |
| | 2.5 | 0.5729 | 2.9676 | 5.8572 | 0.5917 | 3.1410 | 5.9693 | 0.6036 | 3.1685 | 5.9286 | 0.6317 | 3.3965 | 6.7142 | 0.6453 | 3.5965 | 6.8442 | 0.6538 | 3.6273 | 6.7966 |

# UPROŠĆEN MODEL ZA DINAMIČKU ANALIZU KABLOVIMA PODRŽANOG MOSTA

**George T. Michaltsos**

*Ovaj rad se bavi modelom koji uprošćava nalaženje prirodnih sopstvenih frekfencija kablovima podržanog mosta sa raspodelom gustine kablova. Tada je moguće proučiti vertikalne prinudne oscilacije mosta. Konačno, dat je primer koji jasno prikazuje jednostavnost metode.*