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CHANGE OF GEOMETRIC MAGNITUDES UNDER INFINITESIMAL BENDING

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Abstract. *In this paper variations of some geometric magnitudes under infinitesimal bending are studied. Infinitesimal bending of a curve is considered and infinitesimal bending field is determined. Specially, the infinitesimal bending field that plane curve includes in a family of plane curves is given. It is also proved that the area of the region in the plane bounded by a closed curve is stationary under the infinitesimal bending of a curve remaining plane. The variation of the volume bounded by rotational surface, under infinitesimal bending of the meridian, remaining closed plane curve, is given.*

Key words: *Infinitesimal bending, curve, infinitesimal bending of meridian, variation, area, volume.*

0. INTRODUCTION

It is well-known that variations of some geometric magnitudes that depend on coefficients of the first fundamental form of the surface are zero under infinitesimal bending of that surface at $\mathfrak{R}^3$ [8]. For example variation of the arc length of a curve on the surface is zero and also variation of the angle between curves on the surface is zero under infinitesimal bending of that surface.

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V. A. Aleksandrov [2] has proved that variation of the volume, bounded by closed rotational surface of the type 0 or 1 without plane parts, is zero. The meridian of such a surface is C^1 smooth, non-containing a segment perpendicular to the axis of rotation.

We are here calculating variation of the volume of rotational surface generated by meridian that is closed plane curve under infinitesimal bending of that meridian.

It is known that area of a region on the surface is stationary under infinitesimal bending of that surface, i.e. that variation of the area is zero. We are calculating variation of area of the region in the plane under infinitesimal bending of a closed curve that bounds that region. This work is based on works [9, 10, 11].

1. INFINITESIMAL BENDING OF A CURVE IN $\Re^3$

We begin by studying infinitesimal bending of a curve. More information about infinitesimal bending of the curves and surfaces one can get from [1], [4], [8], [6], [7], [9].

Definition 1.1. *Let us consider continuous regular curve*

$$C : \bar{r} = \bar{r}(u), \tag{1.1}$$

included in a family of the curves

$$C_\varepsilon : \bar{r}_\varepsilon = \bar{r}(u) + \varepsilon \bar{z}(u), (\varepsilon \geq 0, \varepsilon \to 0, \varepsilon \in \Re) \tag{1.2}$$

where u is a real parameter and we get C for $\varepsilon = 0$, $(C = C_0)$. Family of curves C_ε is **infinitesimal bending of a curve** *C if*

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon), \tag{1.3}$$

where $\bar{z} = \bar{z}(u)$ is **infinitesimal bending field** *of the curve C.*

Theorem 1.1. *[4] Necessary and sufficient condition for $\bar{z}(u)$ to be an infinitesimal bending field of a curve C is to be*

$$d\bar{r} \cdot d\bar{z} = 0. \quad \square \tag{1.4}$$

The next theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 1.2. *Infinitesimal bending field for curve C (1.1) is*

$$\bar{z}(u) = \int [p(u)\bar{n}(u) + q(u)\bar{b}(u)]du, \tag{1.5}$$

where $p(u), q(u)$, are arbitrary integrable functions, and vectors $\bar{n}(u)$, $\bar{b}(u)$ are unit principal normal and binormal vector field of a curve C.

Proof. As

$$d\bar{r} = \dot{\bar{r}}(u)du, \quad d\bar{z} = \dot{\bar{z}}(u)du,$$

according to (1.4) for infinitesimal bending field of a curve C we have

$$\dot{\bar{r}} \cdot \dot{\bar{z}} = 0, \quad i.e. \quad \dot{\bar{z}} \perp \dot{\bar{r}}. \tag{1.6}$$

Based on that we conclude that $\dot{\bar{z}}$ lies in the normal plane of the curve C, i.e.

$$\dot{\bar{z}}(u) = p(u)\bar{n}(u) + q(u)\bar{b}(u), \tag{1.7}$$

where $p(u), q(u)$ are arbitrary integrable functions. Integrating (1.7) we have (1.5).

As

$$\bar{b} = \frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{|\dot{\bar{r}} \times \ddot{\bar{r}}|}, \quad \bar{n} = \frac{(\dot{\bar{r}} \cdot \dot{\bar{r}})\ddot{\bar{r}} - (\dot{\bar{r}} \cdot \ddot{\bar{r}})\dot{\bar{r}}}{|\dot{\bar{r}}||\dot{\bar{r}} \times \ddot{\bar{r}}|} \tag{1.8}$$

infinitesimal bending field can be written in the form

$$\bar{z}(u) = \int [p(u)\frac{(\dot{\bar{r}} \cdot \dot{\bar{r}})\ddot{\bar{r}} - (\dot{\bar{r}} \cdot \ddot{\bar{r}})\dot{\bar{r}}}{|\dot{\bar{r}}||\dot{\bar{r}} \times \ddot{\bar{r}}|} + q(u)\frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{|\dot{\bar{r}} \times \ddot{\bar{r}}|}]du$$

where $p(u), q(u)$ are arbitrary integrable functions, or in the form

$$\bar{z}(u) = \int [P_1(u)\dot{\bar{r}} + P_2(u)\ddot{\bar{r}} + Q(u)(\dot{\bar{r}} \times \ddot{\bar{r}})]du \tag{1.5$'$}$$

where $P_i(u), i = 1, 2, \quad Q(u)$ are arbitrary integrable functions, too. □

Example 1.1. Let us examine infinitesimal bending of a circle

$$\bar{r} = (\cos u, \sin u, 0), \quad (\text{or} \quad x^2 + y^2 = 1). \tag{1.9}$$

Here $R = 1$ and $u = s$, i.e. the curve can be parameterized by the arc length and we have

$$\dot{\bar{r}} = \bar{r}' = \bar{t}, \quad \ddot{\bar{r}} = \bar{r}'' = K\bar{n},$$
$$|\bar{r}'| = 1, \quad |\bar{r}' \times \bar{r}''| = |\bar{r}''| = K,$$
$$\Rightarrow \bar{n} = \frac{\bar{r}''}{K} = R\bar{r}''. \tag{1.10}$$

From (1.9) we have

$$\bar{b} = \frac{\bar{r}' \times \bar{r}''}{|\bar{r}''|} = R(\bar{r}' \times \bar{r}''). \tag{1.11}$$

As

$$\bar{r}' = (-\sin u, \cos u, 0); \quad \bar{r}'' = (-\cos u, -\sin u, 0), \quad \bar{r}' \times \bar{r}'' = \sin^2 u\bar{k} + \cos^2 u\bar{k} = \bar{k}$$

we obtain

$$\bar{n} = 1 \cdot \bar{r}'' = (-\cos u, -\sin u, 0) = -\bar{r}(u).$$

According to (1.5) and previous relations

$$\bar{z}(u) = \int [p(u)R(u)\bar{r}'' + q(u)R(u)(\bar{r}'(u) \times \bar{r}''(u))]du$$

$$\bar{z}(u) = \int [p(u)(-\cos u\bar{i} - \sin u\bar{j}) + q(u)\bar{k}]du + C, \tag{1.12}$$

where $p(u), q(u)$ are arbitrary functions.

We will not study infinitesimal bending only analytically but we will do this by drawing curves. Computer program *Mathematica* [5], [12] permits us to reproduce any infinitesimal bending of a curve.

Let us consider some characteristic cases.

1) For $p(u) = C = 0, \quad q(u) = 1,$
infinitesimal bending field is

$$\bar{z}(u) = u\bar{k}. \tag{1.13}$$

Infinitesimal bending of circle C (1.9) is

$$C_\varepsilon : \bar{r}_\varepsilon = \bar{r}(u) + \varepsilon\bar{z}(u) = \bar{r}(u) + \varepsilon u\bar{k}, \tag{1.14}$$

i.e. the circle is by infinitesimal bending included in a family of helices

$$\bar{r}_\varepsilon = (\cos u, \sin u, \varepsilon u) \tag{1.15}$$

(see Fig. 1.).

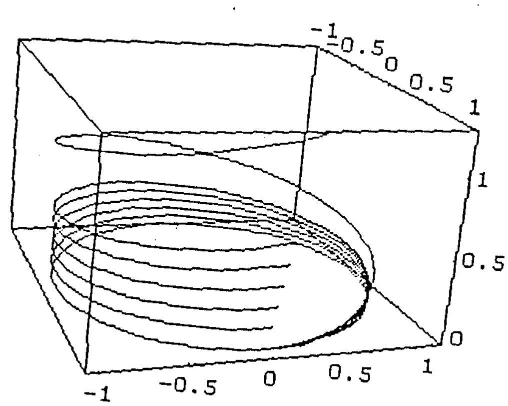

Fig. 1. Infinitesimal bending of a circle 1)

Let us examine if the relation (1.4) is in force i.e. if a field (1.13) is a field of infinitesimal bending.

Calculating

$$d\bar{r}\cdot d\bar{z} = (-\sin u du\bar{i} + \cos u du\bar{j})\cdot(du\bar{k}) = 0$$

we confirm this. Also, according to

$$ds^2 = dx^2 + dy^2 = (-\sin u du)^2 + (\cos u du)^2 = du^2$$

and

$$ds_\varepsilon^2 = ds^2 + \varepsilon^2 du^2$$

we have

$$ds_\varepsilon^2 - ds^2 = \varepsilon^2 du^2 = o(\varepsilon),$$

i.e. we have (1.3).

For the circle (1.9) is $K = \frac{1}{R} = 1, \quad \tau = 0$ (plane curve). For deformed circle (1.15) is

$$K_\varepsilon = \frac{1}{1+\varepsilon^2}, \quad \tau_\varepsilon = \frac{\varepsilon}{1+\varepsilon^2}.$$

Variation of curvature of a circle under infinitesimal bending is

$$\delta K = \frac{\partial K_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = \frac{-2\varepsilon}{(1+\varepsilon^2)^2}|_{\varepsilon=0} = 0,$$

and variation of torsion

$$\delta\tau = \frac{\partial \tau_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = \frac{1+\varepsilon^2-2\varepsilon^2}{(1+\varepsilon^2)^2}|_{\varepsilon=0} = 1,$$

i.e. the circle does not remain plane curve.

2) For $p(u) = C = 0, \quad q(u) = 2\pi - u$
according to (1.12) we get infinitesimal bending field

$$\bar{z}(u) = u(2\pi - u)\bar{k}. \tag{1.16}$$

The curve we get under infinitesimal bending of a circle with this infinitesimal bending field is

$$C_\varepsilon : \bar{r}_\varepsilon = \cos u\bar{i} + \sin u\bar{j} + \varepsilon u(2\pi - u)\bar{k}.$$

This curve is on cylinder $x^2 + y^2 = 1$, but as $z_\varepsilon(u=0) = z_\varepsilon(u=2\pi) = 0$, the curve is closed. This curve is not a helix (Fig. 2). As

$$d\bar{r}\cdot d\bar{z} = 0,$$

and also

$$ds_\varepsilon^2 - ds^2 = \varepsilon^2 u^2 (2\pi - u)^2 du^2 = o(\varepsilon)$$

a field (1.16) is a field of infinitesimal bending of the circle (1.9).

Let us calculate variation of curvature and torsion. As

$$\tau_\varepsilon = \frac{2\varepsilon(\sin 2u + \pi - u)}{1 + 4\varepsilon^2(1+\pi)^2 + u^2 + 2(\pi - u)\sin 2u - \pi u},$$

$$K_\varepsilon = \frac{\sqrt{1 + 4\varepsilon^2(1 + \pi^2 + \ldots)}}{(1 + 4\pi^2\varepsilon^2 - 8\pi\varepsilon^2 u + 4\varepsilon^2 u^2)^{\frac{3}{2}}},$$

variation of curvature and torsion is

$$\delta\tau = \frac{\partial \tau_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = 2(\sin 2u + \pi - u) \neq 0, \quad \delta K = 0,$$

i.e. a circle deforms to a curve that is not plane.

3) For $p(u) = 1, \quad q(u) = 1$

$$\bar r_\varepsilon = \bar r + \varepsilon \bar z = (\cos u - \varepsilon \cos u)\bar i + (\sin u + \varepsilon \cos u)\bar j + \varepsilon u \bar k$$

i.e. circle is included in a family of helices that are not on cylinder $x^2 + y^2 = 1$.

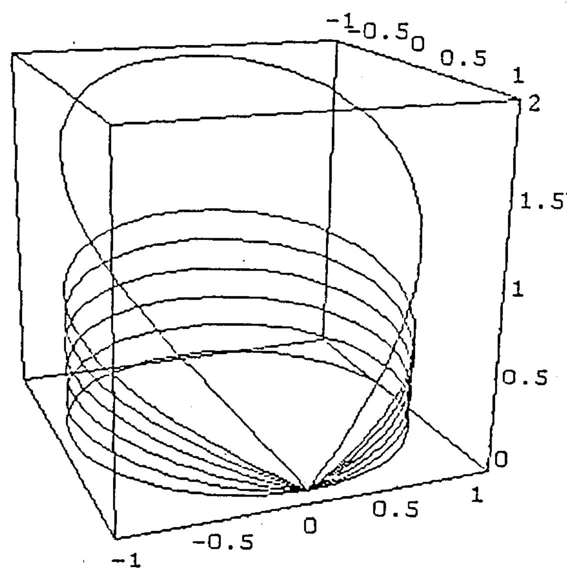

Fig. 2. Infinitesimal bending of circle 2)

2. VARIATION OF REGION IN THE PLANE BOUNDED BY A CLOSED PLANE CURVE UNDER INFINITESIMAL BENDING OF THIS CURVE

2.0. It is known that variation of geometric magnitudes that depend on coefficients of the first fundamental form are zero. Variation of area of a region on the surface under infinitesimal bending of that surface is also zero, as area depends on the coefficients of the first fundamental form of a surface that is infinitesimally bending. We are asking a question:

How is the area of a region in the plane bounded by a plane curve changing under infinitesimal bending of this curve, staying in the plane?

The change of geometric magnitudes under infinitesimal bending is expressed by variation and this question can be precised:

What is the variation of area in the plane under infinitesimal bending of a curve that bounds this region?

2.1. We will consider closed plane curve at polar coordinates

$$K : \rho = \rho(\theta), \quad \theta \in [0, 2\pi]. \tag{2.1}$$

Under infinitesimal bending this curve is included in family of curves

$$K_\varepsilon : \rho_\varepsilon = \rho_\varepsilon(\theta), \quad \theta \in [0, 2\pi], (\varepsilon \geq 0, \quad \varepsilon \to 0). \tag{2.2}$$

Equations of this curves are in vector form:

$$K : \bar{r} = \bar{r}(\theta), \quad \theta \in [0, 2\pi] \tag{2.3}$$

$$K_\varepsilon : \bar{r}_\varepsilon = \bar{r}(\theta) + \varepsilon\bar{z}(\theta), \quad \theta \in [0, 2\pi], \tag{2.4}$$

where $\bar{z}(\theta)$ is a field of infinitesimal bending. We shall consider piecewise smooth curve. At the points where the curve is not regular we choose infinitesimal bending field continuous along a curve, i.e.

$$\bar{z}(\theta - 0) = \bar{z}(\theta + 0). \tag{2.5}$$

Theorem 2.1. *Infinitesimal bending field that plane curve*

$$K : \rho = \rho(\theta)$$

under infinitesimal bendings includes in a family of plane curves

$$K_\varepsilon : \rho_\varepsilon = \rho_\varepsilon(\theta), \quad (\varepsilon \geq 0, \quad \varepsilon \to 0),$$

is

$$\bar{z}(\theta) = \int p(\theta)\bar{n}(\theta)d\theta + \bar{c}. \tag{2.6}$$

Proof. Necessary and sufficient condition for a field $\bar{z}(\theta)$ to be infinitesimal bending field of a curve $\bar{r}(\theta)$ is

$$d\bar{r} \cdot d\bar{z} = 0 \tag{2.7}$$

i.e.

$$\dot{\bar{r}}(\theta) \cdot \dot{\bar{z}}(\theta) = 0. \tag{2.8}$$

From here we conclude that $\dot{\bar{z}} \perp \dot{\bar{r}}$ i.e. that $\dot{\bar{z}}$ is in a normal plane of a curve i.e.

$$\dot{\bar{z}}(\theta) = p(\theta)\bar{n} + q(\theta)\bar{b}, \tag{2.9}$$

where $p(\theta)$ and $q(\theta)$ are arbitrary functions. In order to stay in the plane of the curve we choose

$$q(\theta) = 0. \tag{2.10}$$

So, we have

$$\dot{\bar{z}}(\theta) = p(\theta)\bar{n}(\theta)$$

i.e. (2.6)

We will take $\bar{c} = 0$.

As the equation of a curve K in vector form is

$$K : \bar{r}(\theta) = \rho(\theta)\cos\theta\bar{i} + \rho(\theta)\sin\theta\bar{j} \tag{2.11}$$

we calculate infinitesimal bending field of this curve according to (2.6).

Substituting $\bar{n}$ with respect to (1.8) at (2.6) we have

$$\bar{z} = \int p(\theta)\frac{(\rho\ddot{\rho} - \rho^2 - 2\dot{\rho}^2)[(\rho\cos\theta + \dot{\rho}\sin\theta)\bar{i} + (\rho\sin\theta - \dot{\rho}\cos\theta)\bar{j}]}{\sqrt{\rho^2 + \dot{\rho}^2}|\rho\ddot{\rho} - \rho^2 - 2\dot{\rho}^2|}d\theta.$$

As

$$\rho(\theta)\cos\theta + \dot{\rho}(\theta)\sin\theta = (\rho(\theta)\sin\theta)^{\dot{}}, \quad \rho(\theta)\sin\theta - \dot{\rho}(\theta)\cos\theta = -(\rho(\theta)\cos\theta)^{\dot{}},$$

we choose

$$p(\theta) = |\dot{\bar{r}}| = \sqrt{\rho(\theta)^2 + \dot{\rho}(\theta)^2},$$

that gives

$$\bar{z}(\theta) = \int [(\rho(\theta)\sin\theta)^{\dot{}}\bar{i} - (\rho(\theta)\cos\theta)^{\dot{}}\bar{j}]d\theta = \rho(\theta)\sin\theta\bar{i} - \rho(\theta)\cos\theta\bar{j}. \tag{2.12}$$

So we have determined infinitesimal bending field that plane curve includes in a family of plane curves under infinitesimal bending. □

Corollary 2.1. *If the plane curve*

$$K : \rho = \rho(\theta), \quad \theta \in [0, 2\pi],$$

under infinitesimal bending stays at original plane, the equation of deformed curve will be

$$K_\varepsilon : \rho_\varepsilon(\theta) = \rho(\theta)\sqrt{1 + \varepsilon^2} \tag{2.13}$$

Proof. According to previous theorem we have

$$\bar{r}_\varepsilon = \bar{r} + \varepsilon \bar{z} = \rho(\theta)(\cos\theta + \varepsilon\sin\theta)\bar{i} + \rho(\theta)(\sin\theta - \varepsilon\cos\theta)\bar{j}$$

i.e.

$$\rho_\varepsilon(\theta) = \rho(\theta)\sqrt{1+\varepsilon^2}. \quad \square \tag{2.14}$$

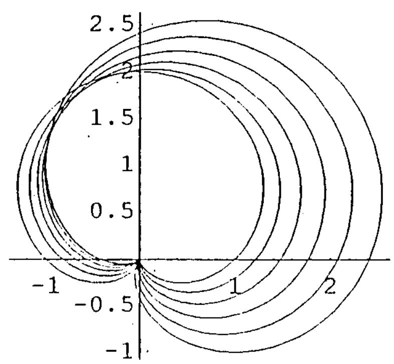

Fig. 3. Infinitesimal bending of the cardioid

Example 2.1. Infinitesimal bending field of the circle
$\rho(\theta) = a, \quad \theta \in [0, 2\pi]$ i.e.

$$K : \bar{r}(\theta) = a\cos\theta\bar{i} + a\sin\theta\bar{j},$$

is

$$\bar{z}(\theta) = \rho(\theta)\sin\theta\bar{i} - \rho(\theta)\cos\theta\bar{j} = a(\sin\theta\bar{i} - \cos\theta\bar{j}).$$

A family of curves that are infinitesimal bendings of the circle K is

$$K_\varepsilon : \bar{r}_\varepsilon(\theta) = \bar{r}(\theta) + \varepsilon\bar{z}(\theta) = a(\cos\theta + \varepsilon sin\theta)\bar{i} + a(\sin\theta - \varepsilon\cos\theta)\bar{j}.$$

The curves K_ε are concentric circles, because from (2.14):

$$\rho_\varepsilon = \rho\sqrt{1+\varepsilon^2} \quad \theta \in [0, 2\pi].$$

Example 2.2. For cardioid

$$\rho(\theta) = 1 + \sin\theta, \quad \theta \in [0, 2\pi],$$

i.e.

$$K : \bar{r} = (1 + \sin\theta)\cos\theta\bar{i} + (1 + \sin\theta)\sin\theta\bar{j}$$

infinitesimal bending field is

$$\bar{z}(\theta) = (1 + \sin\theta)\sin\theta\bar{i} - (1 + \sin\theta)\cos\theta\bar{j}$$

Curves that are infinitesimal bendings of the cardioid K are (Fig. 3.)

$$K_\varepsilon : \bar{r} = (1 + \sin\theta)(\cos\theta + \varepsilon\sin\theta)\bar{i} + (1 + \sin\theta)(\sin\theta - \varepsilon\cos\theta)\bar{j},$$

i.e. we have

$$\rho_\varepsilon = (1 + \sin\theta)\sqrt{1 + \varepsilon^2}, \quad \theta \in [0, 2\pi].$$

2.2. We will examine the change of area of a region bounded by a plane curve under infinitesimal bending of this curve (staying plane).

We will prove the next theorem:

Theorem 2.2. *Area of the region determined by a plane curve being infinitesimally bent staying plane is stationary.*

Proof. Area determined by a curve (2.1) is

$$P = \frac{1}{2}\int_0^{2\pi} \rho^2(\theta)d\theta. \tag{2.15}$$

Area bounded by deformed curve K_ε is

$$P_\varepsilon = \frac{1}{2}\int_0^{2\pi} \rho_\varepsilon^2(\theta)d\theta. \tag{2.16}$$

As from

$$\bar{r}_\varepsilon = x_\varepsilon\bar{i} + y_\varepsilon\bar{j} = (x + \varepsilon z_1)\bar{i} + (y + \varepsilon z_2)\bar{j}$$

$${\rho_\varepsilon}^2 = (x + \varepsilon z_1)^2 + (y + \varepsilon z_2)^2 = \rho^2 + 2\varepsilon\bar{r}\bar{z} + \varepsilon^2|\bar{z}|^2$$

area determined by deformed curve is

$$P_\varepsilon = \frac{1}{2}\int_0^{2\pi} \rho^2(\theta)d\theta + \varepsilon\int_0^{2\pi} \bar{r}\bar{z}d\theta + \frac{\varepsilon^2}{2}\int_0^{2\pi} |\bar{z}|^2 d\theta \tag{2.17}$$

and variation of the area is

$$\delta P = \lim_{\varepsilon\to 0}\frac{P_\varepsilon - P}{\varepsilon} = \lim_{\varepsilon\to 0}\int_0^{2\pi} \bar{r}\bar{z}d\theta.$$

As

$$\bar{r} \cdot \bar{z} = (\rho(\theta) \cos\theta \bar{i} + \rho(\theta) \sin\theta \bar{j}) \cdot (\rho(\theta) \sin\theta \bar{i} - \rho(\theta) \cos\theta \bar{j}) = 0,$$

we have

$$\delta P = 0,$$

i.e. area of the region in the plane, bounded by curve that stays plane under infinitesimal bendings, is stationary. □

3. VARIATION OF THE VOLUME OF A ROTATIONAL SURFACE UNDER INFINITESIMAL BENDING OF MERIDIAN

In this part we confront following question:

How does the volume, bounded by a rotational surface, generated by a meridian being infinitesimally bent, changes?

For infinitesimal bending field of a plane curve, which is a meridian of rotational surface we choose infinitesimal bending field that meridian makes staying in the plane.

We shall consider plane, closed curve at xOy plane.

If ρ and θ are polar coordinates, vector form of this curve is

$$K : \bar{r}(\theta) = \rho(\theta) \cos\theta \bar{i} + \rho(\theta) \sin\theta \bar{j}, \quad \theta \in [0, 2\pi]. \tag{3.1}$$

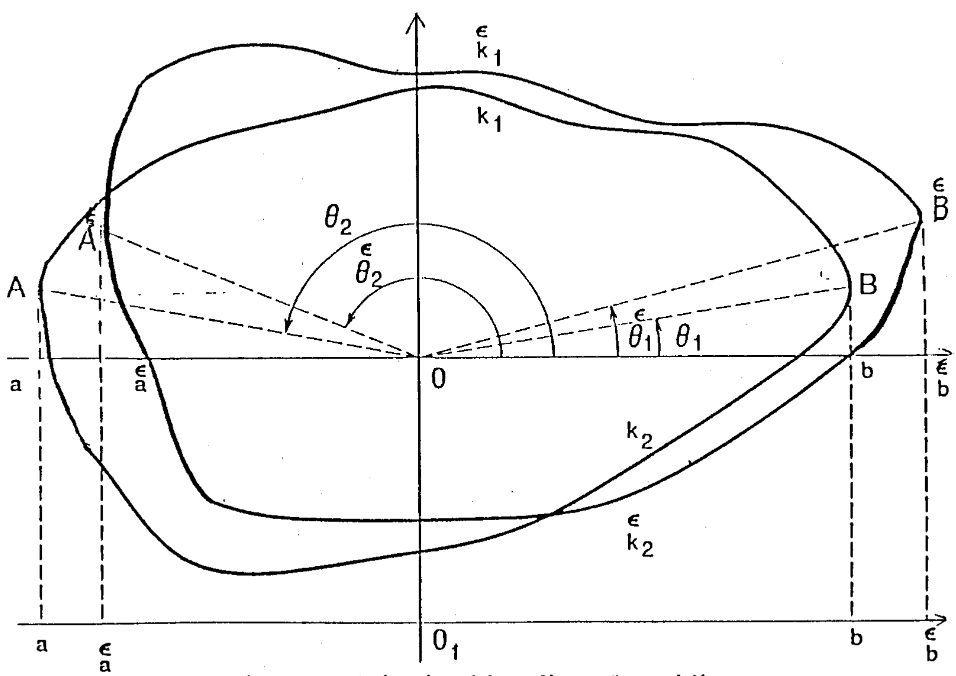

Fig. 4. Infinitesimal bending of meridian

The equation of a curve that remains at xOy-plane according to the Theorem 2.1 and Corollary 2.1 is

$$\overset{\varepsilon}{K} : \overset{\varepsilon}{\bar{r}}(\theta) = \rho(\theta)(\cos\theta + \varepsilon \sin\theta)\bar{i} + \rho(\theta)(\sin\theta - \varepsilon\cos\theta)\bar{j}, \quad \theta \in [0, 2\pi].$$

According to (3.1), parametric form of the equation of the curve K is:

$$K : x = \rho(\theta)\cos\theta, \quad y = \rho(\theta)\sin\theta. \tag{3.1'}$$

Let us introduce a new coordinate system XO_1Y,where $OO_1 = h$, $\quad Y = y+h$, $\quad X = x$ (Fig. 4.). Parametric equations of the curve K are

$$K : X = \rho(\theta)\cos\theta, \quad Y = \rho(\theta)\sin\theta + h. \tag{3.2}$$

Let us divide the curve K by end points A and B in the curves K_1 and K_2. Then for the volume generated by the curve K we have

$$V = \pi\int_a^b (Y_2^2 - Y_1^2)dX, \quad dX = dx = (\dot{\rho}\cos\theta - \rho\sin\theta)d\theta, \tag{3.3}$$

where

$$\begin{aligned} Y &= Y_2, \quad \theta \in [\theta_1, \theta_2] \quad \text{for} \quad K_2 \\ Y &= Y_1, \quad \theta \in [\theta_2, 2\pi + \theta_1] \quad \text{for} \quad K_1 \end{aligned} \tag{3.4}$$

Further, we have :

$$V = \pi\int_{\theta_1}^{\theta_2} \varphi(\theta)d\theta - \pi\int_{\theta_2}^{2\pi+\theta_1} \varphi(\theta)d\theta,$$

where

$$\varphi(\theta) = (\rho\sin\theta + h)^2(\dot{\rho}\cos\theta - \rho\sin\theta).$$

Deformed curve $\overset{\varepsilon}{K}$ has the equations:

$$\overset{\varepsilon}{x} = \rho(\cos\theta + \varepsilon\sin\theta), \quad \overset{\varepsilon}{y} = \rho(\sin\theta - \varepsilon\cos\theta).$$

At the new coordinate system the equations of the curve $\overset{\varepsilon}{K}$ are:

$$\overset{\varepsilon}{X} = \rho(\cos\theta + \varepsilon\sin\theta), \quad \overset{\varepsilon}{Y} = \rho(\sin\theta - \varepsilon\cos\theta) + h.$$

The volume bounded by the surface generated by rotation of deformed curve $\overset{\varepsilon}{K}$ is:

$$\overset{\varepsilon}{V} = \pi\int_{\overset{\varepsilon}{a}}^{\overset{\varepsilon}{b}} (\overset{\varepsilon}{Y}{}_2^2 - \overset{\varepsilon}{Y}{}_1^2)d\overset{\varepsilon}{X}$$

i.e.

$$\overset{\varepsilon}{V} = \pi \int_{\overset{\varepsilon}{\theta}_1}^{\overset{\varepsilon}{\theta}_2} \overset{\varepsilon}{\varphi}(\theta)d\theta - \pi \int_{\overset{\varepsilon}{\theta}_2}^{2\pi+\overset{\varepsilon}{\theta}_1} \overset{\varepsilon}{\varphi}(\theta)d\theta,$$

where

$$\overset{\varepsilon}{\varphi}(\theta) = (\rho\sin\theta - \varepsilon\rho\cos\theta + h)^2(\dot\rho\cos\theta - \rho\sin\theta + \varepsilon\dot\rho\sin\theta + \varepsilon\rho\cos\theta).$$

If we take that $\theta_1 < \theta_1 + \varepsilon_1 = \overset{\varepsilon}{\theta}_1 < \overset{\varepsilon}{\theta}_2 = \theta_2 - \varepsilon_2 < \theta_2$, where $\varepsilon_1, \varepsilon_2 > 0, \varepsilon_1, \varepsilon_2 \to 0$, when $\varepsilon \to 0$ (in this case we take $\varepsilon_1, \varepsilon_2 > 0$, but we can take $\varepsilon_1, \varepsilon_2$ grater or less then 0). We can express $\overset{\varepsilon}{V}$ in the following way

$$\begin{aligned}\overset{\varepsilon}{V} =& [\pi \int_{\theta_1}^{\theta_2} \overset{\varepsilon}{\varphi}(\theta)d\theta - \pi \int_{\theta_1}^{\theta_1+\varepsilon_1} \overset{\varepsilon}{\varphi}(\theta)d\theta - \pi \int_{\theta_2-\varepsilon_2}^{\theta_2} \overset{\varepsilon}{\varphi}(\theta)d\theta] \\ &-[\pi \int_{\theta_2}^{2\pi+\theta_1} \overset{\varepsilon}{\varphi}(\theta)d\theta + \pi \int_{\theta_2-\varepsilon_2}^{\theta_2} \overset{\varepsilon}{\varphi}(\theta)d\theta + \pi \int_{\theta_1}^{\theta_1+\varepsilon_1} \overset{\varepsilon}{\varphi}(\theta)d\theta]\end{aligned} \tag{3.5}$$

where

$$\overset{\varepsilon}{\varphi}(\theta) = \varphi(\theta) + \varepsilon\lambda(\theta) + \varepsilon^2\mu(\theta) + \varepsilon^3\nu(\theta). \tag{3.6}$$

Substituting at (3.5) we get

$$\begin{aligned}\overset{\varepsilon}{V} =& V - 2\pi[\int_{\theta_1}^{\theta_1+\varepsilon_1} \varphi(\theta)d\theta + \int_{\theta_2-\varepsilon_2}^{\theta_2} \varphi(\theta)d\theta] \\ &+\varepsilon\pi[\int_{\theta_1}^{\theta_2} \lambda(\theta)d\theta - \int_{\theta_2}^{2\pi+\theta_1} \lambda(\theta)d\theta - 2\int_{\theta_1}^{\theta_1+\varepsilon_1} \lambda(\theta)d\theta - 2\int_{\theta_2-\varepsilon_2}^{\theta_2} \lambda(\theta)d\theta] \\ &+\varepsilon^2\pi[\int_{\theta_1}^{\theta_2} \mu(\theta)d\theta - \int_{\theta_2}^{2\pi+\theta_1} \mu(\theta)d\theta - 2\int_{\theta_1}^{\theta_1+\varepsilon_1} \mu(\theta)d\theta - 2\int_{\theta_2-\varepsilon_2}^{\theta_2} \mu(\theta)d\theta] \\ &+\varepsilon^3\pi[\int_{\theta_1}^{\theta_2} \nu(\theta)d\theta - \int_{\theta_2}^{2\pi+\theta_1} \nu(\theta)d\theta - 2\int_{\theta_1}^{\theta_1+\varepsilon_1} \nu(\theta)d\theta - 2\int_{\theta_2-\varepsilon_2}^{\theta_2} \nu(\theta)d\theta].\end{aligned}$$

The variation of the volume in this case is

$$\delta V = \pi[\int_{\theta_1}^{\theta_2} \lambda(\theta)d\theta - \int_{\theta_2}^{2\pi+\theta_1} \lambda(\theta)d\theta] \tag{3.7}$$

where

$$\lambda(\theta) = (\rho\sin\theta + h)(-2\rho\dot\rho\cos^2\theta + 3\rho^2\sin\theta\cos\theta + h\dot\rho\sin\theta + h\rho\cos\theta + \rho\dot\rho\sin^2\theta). \tag{3.8}$$

We have the next theorem:

Theorem 3.1. *The variation of the volume, bounded by rotational surface, under infinitesimal bending of the meridian doesn't have to be zero and it is given by (3.7) and (3.8).*

REFERENCES

[1] Aleksandrov, A. D., *O beskonechno malyh izgibaniyah neregulyarnyh poverhnostei*, Matem. sbornik, 1(43), 3 (1936), 307-321.

[2] Aleksandrov, V. A. , *Remarks on Sabitov's conjecture that volume is stationary under an infinitesimal bending of a surface*, Sib. Mat. Zh. N°5 (1989), 16-24.

[3] , *Issledovaniya po metricheskoi teorii poverhnostei*, Mir, Moskva (1980), 280.

[4] Efimov, N. V., *Kachestvennye voprosy teorii deformacii poverhnostei* , UMN 3.2 (1948), 47-158.

[5] Gray, A., *Modern differential geometry of curves and surfaces*, CRC Press, Boca Raton, 1993.

[6] Ivanova-Karatopraklieva, I.; Sabitov, I. Kh., *Surface deformation.*, J. Math. Sci., New York 70 N°2 (1994), 1685–1716.

[7] Ivanova-Karatopraklieva, I.; Sabitov, I. Kh., *Bending of surfaces II*, J. Math. Sci., New York 74 N°3 (1995), 997–1043.

[8] Vekua I. N., *Obobschennye analiticheskie funkcii*, Moskva, 1959.

[9] Velimirović, L. S., *On variation of the volume under infinitesimal bending of a closed rotational surface*, Novi Sad J. Math. vol. 29, No. 3 (1999), 377-386.

[10] Velimirović, L. S., *Change of area under infinitesimal bending of border curve*, Bulletins for Applied Mathematics (to appear).

[11] Velimirović, L. S., On variation of the volume bounded by rotational surface under infinitesimal bending of meridian, Buletinul Stiintific (to appear).

[12] Wolfram S., *Mathematica*, Addison-Wesley, Redwood City, 1990.

PROMENA GEOMETRIJSKIH VELIČINA PRI BESKONAČNO MALOM SAVIJANJU

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U ovom radu se razmatraju varijacije nekih geometrijskih veličina pri beskonačno malom savijanju. Beskonačno malo savijanje krive se razmatra i odredjuje se polje beskonačno malog savijanja. Specijalno, dato je polje savijanja koje ravnu krivu uključuje u familiju ravnih krivih. Dokazuje se da je površina oblasti u ravni ograničena zatvorenom krivom stacionarna pri beskonačno malom savijanju krive koja ostaje u ravni. Odredjena je varijacija zapremine ograničena rotacionom površi, pri beskonačno malom savijanju meridijana, koja ostaje u ravni.