FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 11, 2001, pp. 127 - 133

VISUALIZATION OF DIFFERENTIAL GEOMETRY

UDC 514.7(045)514.752.6:519.688:519.673

Eberhard Malkowsky[1], Vesna Veličković[2]

[1] Department of Mathematics, University of Giessen, Germany
Arndtstrasse 2, D-35392 Giessen, Germany
[2] Prirodno-matematički fakultet, Univerzitet u Nišu, Ćirila i Metodija 2, 18000 Niš

Abstract. *We apply our own open software in PASCAL [6], [7] to visualise some results in differential geometry. In particular, we deal with the graphic representation of parallel and focal surfaces of surfaces of revolution, of potetial surfaces and their Gaussian and mean curvatures.*[1]

Key words: *Computer Graphics, Geometry, Differential Geometry*

1. INTRODUCTION

Throughout this paper, we assume that surfaces are given by a parametric representation

$$\vec{x}(u^i) = (x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2)) \quad ((u^1,u^2) \in D) \tag{1}$$

where the component functions have continuous partial derivatives of order $r \geq 1$ and the surface (unit) vector $\vec{N}$ exists at every point. The normal curvature of a curve γ on a surface is the projection of the vector of curvature of γ into the tangent plane of the surface in the direction of the vector product the surface normal vector and the tangent of γ. It is well known that at every point of a surface, there corresponds one and only one value of the normal curvature to every direction. The extreme values of the normal curvature at a point of a surface are called *principal curvatures*, denoted by κ_1 and κ_2. The *Gaussian* and *mean curvatures* are defined by $K = \kappa_2\kappa_2$ and $H = (\kappa_1 + \kappa_2)/2$ (cf. [1], [4]). In general they are real-valued functions of the parametres of the surface.

In this paper, we apply our own software to represent *parallel* and *focal surfaces* of *surfaces of revolution*, *potential surfaces* and their *Gaussian* and *mean curvatures*.

Received August 15, 2000

[1] 1991 *Mathematics Subject Classification*. 53A05, 68N05.

2. PARALLEL SURFACES

In this section, we consider *parallel surfaces* and their Gaussian and mean curvatures. They play an important role in the theory of *minimal surfaces*, that is surfaces with $H \equiv 0$, due to the relation between their mean curvatures.

Let S be a surface with parametric representation (1) and normal vector $\vec{N}(u^i)$. Then, for $\varepsilon > 0$, the surface S^* with parametric representation $\vec{y}(u^i) = \vec{x}(u^i) + \varepsilon \cdot \vec{N}(u^i)$ is called *parallel surface of S*. If K, H and K^*, H^* denote the Gaussian and mean curvatures of S and S^*, respectively, then it is well known that

$$K^* = \frac{K}{1 - 2\varepsilon H + \varepsilon^2 K} \quad \text{and} \quad H^* = \frac{H - \varepsilon K}{1 - 2\varepsilon H + \varepsilon^2 K}$$

([1,3.5, Problem 11, p. 162]).

First we consider surfaces of revolution. They are given by

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)) \qquad ((u^1, u^2) \in I \times (0, 2\pi)) \tag{2}$$

where $r(u^1) > 0$ on I. Their parallel surfaces again are surfaces of revolution with parametric representation

$$\vec{y}(u^i) = (\rho(u^1)\cos u^2, \rho(u^1)\sin u^2, \psi(u^1)), \quad \text{where}$$

$$\rho(u^1) = r(u^1) - \frac{\varepsilon h'(u^1)}{\phi(u^1)} \quad \text{and} \quad \psi(u^1) = h(u^1) + \frac{\varepsilon r'(u^1)}{\phi(u^1)} \quad \text{where}$$

$$\phi(u^1) = \sqrt{(r'(u^1))^2 + (h'(u^1))^2} \quad \text{for} \quad r(u^1)\phi(u^1) > \varepsilon h'(u^1).$$

Furthermore, the Gaussian and mean curvatures of surfaces of revolution depend on the parameter u^1 only. Therefore we may represent K or H as surfaces of revolution with $r(u^1) = u^1$ and $h(u^1) = K(u^1)$ or $h(u^1) = H(u^1)$, respectively, in (1).

If the original surface of revolution is a catenoid with $r(u^1) = \cosh u^1$ and $h(u^1) = u^1$ for $u^1 \in$ R, then the parallel surfaces are given by

$$\rho(u^1) = \cosh u^1 - \frac{\varepsilon}{\cosh u^1} \quad \text{and} \quad \psi(u^1) = u^1 + \varepsilon \tanh u^1$$

for $\cosh^2 u^1 > \varepsilon$ that is, for all $u^1 \in$ R whenever $\varepsilon < 1$.

Furthermore we have

$$K = -\frac{1}{\cosh^2 u^1}, \quad H = 0,$$

$$K^* = -\frac{1}{\cosh^4 u^1 - \varepsilon^2} \quad \text{and} \quad H^* = \frac{\varepsilon}{\cosh^4 u^1 - \varepsilon^2}$$

for the Gaussian and mean curvatures of the catenoid and its parallel surfaces.

As a second example, we consider the pseudo-sphere, a surface of revolution, given by

$$r(u^1) = \mathrm{e}^{-u^1}, \; h(u^1) = \int \sqrt{1 - \mathrm{e}^{-2u^1}}\, du^1 \qquad (u^1 > 0).$$

It is a surface of constant Gaussian curvature $K = -1$. Furthermore we have

$$H = \frac{1}{2e^{-u^1}\sqrt{1-e^{-2u^1}}}(1 - 2e^{-2u^1})$$

$$K^* = -\frac{1}{1-2\varepsilon H - \varepsilon^2} \quad \text{and} \quad H^* = \frac{H+\varepsilon}{1-2\varepsilon H - \varepsilon^2}$$

for its mean curvature and the Gaussian and mean curvatures of its parallel surfaces.

As a last example in this section, we consider explicit surfaces with parametric representation

$$\vec{x}(u^i) = (u^1, u^2, f(u^1, u^2)) \quad ((u^1, u^2) \in D \subset \mathrm{R}^2).$$

Their Gaussian and mean curvatures are given by

$$K = \frac{f_{11}f_{22} - f_{12}}{\phi^4} \text{ and } H = \frac{1}{2\phi^3}((1+f_2^2)f_{11} - 2f_1 f_2 f_{12} + (1+f_1^2)f_{22}),$$

where $\phi = \sqrt{1+(f_1)^2+(f_2)^2}$. The figures below show the Gaussian and mean curvatures of a hyperbolic paraboloid and its parallel surface.

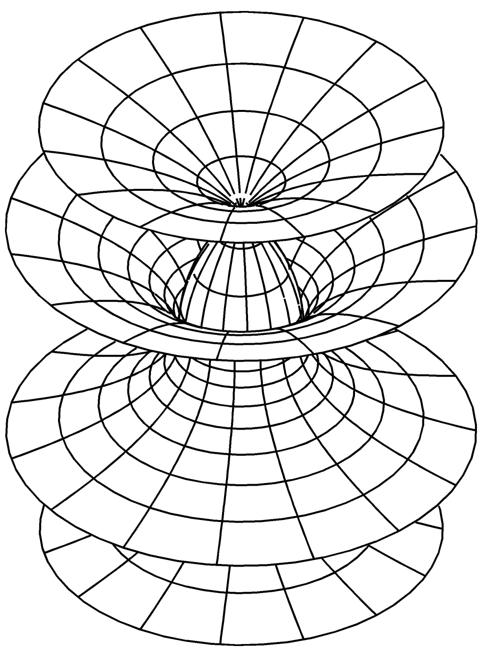

Fig. 1. Catenoid and parallel surface

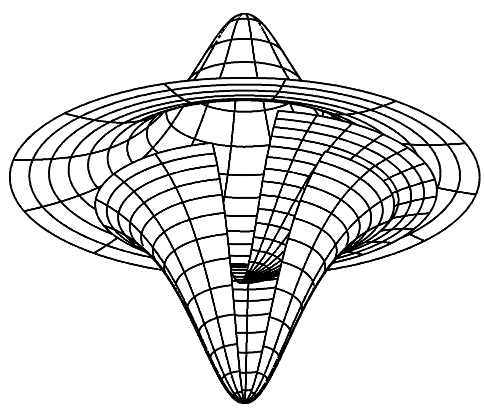

Fig. 2. Gaussian curvature of a catenoid and Gaussian and mean curvature of its parallel surface

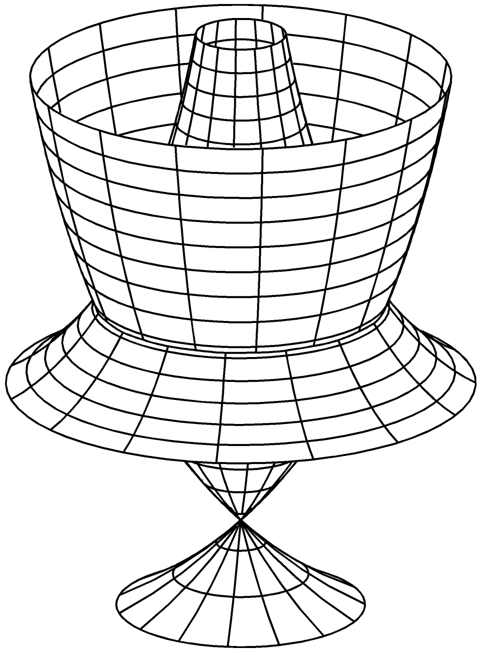

Fig. 3. Pseudo-sphere and parallel surface

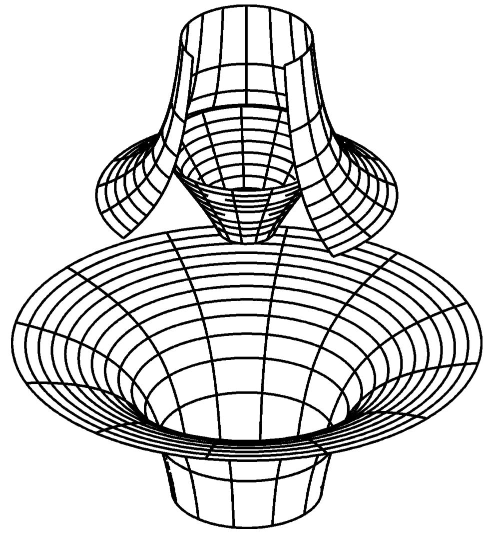

Fig. 4. Mean curvature of pseudo-sphere and Gaussian and mean curvature of its parallel surface

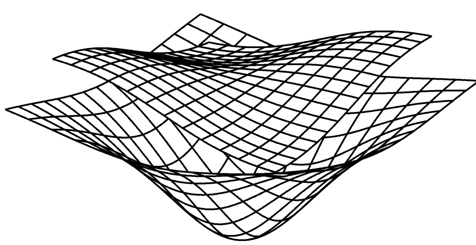

Fig. 5. Gaussian and mean curvature of a hyperbolic parabloid

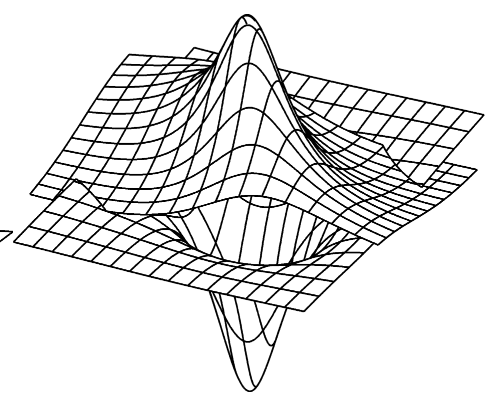

Fig. 6. Gaussian and mean curvature of a parallel surface of a hyperbolic paraboloid

3. Focal or central surfaces

In this section, we consider *focal* or *central surfaces* which, in some cases, are generalizations of parallel surfaces.

Let S be a surface without *parabolic points*, that is points of vanishing Gaussian curvature, and without *umbilical points*, that is points with $\kappa_1 = \kappa_2$ for the principal curvatures κ_1 and κ_2. If $\vec{x}(u^i)$ is a parametric representation of S such that the *lines of curvature* coincide with the parameter lines, which is the case if and only if the *first* and *second fundamental coefficients* g_{ik} and L_{ik} $(i,k = 1,2)$ of S satisfy the condition $g_{12} = L_{12}$, then the surfaces S_1 and S_2 with parametric representations

$$\vec{y}(u^i) = \vec{x}(u^i) + \frac{1}{\kappa_1(u^i)}\vec{N}(u^i) \quad \text{and} \quad \vec{z}(u^i) = \vec{x}(u^i) + \frac{1}{\kappa_2(u^i)}\vec{N}(u^i)$$

are called the *focal* or *central surfaces of S*. The name originates from the fact that $\vec{y}$ and $\vec{z}$ are the position vectors of the centres of the osculating circles of the normal sections of the surface S that correspond to the principal curvatures κ_1 and κ_2.

In the case of surfaces of revolution, we have $g_{12} = L_{12} = 0$, and $\kappa_1 = g_{11}L_{22}$ and $\kappa_2 = g_{22}L_{11}$. Thus the focal surfaces of a surface of revolution without parabolic and umbilical points are again surfaces of revolution given by

$$\rho_j(u^1) = r - \frac{h'}{\phi\kappa_j} \text{ and } \psi_j(u^1) = h - \frac{r'}{\phi\kappa_j} \qquad \text{for } j=1,2,$$

where $\phi = \sqrt{(r')^2 + (h')^2}$.

If the surface of revolution is the catenoid of Section 2, then

$$\rho_2(u^1) = 2\cosh u^1 \qquad \text{and}$$
$$\psi_2(u^1) = u^1 - \sinh u^1 \cosh u^1 .$$

If the surface of revolution is the pseudo-sphere of Section 2, then

$$\rho_1(u^1) = -2\sinh u^1 ,$$
$$\psi_1(u^1) = \int \sqrt{1 - e^{-2u^1}}\, du^1 - \frac{e^{-2u^1}}{\sqrt{1 - e^{-2u^1}}} ,$$
$$\rho_2(u^1) = e^{u^1} ,$$
$$\psi_2(u^1) = \int \sqrt{1 - e^{-2u^1}}\, du^1 + \sqrt{1 - e^{-2u^1}} .$$

Finally, for the hyperboloid of rotation with $r(u^1) = \cosh u^1$, $h(u^1) = \sinh u^1$ and $\phi(u^1) = \sqrt{\cosh u^1 + \sinh u^1}$, we obtain

$$\rho_2(u^1) = 2\cosh^3 u^1 \qquad \text{and}$$
$$\psi_2(u^1) = -2\sinh^3 u^1 .$$

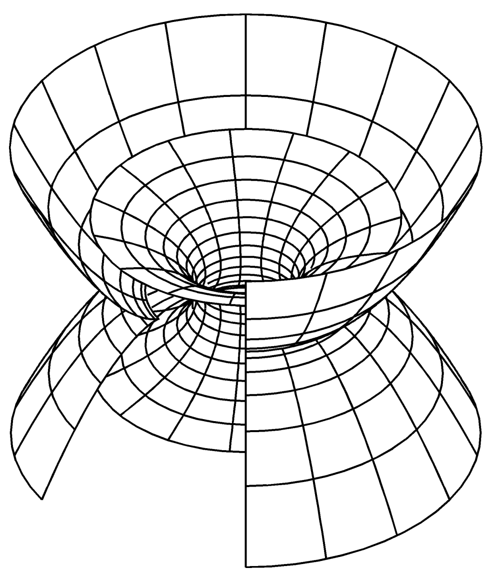

Fig. 7. Catenoid and its focal surface

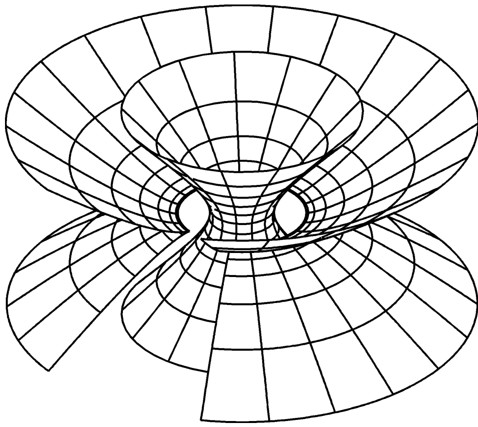

Fig. 8. Hyperboloid and ist focal surface

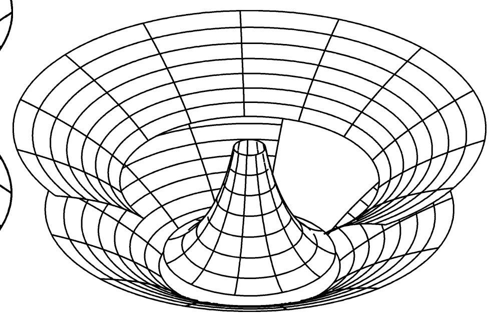

Fig. 9. Pseudosphere and its two focal surfaces

4. THE GAUSSIAN AND MEAN CURVATURES OF POTENTIAL SURFACES

In this section we consider the Gaussian and mean curvatures of so-called *potential surfaces* given by a parametric representation

$$\vec{y}(u^1,u^2) = h(u^1,u^2)\vec{x}(u^i) \quad ((u^1,u^2) \in (-\pi/2,\pi/2)\times(0,2\pi)),$$

where $\vec{x}(u^i) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1)$. Potential surfaces play an important role in physics, chemistry and crystallography.

A lengthy but otherwise straightforward computation yields

$$g^*_{11} = h_1^2 = h^2 g_{11} = h_1^2 + h^2,\; g^*_{12} = h_1h_2 + h^2 g_{12} = h_1h_2,$$
$$g^*_{22} = h_2^2 + h^2 g_{22} = h_2^2 + h^2\cos^2 u^1,$$
$$g^* = h^2((h_1^2 + h_2^2)\cos^2 u^2 + h_2^2)$$

and

$$L^*_{11} = \frac{1}{\sqrt{g^*}} h\cos u^1 (h^2 - hh_{11} + 2h_1^2),$$

$$L^*_{12} = \frac{1}{\sqrt{g^*}} h(\cos u^1 (2h_1h_2 - hh_{12}) - hh_2 \sin u^1),$$

$$L^*_{22} = \frac{1}{\sqrt{g^*}} h\cos u^1 (2h_2^2 - hh_{22} + h^2\cos^2 u^1 + hh_1 \sin u^1 \cos u^1),$$

for the first and second fundamental coefficients of potential surfaces and their Gaussian and mean curvatures $K^* = L^*/g^*$ and $H^* = 1/(2g^*)\cdot(g^*_{11}L^*_{22} - 2g^*_{12}L^*_{12} + g^*_{22}L^*_{11})$.

REFERENCES

1. Carmo, M. P. do, *Differentialgeometrie von Kurven und Flächen*, Vieweg Verlag, Wiesbaden, Braunschweig, 1983
2. M. Failing, *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallographie*, Ph.D. Thesis, Giessen, 1996, Shaker Verlag, Aachen,1996
3. M. Failing, E. Malkowsky, Ein effizienter Nullstellenalgorithmus zur computergrafischen Darstellung spezieller Kurven und Flächen, *Mitt. Math. Sem. Giessen*, **229**, (1996), 11-25
4. J. Jost, *Differentialgeometrie der Minimalflächen*, Springer-Verlag, 1994
5. E. Kreyszig, *Differentialgeometrie*, Akademische Verlagsgesellschaft Leipzig, 1957
6. E. Malkowsky, An open software in OOP for computer graphics and some applications in differential geometry, *Proceedings of the 20th South African Symposium on Numerical Mathematics*, (1994), 51-80
7. E. Malkowsky, W. Nickel, *Computergrafik in der Differentialgeometrie*, Vieweg-Verlag Wiesbaden, Braunschweig, 1993

VIZUALIZACIJA DIFERENCIJALNE GEOMETRIJE

Eberhard Malkowsky, Vesna Veličković

Za predstavljanje rezultata iz diferencijalne geometrije primenjujemo svoj sopstveni otvoreni softver pisan u PASCAL-u [6,7]. Posebno istražujemo grafičke reprezentacije paralelnih i fokalnih površi rotacionih i potencijalnih površi i njihove Gausove i srednje krivine.