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ASYMPTOTICAL PROPERTIES OF NONLINEAR DIFFERENTIAL EQUATION

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Abstract. *Asymptotic property of solutions has been considered for some nonlinear differential equations. The paper deals with investigation of bounded solutions, of prolongation of solutions, oscillatory solutions and another asymptotic property. The examples have been stated which illustrate the given methods and have got physical interest.*

The paper is divided in three parts and each of them investigating some of asymptotic properties for certain differential equation. For general information is one referred to a short reference

The part one deals with asymptotic behavior of positive solutions. They are also related to oscillation theory.

In part two we apply the fixed point method to oscillation theory. We consider some equation and we obtain necessary conditions and sufficient conditions for existence of certain monotonic, nonoscillatory solutions.

The part three deals with the growth and bounds solutions and with existence and unique solutions.

1.

We wish to study the asymptotic behavior of the solution of the differential equation

$$u^{(n)} - qu = 0 \tag{1}$$

on [0,∞), particularly with respect to oscillation. A continuous function f from [0,∞) to (−∞,∞) is called oscillatory if and only if the set $\{t\colon t \geq 0 \text{ and } f(t) = 0\}$ is unbounded. Let q be a continuous function from [0,∞) to (0,∞), and let n be an integer $n \geq 2$.

It is clear that (1) has nonoscillatory solutions. In particular, if k is an integer in [1,n]

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and z_k solves

$$z_k(t) = \frac{t^{k-1}}{(k-1)!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} q(s) z_k(s) ds \tag{2}$$

on $[0,\infty)$, then z_k is a nonoscillatory solution of (1).We shall call a solution u of (1) strongly increasing if and only if each of u, u', u'',...$u^{(n-1)}$ is eventually positive.

On the other hand, it is known ([6]) that there is a positive solution u of (1) such that $(-1)^k u^{(k)} > 0$ for each k. We shall call a solution u of (1) strongly decreasing if and only if $(-1)^k u^{(k)}$ is eventually positive for each integer k in $[0,n-1]$. Since we know that there exist strongly increasing and strongly decreasing solutions, the best conclusion one can hope for in an oscillation theorem is that every eventually positive solution is either strongly increasing or strongly decreasing.

Theorem 1. If

$$\int_0^\infty t^{n-1} q(t) dt = \infty \tag{3}$$

or if (3) fails and the solution of the second order equation

$$v''(t) + \frac{1}{(n-3)!}\left(\int_t^\infty (s-t)^{n-3} q(s) ds\right) v(t) = 0 \tag{4}$$

is oscillatory, then every eventually positive solution of (1) is either strongly increasing or strongly decreasing.

2.

We shall investigate the nonoscillatory solutions of the nonlinear equation

$$(r(x)y^{(n)})^{(n)} = yf(x,y) \tag{5}$$

where $r(x)$ is positive and continuous on $[\tau,\infty)$ with $\int_\tau^\infty \frac{dx}{r(x)} = \infty$ and $f(x,y)$ is continuous on $[\tau,\infty) \times R$ with $f(x,y) > 0$ if $y \neq 0$, and for equation

$$(r(x)y'(x))' = yf(x,y)\,. \tag{6}$$

We are interested in necessary and sufficient conditions that (5) have solutions $y_m(x)$, which are asymptotically equivalent to the functions $R_m(x)$, $0 \le m \le 2n-1$, the nonoscillatory solutions of the homogeneous equation $(r(x)y^{(n)}(x))^{(n)} = 0$. Unlike earlier work on this subject, which considered solutions satisfying the asymptotic condition

$$0 < \lim_{x\to\infty} y(x) R_m^{-1} < \infty \tag{7}$$

we impose the stronger asymptotic condition

$$\lim_{x\to\infty} | y(x) - \sum_{k=0}^{m} A_k R_k(x) | = 0 . \tag{8}$$

As a consequence the necessary and sufficient conditions which we obtain are more complex in that they change as *m* changes. We give an example of a linear equation satisfying (7) but not (8).

First we consider the equation

$$(r(x)y^{(n)})^{(n)} = yf(x) \tag{9}$$

where $f(x)$ and $r(x)$ are positive and continuous on $[\tau,\infty)$ and $\int_{\tau}^{\infty} \frac{du}{r(u)} = \infty$.

Definition. Denote $E_k(x,y) = y^{(k)}$, $0 \le k \le n-1$, $E_k(x,y) = (r(x)y^{(n)})^{(k-n)}$ for $n \le k \le 2n$, and $E_k(x) = E_k(x,y(x))$. A solution y(x) of the equation (9) is said to be of the type $2j$, $0 \le j \le n$ if $E_k(x) > 0$, $0 \le k \le 2j$ and $(-1)^k E_k(x) > 0$, $2j \le k \le 2n$ and $\sigma \le x \le \infty$ for some $\sigma \ge \tau$. Denote also

$$R_k(x,t) = \begin{cases} \dfrac{(t-x)^k}{k!}, & \text{for } 0 \le k \le n-1, \\ \displaystyle\int_x^t \frac{(u-x)^{n-1}(t-u)^{k-n}}{(n-1)!(k-n)!r(u)} du, & \text{for } n \le k \le 2n-1 \end{cases} ,\tau \le t, x < \infty$$

We observe that $R_k(x,t) > 0$ for $\tau \le t < s < \infty$, and $(-1)^k R_k(x,,t) > 0$ for $x > t$.

The following facts are known:

1) A solution of (5) which is positive on $[\tau,\infty)$must be of the type $2j$ for some j, $0 \le j \le n$
2) Equation (5) has solutions of the type $2j$ for $j = 0$ and $j = n$ [3].

Theorem 2. A solution $y(x)$ is of type $2j$, $0 < j < n$, of (9) if and only if $E_{2j}(x)\downarrow A_{2j} \ge 0$ as $x \to \infty$. In this case there exist positive constants α, β such that

$$\alpha R_{2j-1}(\rho, x) < y(x) < \beta R_{2j}(\rho, \tau), \ \rho \le x < \infty$$

ρ sufficiently large. Further $y(x) \approx R_{2j}(\tau,x)$ if and only if $A_{2j} > 0$; $y(x) \approx R_{2j-1}(\tau,x)$ if and only if $A_{2j} = 0$ and $E_{2j-1}(x)\uparrow A_{2j-1} > 0$, $x\to\infty$. [5]

We note that a solution of (9) can be written as

$$y(x) = E_0(x) = \sum_{k=0}^{2n-1} (-1)^k E_k(b) R_k(x,b) + \int_x^b R_{2n-1}(x,t) f(t) y(t) dt$$

This formula follows from the Taylor's theorem.

If $A_{2j} = 0$ and E_{2j-1} is bounded and increasing, a similar argument shows $y(x)$ is asymptotic to $R_{2j-1}(\tau_1,x)$.

A type 0 solution satisfies $\lim_{x\to\infty} y(x) = A \ge 0$, $\lim_{x\to\infty} E_k(x) R_k(\tau,x) = 0$ for $1 \le k \le 2n-1$.

Further, $A > 0$ if and only if $\int_{\tau}^{\infty} R_{2n-1}(\tau,t) f(t) dt < \infty$, and in this case the solution is uniquely determined by A.

If y is a type $2j$ solution of (9) $0 < j < n$, and $y(x) \sim R_m(\tau,x)$, $m = 2j$ or $2j - 1$ then $E_k(x) \sim R_m^{(k)}(\tau,x)$, $0 < k < 2j$.

The following example shows that the integral condition is the best possible result of this kind. The equation

$$y'' = \frac{2}{x(x-1)^2}\, y$$

has solutions $y_0(x) = \frac{x}{x-1}$, $y_1(x) = 1 + x - \frac{x}{x-1}\ln x^2$. Clearly $y_1(x)$ is not asymptotic to $C_0 + C_1 x$, and correspondingly $\int_{\infty}^{\infty} R_1(t) f(t)dt = \infty$. However, for $\sigma > 0$, $\int^{\infty} x^{-\sigma} R_1(x) f(x)dx < \infty$.

This results will now be extended to $(r(x)y^{(n)}(x))^{(n)} = yf(x,y)$, to equation (5).

Theorem 3. Equation (1) has a solution y_m of type $2j$ satisfying

$$\lim_{x\to\infty} E_m(x) = A_m > 0 \tag{10}$$

$$\int_{\tau}^{\infty} R_{2n-1}(\tau,t) R_m(\tau,t) f(t, cR_m(\tau,t))dt < \infty \tag{11}$$

for some $c > 0$. Equation (5) has a solution y_m satisfying

$$y_m(x) = \sum_{k=0}^{m} (-1)^k A_k R_k(\tau,x) + \int_{x}^{\infty} R_{2n-1}(t,s) f(s, y_m(s)) y_m(s)ds$$

If y_{1m} and y_{2m} are two solutions of (5) satisfying (10)–(11) then

$$\lim_{x\to\infty} \frac{y_{1m}(x) - y_{2m}(x)}{R_m(\tau,x)} = 0\,.$$

To proof this suppose y_m is a solution of (9) which satisfies (10). Then y_m is a solution of the linear equation

$$(r(x)u^{(n)})^{(n)} = f(x, y_m(x))u\,. \tag{12}$$

By Theorem 2 we have , with

$$f(s) = f(s, y_m(s)), 0 < \int_{\tau}^{\infty} R_{2n-1}(\tau,s) R_m(\tau,s) f(s, y_m(s))ds < \infty$$

and y_m satisfies (11).

Define $T\colon S \to S$ by Tu is the solution y_m of type $2j$ of the linear equation (12). As in [3] we use the contraction mapping theorem to solve the linear problem (12) and Schauder fixed point theorem to extend the linear results to the nonlinear problem (9).

It is interesting to compare the conditions: (9) has a solutions $y_m(x)$ satisfying

$$0 < \lim_{b\to\infty} \left| \frac{y_m(b)}{R_m(b)} \right| < \infty \tag{13}$$

and (9) has a solution $y_m(x)$ satisfying

$$\lim_{b\to\infty} | y_m(b) - \sum_{k=0}^{m} A_k R_k(b) | = 0 \tag{14}$$

Clearly one would expect necessary and sufficient conditions for (14) to be stronger than those for (13). For example, in the case of maximal solutions, corresponding to $m = 2n - 1$, by the Theorem 2 (13) is satisfied if and only if $\int^{\infty} R(t) f(t, cR(t))dt < \infty$. By the Theorem 3 (14) is satisfied if and only if $\int^{\infty} R^2(t) f(t, cR(t))dt < \infty$.

Corollary If (9) has a solution y_m satisfying $\lim_{b\to\infty} | y_m(b) - \sum_{k=1}^{m} A_k R_k | = 0$ for some $m = m_0$, then it has such a solution for all $m \le m_0$.

As special cases of Theorem 3 we will consider generalizations of the Emden-Fowler equation

$$(r(x) y^{(n)})^{(n)} = f(x) | y |^{\gamma} \operatorname{sgn} y \tag{15}$$

where $\gamma > 0$, and the hypotheses of Theorem 3 are satisfied.

Corollary Equation (15) has a solution $y_m(x)$ if and only if

$$\int^{\infty} R(t) f(t) (R_m(t))^{\gamma} dt < \infty$$

3.

We consider a class of nonlinear equations which admit a characteristic equation. If the roots of this algebraic equation are real and distinct, the growth and bounds of solutions of the differential equations exists.

Theorem 6. Consider the n-th order differential equation

$$y^{(n)} + a_{n-1}(y(x)) y^{(n-1)} + ... + a_1(y(x)) y' + a_0(y(x)) y = 0 \tag{16}$$

where $a_i(y(x))$, $0 \le i \le n - 1$, are continuous functions. Let $\lambda_i(y(x))$, $1 \le i \le n$, be the roots of

$$\lambda^n + a_{n-1}(y(x)) \lambda^{n-1} + ... + a_0(y(x)) = 0$$

and suppose that the $\lambda_i(y(x))$ are real valued and that exist $2n$ constants $\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < ... < \alpha_n \le \beta_n$ such that

$$\alpha_i \le \lambda_i(y(x)) \le \beta_i \tag{17}$$

for $(x,y) \in [0,\omega) \times [0,\infty)$. Then (16) has n linearly independent solutions $y_1(x),...,y_n(x)$ such that

$$y_i(x) > 0 \qquad \alpha_i \le \frac{y_i'(x)}{y_i(x)} \le \beta_i, \quad 1 \le i \le n, \quad 0 \le x \le \omega \tag{18}$$

If, additionally, (17) holds , then for each solution $y(x)$ of (16) there exist n solutions

$y_1(x),..., y_n(x)$ which satisfy (18) and n constants $C_1,...,C_n$ such that

$$y(x) = C_1 y_1(x) + ... + C_n y_n(x) \qquad 0 \le x \le \omega \tag{19}$$

Example. Given the equation

$$y'' + 2(6 + a(x, y))y' + (20 + k \text{ arctg } y)y = 0 \tag{20}$$

where $a(x,y)$ is continuous and satisfies $|a(x,y)| \le 1$, $[0,\infty) \times R$, for that values of the constant k we are assured that solutions decay exponentially to zero? Physically this equation may represent the motion of a nonlinear spring immersed in a liquid. The damping cannot be precisely measured yet we want to be sure that the motion dies out.

One can verify that if $0 \le k < 6/\pi$, then the theorem may be applied with

$$\alpha_1 = -7 - \sqrt{29 + k\frac{\pi}{2}}, \qquad \beta_1 = -5 - \sqrt{5 - k\frac{\pi}{2}}$$

$$\alpha_2 = -7 + \sqrt{5 - k\frac{\pi}{2}}, \qquad \beta_2 = -5 + \sqrt{29 + k\frac{\pi}{2}} < 0$$

Example. A standard method of solving second order differential equation of the type

$$\frac{d^2y}{dx^2} + f(y) = 0, \qquad y(0) = C_1, \quad \frac{dy(0)}{dx} = C_2 \tag{21}$$

is to multiply the equation by dy/dx. Than it can be rewritten in the form $\frac{1}{2}\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + f(y)\frac{dy}{dx} = 0$. This last equation can be integrated to obtain

$$\left(\frac{dy}{dx}\right)^2 + F(y) = C_2^2 + F(C_1), \qquad y(0) = C_1 \tag{22}$$

where $dF(y)/dy = 2f(y)$. In this fashion the second order equation (21) has been reduced to a first order equation (22). Presumably once (22) has been solved (21) has also been solved.

But it can happen that (21) is of a type which has unique solutions, whereas (22) does not. For example, the problem

$$\frac{d^2y}{dx^2} + y = 0, \qquad y(0) = 0, \quad \frac{dy(0)}{dx} = 1 \tag{23}$$

can be reduced to $(dy/dx)^2 + y^2 = 1$, $y(0) = 0$. One can easily check that each of the following is n acceptable solution of last equation:

$$\begin{array}{llll} y_1 = \sin x & 0 \le x < \infty & y_3 = \sin x & 0 \le x < \pi/2 \\ y_2 = \sin x & 0 \le x < \pi/2 & \quad = 1 & \pi/2 \le x < T \\ \quad = 1 & \pi/2 \le x < \infty & \quad = \cos(x - T) & T \le x < \infty \end{array}$$

Only y_1, however, also satisfies (23). The reason for the multiplicity of solutions is related to the fact that whenever $y^2 = 1$ neither value of dy/dx from reduced equation satisfies a Lipschitz condition. The question therefore rises, as to what additional conditions must be imposed on reduced equation in order to be sure that selected solution also satisfies (23). We must select a solution of reduced equation which is a least doubly differentiable everywhere. For (23) and reduced equation y_1 evidently is distinguished from the others by this requirement. Actually all solutions of (23) mast be analytic.

This problems have much physical interest since in (22), $(dy/dx)^2$ can be considered as a kinetic energy term and $F(y)$ a potential energy term so that (22) is a mathematical statement of the low of conservation of energy. (21) is the corresponding force equation.

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ASIMPTOTSKA SVOJSTVA NELINEARNIH DIFERENCIJALNIH JEDNAČINA

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Asimptotska svojstva rešenja su razmatrana za neke diferencijalne jednačine. Članak se bavi istraživanjem graničnih rešenja, produženjem rešenja, oscilatornim rešenjima i drugim asimptotskim svojstvima. Studirana su rešenja koja ilustruju dobijene metode i koja su od interesa za primenu u fizici.