FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 11, 2001, pp. 55 - 70

CONTRIBUTION TO THE DISCUSSION ON ABSOLUTE INTEGRATION OF DIFFERENTIAL EQUATIONS OF GEODESICS IN NON-EUCLIDEAN SPACE

UDC 517.93:514.13:528.232.22(045)

Zoran Drašković

Military Technical Institute, Katanićeva 15, 11000 Belgrade, Yugoslavia

Abstract. *In this paper: a) the existence of parallel propagation operators with respect to a surface, along a curve given on that surface, is once again pointed out; b) a closed form of these operators, in the case of a parallel transport with respect to a spherical surface and along its great circles, is obtained; c) the opinion that an absolute integration of differential equations of geodesics in non-Euclidean spaces is not possible in principle (because of the very method of introducing the covariant differentiation in these spaces) is stated.*

Key words: *Absolute integral, Geodesic lines,* "*Riemannian shifters*".

INTRODUCTION

In papers [8] and [11], V. Vujičić *postulated* the absolute integral of a tensor as an integral operator "*... by which it is possible to obtain the initial tensor from its absolute differential*" ([11], p. 375). For example, for an absolute integral of an absolute differential of a sufficiently smooth vectorial function **V**, from the point P_o to the point P on an *arbitrary* curve, the following formula was quoted

$$\overset{\nabla}{\int\limits_{P_o,P}} DV^\alpha = V^\alpha(P) - A^\alpha(P_o,P)\,, \tag{1}$$

where **A** is a covariantly constant vector field. It was demonstrated in paper [16] that this operation in Euclidean space reduces to an integration in accordance with Ericksen's concept of integration of vector and tensor fields in curvilinear coordinates (s. p. 808 in [4]) and that (1) can be rewritten in the form

Received October 05, 2000
Supported by the Serbian Ministry of Science and Technology, under Project No. 04M03A, through Mathematical Institute, Kneza Mihaila 35, 11000 Belgrade, Yugoslavia.
Presented at 5th YUSNM Niš 2000, Nonlinear Sciences at the Threshold of the Third Millenium, October 2-5, 2000, Faculty of Mechanical Engineering University of Niš

$$\left(\overset{\nabla}{\int\limits_{P_oP}} DV^{\alpha} \equiv\right) \int\limits_{P_oP} g^{\cdot\alpha}_{\beta}(M,P)\, DV^{\beta}(M) = V^{\alpha}(P) - V^{\beta}(P_o)\, g^{\cdot\alpha}_{\beta}(P_o,P), \tag{2}$$

where M is the "current" point of integration, and $g^{\cdot\alpha}_{\beta}$ are the shifting operators ("Euclidean shifters"; [4], p. 806); Einstein's summation convention for diagonally repeated indices is used, and Greek indices have the range $\{1,2\}$, while Latin indices will have the range $\{1,2,3\}$. The vector $V^{\beta}(P_o)\, g^{\cdot\alpha}_{\beta}(P_o,P)$, having been obtained by the parallel transport of the vector $V^{\beta}(P_o)$, represents a covariantly constant vector field.

However, when non-Euclidean spaces are in question, the doubt in the sense of the introduction of such an integral operator was still present in the audience on some communications of V. Vujičić. This was unintentionally due, perhaps, to the statement (in the paper [17], proposing the use of the idea of an absolute integral to solve some problems of analytical mechanics) that still "*... the problem of the covariantly constant tensor* [...**A**...] *in Riemannian spaces is not solved generally*" ([17], p. 1307).

But, in the meantime (when paper [1] was obtained), it was discovered that the introduction of the above mentioned operator − *a'priori* declared to be *nonsens* − was the subject of a communication on one of the sessions of the French Academy of Sciences as far back as 1929!

Namely, paper [1] considers the determination of a vector field **V** such that, along a curve K

$$u^{\alpha} = u^{\alpha}(t), \tag{3}$$

in a space equipped with linear connection, the absolute differential of this field is equal to

$$\frac{DV^{\alpha}}{Dt} = v^{\alpha} \qquad (Dt \equiv dt), \tag{4}$$

where $v^{\alpha}(t)$ is the field given at the points of the curve K. In the next step, Horák introduced "*un symbole d'intégration absolue le long d'une courbe*"

$${}_{t_o}\!\overset{-\,t}{\underline{|}}\, v^{\alpha} d\tau = K^{\alpha}_{\beta}(t_o,t) \int\limits_{t_o}^{t} K^{\beta}_{\cdot\gamma}(t_o,\tau)\, v^{\gamma}\, d\tau = \int\limits_{t_o}^{t} K^{\alpha}_{\gamma}(\tau,t)\, v^{\gamma}\, d\tau, \tag{5}$$

such that, after some stipulations quoted in [19] (and having in mind that the equation (4) should be satisfied along the curve K), the absolute integral can be written in the form

$${}_{t_o}\!\overset{-\,t}{\underline{|}}\, v^{\alpha} d\tau \left(= {}_{t_o}\!\overset{-\,t}{\underline{|}}\, DV^{\alpha}\right) = V^{\alpha}(t) - K^{\alpha}_{\gamma}(t_o,t)\, V^{\alpha}(t_o), \tag{6}$$

i.e.

$$\begin{aligned}\int\limits_{t_o}^{t} K^{\alpha}_{\gamma}(\tau,t)\, v^{\gamma}\, d\tau &= \int\limits_{P_oP} K^{\alpha}_{\gamma}(M,P)\, DV^{\gamma}(M) \left(\equiv \overset{\nabla}{\int\limits_{P_oP}} DV^{\alpha}\right), \\ &= V^{\alpha}(t) - K^{\alpha}_{\gamma}(t_o,t)\, V^{\gamma}(t_o)\end{aligned} \tag{7}$$

where the coefficients[1] K^{α}_{γ} represent the fundamental solution of the homogeneous system corresponding to the system (4); the following relations are valid

$$K^{\alpha}_{\gamma} \; K^{\gamma}_{.\beta} = \delta^{\alpha}_{\beta} \quad , \quad K^{.\gamma}_{\beta} \; K^{\alpha}_{.\gamma} = \delta^{\alpha}_{\beta} \, . \tag{8}$$

To be quite precise, the expression quoted in [1] was neither of the form postulated in [8] and [11] (namely, the absolute integral of an absolute differential is not mentioned, but only an "*intégrale absolue du vecteur ... prise le long de K entre les limites t_o et t*"), nor was its geometrical interpretation given, but it was *unambiguously* shown how to determine the coefficients K^{α}_{β} appearing in [1] – they represent the fundamental solution of the corresponding homogeneous system of differential equations. However, only the procedure of the introduction of the *parallel propagator* in [3][2] (p. 59) enabled us to link (in [19]) Vujičić's results with the ones Horák obtained; namely, it was noticed that these coefficients from [1] represent the shifting operator along a curve mentioned in [8], [11] and [17], making possible to evaluate the covariantly constant vector (tensor) **A**, as well as to determine a vector (tensor) field if its absolute differential (along a *given curve*) is known, i.e. to determine the absolute integral[3] introduced in (1).

The presentation of the paper [19] at the 21st Yugoslav Congress of Theoretical and Applied Mechanics (Niš, 1995.) was followed by a discussion between V. Vujičić and B. Jovanović (Mathematical Institute, Belgrade) and Đ. Đukić (Faculty of Engineering Sciences, Novi Sad), concerning the possibility of using the notion of an absolute integral in order to integrate the differential equations of geodesics in non-Euclidean space or, more precisely, concerning the procedure (proposed in [10] and [13]) for the reduction of the order of these equations. The following sections should represent a contribution to this discussion, also pointing out a dilemma which then appears. But, first of all we shall dwell on this procedure for

REDUCING OF THE ORDER OF THE DIFFERENTIAL EQUATIONS OF GEODESICS

The differential equations of geodesic lines in a Riemannian space, i.e. (if we dwell on the two-dimensional case) on a surface were formulated a long time ago

$$\frac{d^2u^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0 \, , \tag{9}$$

where u^{α} are so-called surface coordinates, $\Gamma^{\alpha}_{\beta\gamma}$ are Christoffel symbols of the second kind determined for this surface, and *s* is the arc length of the geodesic line. However, it was also stated a long time ago that, in the general case, the solution of these equations is unknown[4]. Namely, in order to verify the existence of a geodesic line passing through

[1] The first index in $\boldsymbol{K}(t_o,t)$, *either* superscript or subscript, refers to the point on curve *K* determined by the first argument, while the second one refers to the point determined by the second argument.

[2] Although, as we know (s. [15], p. 130), J.L. Synge himself has rejected the notion of an absolute integral.

[3] Of course, an integral defined in this way in non-Euclidean space is not, in general, independent of the chosen curve *K*.

[4] "*Notons qu'en général, on ne sait pas, sauf quelques cas particuliers, résoudre de telles équations différentielles.*" ([5], p. 134).

two points on a surface, i.e. through two points in a Riemannian space[5], a particular examination is necessary in each single case.

Hence, papers [10] and [13] must have drawn particular attention since, due to the introduction of the notion of an absolute integral, a simple procedure for the reduction of the order of the differential equations (9) was proposed.

The procedure is based on the possibility of rewriting the system (9) in the form

$$\frac{d}{ds}\left(\frac{du^{\alpha}}{ds}\right)+\Gamma^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds}=0, \tag{10}$$

i.e. in the form ($Ds \equiv ds$)

$$\frac{D}{Ds}\left(\frac{du^{\alpha}}{ds}\right)=0, \tag{11}$$

and hence

$$D\left(\frac{du^{\alpha}}{ds}\right)=0\cdot \tag{12}$$

If we knew that - for the vector field in the parentheses - the relation (12) holds along a *given* curve, then, in accordance with (7) and (4), we should write

$$\left(0=\int_{t_o}^{t}K^{\alpha}_{\beta}(\tau,t)\;0\;d\tau=\int_{P_oP}K^{\alpha}_{\beta}(M,P)\;D\left(\frac{du^{\beta}}{ds}\right)\bigg|_{M}=\right)$$

$$\overset{\nabla}{\int_{P_oP}}D\left(\frac{du^{\alpha}}{ds}\right)=\frac{du^{\alpha}}{ds}-K^{\alpha}_{\beta}(P_o,P)\;\frac{du^{\beta}}{ds}\bigg|_{P_o}=0\qquad, \tag{13}$$

i.e.

$$\frac{du^{\alpha}}{ds}=K^{\alpha}_{\beta}(P_o,P)\;\frac{du^{\beta}}{ds}\bigg|_{P_o}, \tag{14}$$

and the coefficients K^{α}_{β} would form the fundamental solution of the starting homogeneous system (11), the satisfying of which is required along this known curve.

However, the situation here is quite different from the one in (7) − instead of a *given* curve, now an *unknown* curve (a geodesic line), which should be determined from the condition that (11) is satisfied, is in question! But, an implicit supposition in the previous procedure is that in the considered space, i.e. on the considered surface there *exists* a geodesic line between the chosen points P_o and P (this follows from the classical theory of differential equations or from the calculus of variation), so (although this line is not known) the above mentioned absolute integration along the geodesic line is possible in principle (and, in principle, exists the corresponding fundamental system, i.e. the operators K^{α}_{β} of the parallel transport along this unknown geodesic line) − the

[5] S. [7], §17.3-12 and §17.4-2.

differential equations of geodesics of the first order[6] (14) are obtained in that very way, as the first integrals of the equations (9).

Notwithstanding all this, the further integration of the equations (14) "*is not solved generally*" ([15], p. 40) because "*the explicit form of the function ...* [K^{α}_{β}] *is not known*" ([13], p. 260), i.e. because "*the covariantly constant vector* **A** [$A^{\alpha} = K^{\alpha}_{\beta}\,(du^{\beta}/ds)|_{P_o}$] *is not determined in the general case*" ([15], p. 40). And at this moment, in an example in [15] (p. 41), the author resorted to the use of the result of Clairaut's theorem in order to determine the covariant coordinates of the vector **A** and then to solve the differential equations of geodesics[7], while the very problem of determining the coefficients K^{α}_{β}, i.e. the operators of parallel transport with respect to a surface (along a geodesic line lying on it) is put aside (with a comment that they cannot be obtained by extracting the "surface" part from the shifting operators of the corresponding enveloping Euclidean space; [15], p. 130. It seems that this is reason enough to say a few more words

ON THE DETERMINATION OF SHIFTING OPERATORS IN RIEMANNIAN SPACES[8]

For the sake of simplicity, we shall stay on the case of a surface in three-dimensional Euclidean space. As we know, the vectors $\mathbf{v}(P_o)$ and $\mathbf{v}(P)$ in a plane are parallel[9] if they form equal angles with the line connecting the points P_o and P. Similarly, one can say that the vectors $\mathbf{v}(P_o)$ and $\mathbf{v}(P)$, in tangent planes at the points P_o and P of a surface, are parallel if they form equal angles with the tangents (in P_o and P) of a geodesic line connecting these points on this surface (s. p. 143 in [6]).

Hence, in order to establish the relation between the coordinates of the vector **v** before and after its parallel transport with respect to a surface *along the geodesic line* connecting the points P_o and P (at the finite distance), we shall proceed in the following manner: let us introduce a surface coordinate system $\bar{u}^{\alpha}$, but in such a way[10] that the above mentioned geodesic line belongs, for example, to the $\bar{u}^1$- family of coordinate lines, while the lines of the $\bar{u}^2$ - family are *orthogonal* to the $\bar{u}^1$ ones. Bearing in mind that the vectors $\mathbf{v}(P_o)$ and $\mathbf{v}(P)$ have the same modulus and form equal angles with the coordinate line $\bar{u}^1$ at the points P_o and P, the equality of their projections at these points on the axes of the curvilinear coordinates $\bar{u}^{\alpha}$ follows

$$\mathbf{v}(P_o)\otimes\bar{\mathbf{t}}_{\alpha}(P_o)=\mathbf{v}(P)\otimes\bar{\mathbf{t}}_{\alpha}(P)\,, \tag{15}$$

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[6] A further step is made in paper [13], where the finite equations of geodesics are obtained under the supposition of the existence of a vector ρ^{α} such that $du^{\alpha}\big/ds = D\rho^{\alpha}\big/Ds$!

[7] When the concept of the absolute integral is not used to obtain the equations of geodesics with respect to surfaces, a resort to this theorem is made, too (s. e.g. p. 324 in [18]).

[8] This section is contained in the note "Contribution to an attempt of introduction of shifting operators in Riemannian spaces" (*private communication*, 1976), resulted from the first encounter with the notion of an absolute integral at V. Vujičić's communications, and this note was presented to him for inspection. Now ─ when there is no reason to doubt in the existence of shifting operators along a given curve (and hence along a geodesic line, too) on a surface ─ it seems to be the right moment to quote the subsequent results, which will be used in the next section.

[9] Here we take parallelism in a narrow sense, since vectors of equal intensities are considered.

[10] Cf. with geodesic polar coordinates in [2] (p. 177) and with Riemannian coordinates in [6] (pp. 166-167).

where

$$\bar{\mathbf{t}}_\alpha = \frac{\bar{\mathbf{a}}_\alpha}{|\bar{\mathbf{a}}_\alpha|}, \quad \bar{\mathbf{a}}_\alpha = \frac{\partial \mathbf{r}}{\partial \bar{u}^\alpha}; \qquad |\bar{\mathbf{a}}_\alpha| = \sqrt{\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\alpha} = \sqrt{\bar{a}_{\alpha\alpha}} \tag{16}$$

the following holds

$$\frac{\bar{v}_\alpha(P_o)}{\sqrt{\bar{a}_{\alpha\alpha}(P_o)}} = \frac{\bar{v}_\alpha(P)}{\sqrt{\bar{a}_{\alpha\alpha}(P)}} \qquad (\, non\ \Sigma_\alpha) \cdot \tag{17}$$

If we now introduce some other arbitrary surface coordinates u^α

$$\left.\begin{aligned} u^\alpha &= u^\alpha(\bar{u}^\beta) \\ \bar{u}^\alpha &= \bar{u}^\alpha(u^\beta) \end{aligned}\right\}, \tag{18}$$

it will be

$$\bar{v}_\alpha = \frac{\partial u^\beta}{\partial \bar{u}^\alpha} v_\beta, \tag{19}$$

of course, in the point of the coordinate transformation; hence it follows

$$\frac{v_\beta(P_o)}{\sqrt{\bar{a}_{\alpha\alpha}(P_o)}} \left.\frac{\partial u^\beta}{\partial \bar{u}^\alpha}\right|_{P_o} = \frac{v_\beta(P)}{\sqrt{\bar{a}_{\alpha\alpha}(P)}} \left.\frac{\partial u^\beta}{\partial \bar{u}^\alpha}\right|_{P} \qquad (\, non\ \Sigma_\alpha), \tag{20}$$

and (bearing in mind that α is a free index), after the composition with $\left.\frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\right|_{P_o}$, we obtain[11]

$$v_\gamma(P_o) = \sqrt{\frac{\bar{a}_{(\alpha)(\alpha)}(P_o)}{\bar{a}_{(\alpha)(\alpha)}(P)}} \left.\frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\right|_{P_o} \left.\frac{\partial u^\beta}{\partial \bar{u}^\alpha}\right|_{P} v_\beta(P) \cdot \tag{21}$$

This expression can be rewritten in the form

$$v_\gamma(P_o) = K_\gamma^{\cdot\beta}(P_o, P)\, v_\beta(P), \tag{22}$$

where the quantities (let us call them "Riemannian shifters")

$$K_\gamma^{\cdot\beta}(P_o, P) = \sqrt{\frac{\bar{a}_{(\alpha)(\alpha)}(P_o)}{\bar{a}_{(\alpha)(\alpha)}(P)}} \left.\frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\right|_{P_o} \left.\frac{\partial u^\beta}{\partial \bar{u}^\alpha}\right|_{P} \tag{23}$$

obviously establish a relation between the coordinates of parallel surface vectors with respect to an arbitrary surface system u^α, i.e. they take the role of the above introduced operators of parallel transport with respect to a surface[12]; therefore, we have obtained their analytical expressions – of course, on the condition that the geodesic lines on the

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[11] The placement of an index in parentheses means that the summation convention is not applied to the corresponding member - for example in summation over α in (21) this member is simply associated to the other members with this index.

[12] It is noticeable that the expression (23), obtained for the operators of parallel transport along the geodesics on a certain surface, is analogous to that for "Euclidean shifters" ([4], p. 808), where the coordinates $\bar{u}^\alpha$, introduced in the above described manner, now play the role of Cartesian coordinates.

surface under consideration are *known* (these expressions will be used in the next section for determining the shifting operators on a spherical surface).

Let us mention that it is easy to show that, for the inverse operators, we have

$$K^{\cdot\beta}_{\gamma}(P_o,P)=\sqrt{\frac{\overline{a}_{(\alpha)(\alpha)}(P)}{\overline{a}_{(\alpha)(\alpha)}(P_o)}}\left.\frac{\partial u^{\beta}}{\partial \overline{u}^{\alpha}}\right|_{P_o}\left.\frac{\partial \overline{u}^{\alpha}}{\partial u^{\gamma}}\right|_{P} \tag{24}$$

and it holds that (s. (8))

$$K^{\cdot\beta}_{\gamma}(P_o,P)\ K^{\alpha}_{\cdot\beta}(P_o,P)=\delta^{\alpha}_{\gamma}\,. \tag{25}$$

OPERATORS OF PARALLEL TRANSPORT ON A SPHERICAL SURFACE

Bearing in mind that the geodesic lines on a spherical surface are its *great circles*, we shall choose the coordinates $\overline{u}^{\alpha}$ (appearing in expression (23) for the coordinates of the shifting operators) to be the geographical coordinates ($\overline{u}^1\equiv\overline{\varphi},\ \overline{u}^2\equiv\overline{\vartheta}$) in a spherical polar system $\{\overline{r},\overline{\varphi},\overline{\vartheta}\}$ corresponding to the Cartesian system $\overline{z}^i$ ($\overline{z}^1\equiv\overline{x},\ \overline{z}^2\equiv\overline{y},\ \overline{z}^3\equiv\overline{z}$) whose plane $O\overline{z}^1\overline{z}^2$ (i.e. $O\overline{xy}$) coincides with the plane OP_oP, where P_o and P are arbitrary points on the spherical surface. In this way, we managed to make the geodesic line, i.e. the great circle passing through the points P_o and P, belong to the $\overline{u}^1\equiv\overline{\varphi}$ - family of coordinate lines (more precisely, lie on the equator). Therefore, the expressions (23) can be used, but now (knowing that the diagonal coordinates of the fundamental metric tensor in the system $\{\overline{\varphi},\overline{\vartheta}\}$ are $\overline{a}_{11}=\overline{r}^2\cos^2\overline{\vartheta},\ \overline{a}_{22}=\overline{r}^2$ as well as that $\overline{\vartheta}_P=\overline{\vartheta}_o=0$) they reduce to

$$K^{\cdot\beta}_{\gamma}(P_o,P)=\left.\frac{\partial \overline{u}^{\alpha}}{\partial u^{\gamma}}\right|_{P_o}\left.\frac{\partial u^{\beta}}{\partial \overline{u}^{\alpha}}\right|_{P}\,. \tag{26}$$

However, in order to obtain the effective expressions for the operators of parallel transport with respect to a spherical surface (along its great circle), i.e. to determine the partial derivatives in (26), one should establish the relations (18) between the surface coordinates u^{α} and $\overline{u}^{\alpha}$. To realize this, and bearing in mind that $\overline{u}^{\alpha}$ (i.e. $\{\overline{\varphi},\overline{\vartheta}\}$) are the geographical coordinates on a spherical surface, we shall choose u^{α} as the geographical coordinates on this surface too (i.e. $u^1\equiv\varphi,\ u^2\equiv\vartheta$), but corresponding to another Cartesian system z^i (a "fixed" one, in which the points P_o and P are given); then the expressions (26) can be rewritten in a developed form (using $u^1\equiv\varphi,\ u^2\equiv\vartheta,$ $\overline{u}^1\equiv\overline{\varphi},\ \overline{u}^2\equiv\overline{\vartheta}$)

$$\begin{aligned}
K^{\cdot 1}_{1}(P_o,P)&=\left.\frac{\partial\overline{\varphi}}{\partial\varphi}\right|_{P_o}\left.\frac{\partial\varphi}{\partial\overline{\varphi}}\right|_{P}+\left.\frac{\partial\overline{\vartheta}}{\partial\varphi}\right|_{P_o}\left.\frac{\partial\varphi}{\partial\overline{\vartheta}}\right|_{P}\ , \quad K^{\cdot 1}_{2}(P_o,P)=\left.\frac{\partial\overline{\varphi}}{\partial\vartheta}\right|_{P_o}\left.\frac{\partial\varphi}{\partial\overline{\varphi}}\right|_{P}+\left.\frac{\partial\overline{\vartheta}}{\partial\vartheta}\right|_{P_o}\left.\frac{\partial\varphi}{\partial\overline{\vartheta}}\right|_{P}\ ,\\
K^{\cdot 2}_{1}(P_o,P)&=\left.\frac{\partial\overline{\varphi}}{\partial\varphi}\right|_{P_o}\left.\frac{\partial\vartheta}{\partial\overline{\varphi}}\right|_{P}+\left.\frac{\partial\overline{\vartheta}}{\partial\varphi}\right|_{P_o}\left.\frac{\partial\vartheta}{\partial\overline{\vartheta}}\right|_{P}\ , \quad K^{\cdot 2}_{2}(P_o,P)=\left.\frac{\partial\overline{\varphi}}{\partial\vartheta}\right|_{P_o}\left.\frac{\partial\vartheta}{\partial\overline{\varphi}}\right|_{P}+\left.\frac{\partial\overline{\vartheta}}{\partial\vartheta}\right|_{P_o}\left.\frac{\partial\vartheta}{\partial\overline{\vartheta}}\right|_{P}\ .
\end{aligned} \tag{27}$$

However, in order to determine the partial derivatives

$$\left.\frac{\partial\overline{\varphi}}{\partial\varphi}\right|_{P_o},\ \left.\frac{\partial\overline{\varphi}}{\partial\vartheta}\right|_{P_o},\ \left.\frac{\partial\overline{\vartheta}}{\partial\varphi}\right|_{P_o},\ \left.\frac{\partial\overline{\vartheta}}{\partial\vartheta}\right|_{P_o} \quad \text{and} \quad \left.\frac{\partial\varphi}{\partial\overline{\varphi}}\right|_{P},\ \left.\frac{\partial\varphi}{\partial\overline{\vartheta}}\right|_{P},\ \left.\frac{\partial\vartheta}{\partial\overline{\varphi}}\right|_{P},\ \left.\frac{\partial\vartheta}{\partial\overline{\vartheta}}\right|_{P}, \tag{28}$$

but, not having the explicit expressions for the relations between the systems $\{r,\varphi,\vartheta\}$ and $\{\overline{r},\overline{\varphi},\overline{\vartheta}\}$, i.e. (because of $r=\overline{r}$) between the systems $\{\varphi,\vartheta\}$ and $\{\overline{\varphi},\overline{\vartheta}\}$, we should use the following relations

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial\overline{\varphi}} &= \frac{\partial\varphi}{\partial z^i}\frac{\partial z^i}{\partial\overline{z}^j}\frac{\partial\overline{z}^j}{\partial\overline{\varphi}} \\
\frac{\partial\varphi}{\partial\overline{\vartheta}} &= \frac{\partial\varphi}{\partial z^i}\frac{\partial z^i}{\partial\overline{z}^j}\frac{\partial\overline{z}^j}{\partial\overline{\vartheta}} \\
\frac{\partial\vartheta}{\partial\overline{\varphi}} &= \frac{\partial\vartheta}{\partial z^i}\frac{\partial z^i}{\partial\overline{z}^j}\frac{\partial\overline{z}^j}{\partial\overline{\varphi}} \\
\frac{\partial\vartheta}{\partial\overline{\vartheta}} &= \frac{\partial\vartheta}{\partial z^i}\frac{\partial z^i}{\partial\overline{z}^j}\frac{\partial\overline{z}^j}{\partial\overline{\vartheta}}
\end{aligned}\right\} \qquad
\begin{aligned}
\frac{\partial\overline{\varphi}}{\partial\varphi} &= \frac{\partial\overline{\varphi}}{\partial\overline{z}^i}\frac{\partial\overline{z}^i}{\partial z^j}\frac{\partial z^j}{\partial\varphi} \\
\frac{\partial\overline{\varphi}}{\partial\vartheta} &= \frac{\partial\overline{\varphi}}{\partial\overline{z}^i}\frac{\partial\overline{z}^i}{\partial z^j}\frac{\partial z^j}{\partial\vartheta} \\
\frac{\partial\overline{\vartheta}}{\partial\varphi} &= \frac{\partial\overline{\vartheta}}{\partial\overline{z}^i}\frac{\partial\overline{z}^i}{\partial z^j}\frac{\partial z^j}{\partial\varphi} \\
\frac{\partial\overline{\vartheta}}{\partial\vartheta} &= \frac{\partial\overline{\vartheta}}{\partial\overline{z}^i}\frac{\partial\overline{z}^i}{\partial z^j}\frac{\partial z^j}{\partial\vartheta}
\end{aligned}\right\}. \tag{29}$$

Namely, on the one hand we know the relations between Cartesian and spherical coordinates

$$\left.\begin{aligned}
z^1 &= r\cos\varphi\cos\vartheta \\
z^2 &= r\sin\varphi\cos\vartheta \\
z^3 &= r\sin\vartheta
\end{aligned}\right\} \qquad
\begin{aligned}
r &= \sqrt{\left(z^1\right)^2+\left(z^2\right)^2+\left(z^3\right)^2} \\
\operatorname{tg}\varphi &= z^2/z^1 \\
\operatorname{tg}\vartheta &= z^3\Big/\sqrt{\left(z^1\right)^2+\left(z^2\right)^2}
\end{aligned}\right\}, \tag{30}$$

as well as the relations

$$\left.\begin{aligned}
&\frac{\partial z^1}{\partial r}=\cos\varphi\cos\vartheta\ , && \frac{\partial z^1}{\partial\varphi}=-r\sin\varphi\cos\vartheta\ , && \frac{\partial z^1}{\partial\vartheta}=-r\cos\varphi\sin\vartheta \\
&\frac{\partial z^2}{\partial r}=\sin\varphi\cos\vartheta\ , && \frac{\partial z^2}{\partial\varphi}=\ r\cos\varphi\cos\vartheta\ , && \frac{\partial z^2}{\partial\vartheta}=-r\sin\varphi\sin\vartheta \\
&\frac{\partial z^3}{\partial r}=\sin\vartheta\ , && \frac{\partial z^3}{\partial\varphi}=\ 0\ , && \frac{\partial z^3}{\partial\vartheta}=\ r\cos\vartheta
\end{aligned}\right\} \tag{31}$$

and their inverse

$$\left.\begin{aligned}
&\frac{\partial r}{\partial z^1}=\ \cos\varphi\cos\vartheta\ , && \frac{\partial r}{\partial z^2}=\ \sin\varphi\cos\vartheta\ , && \frac{\partial r}{\partial z^3}=\sin\vartheta \\
&\frac{\partial\varphi}{\partial z^1}=-\frac{\sin\varphi}{r\cos\vartheta}\ , && \frac{\partial\varphi}{\partial z^2}=\ \frac{\cos\varphi}{r\cos\vartheta}\ , && \frac{\partial\varphi}{\partial z^3}=0 \\
&\frac{\partial\vartheta}{\partial z^1}=-\frac{\cos\varphi\sin\vartheta}{r}\ , && \frac{\partial\vartheta}{\partial z^2}=-\frac{\sin\varphi\sin\vartheta}{r}\ , && \frac{\partial\vartheta}{\partial z^3}=\frac{\cos\vartheta}{r}
\end{aligned}\right\} \tag{32}$$

(analogously is for the relations between $\overline{z}^i$ and $\{\overline{r},\overline{\varphi},\overline{\vartheta}\}$), and, on the other hand, between the Cartesian systems z^i and $\overline{z}^i$ there exist the relations

$$z^i = a^i_{.j}\bar{z}^j \quad , \quad \bar{z}^i = a^{.i}_j z^j \quad (a^{.i}_j \equiv a^j_{.i}), \tag{33}$$

where $a^i_{.j}$ are the cosines of the angles between the axes of these systems and

$$\frac{\partial z^i}{\partial \bar{z}^j} = a^i_{.j} \quad , \quad \frac{\partial \bar{z}^i}{\partial z^j} = a^{.i}_j \cdot \tag{34}$$

As is known, the $a^i_{.j}$'s can be expressed in terms of Euler angles, but the usual relations (due to a suitable choice of the angle of proper rotation, such that $\varphi_{Eu} = 0$, i.e. the axis $\bar{z}^1$ lies in the plane Oz^1z^2) are now reduced and read

$$\begin{array}{lll} a^1_{.1} = \cos\psi_{Eu} & a^1_{.2} = -\sin\psi_{Eu}\cos\vartheta_{Eu} & a^1_{.3} = \sin\psi_{Eu}\sin\vartheta_{Eu} \\ a^2_{.1} = \sin\psi_{Eu} & a^2_{.2} = \cos\psi_{Eu}\cos\vartheta_{Eu} & a^2_{.3} = -\cos\psi_{Eu}\sin\vartheta_{Eu} \\ a^3_{.1} = 0 & a^3_{.2} = \sin\vartheta_{Eu} & a^3_{.3} = \cos\vartheta_{Eu} \end{array} . \tag{35}$$

As for the angles of precession ψ_{Eu} and nutation ϑ_{Eu}, the first of these (as the angle of inclination of the line which represents the intersection of the plane OP_oP and the coordinate plane Oz^1z^2) can be expressed in the form

$$\mathrm{tg}\,\psi_{Eu} = \frac{\sin\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\sin\varphi_P\cos\vartheta_P}{\cos\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\cos\varphi_P\cos\vartheta_P}, \tag{36}$$

and the second (as the angle between the normals to the planes Oz^1z^2 and OP_oP) in the form

$$\cos\vartheta_{Eu} = \frac{\cos\varphi_o\cos\vartheta_o\sin\varphi_P\cos\vartheta_P - \sin\varphi_o\cos\vartheta_o\cos\varphi_P\cos\vartheta_P}{\sqrt{\begin{array}{l}(\sin\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\sin\varphi_P\cos\vartheta_P)^2 + \\ +(\sin\vartheta_o\cos\varphi_P\cos\vartheta_P - \cos\varphi_o\cos\vartheta_o\sin\vartheta_P)^2 + \\ +(\cos\varphi_o\cos\vartheta_o\sin\varphi_P\cos\vartheta_P - \sin\varphi_o\cos\vartheta_o\cos\varphi_P\cos\vartheta_P)^2\end{array}}}; \tag{37}$$

their dependence on the coordinates (φ_o, ϑ_o) and (φ_p, ϑ_p), i.e. of the points P_o and P respectively, is obvious.

Taking into account the expressions (31), (32), (34) and (35), replacing them in (29) and determining the derivatives (28) appearing in (27), we obtain the following explicit expressions, in geographical coordinates, for the operators of parallel transport with respect to a spherical surface along the geodesic line (the great circle) connecting P_o and P

$$\begin{aligned} K^{.1}_1(P_o,P) = \frac{\cos\vartheta_o}{\cos\vartheta_P} \Big\{ & [\sin\bar{\varphi}_P\sin(\varphi_P - \psi_{Eu}) + \cos\bar{\varphi}_P\cos(\varphi_P - \psi_{Eu})\cos\vartheta_{Eu}]\times \\ & \times[\sin\bar{\varphi}_o\sin(\varphi_o - \psi_{Eu}) + \cos\bar{\varphi}_o\cos(\varphi_o - \psi_{Eu})\cos\vartheta_{Eu}] + \\ & + \cos(\varphi_P - \psi_{Eu})\cos(\varphi_o - \psi_{Eu})\sin^2\vartheta_{Eu} \Big\} \\ K^{.1}_2(P_o,P) = \frac{1}{\cos\vartheta_P} \Big\{ & [\sin\bar{\varphi}_P\sin(\varphi_P - \psi_{Eu}) + \cos\bar{\varphi}_P\cos(\varphi_P - \psi_{Eu})\cos\vartheta_{Eu}]\times \\ & \times\{\sin\vartheta_o[\sin\bar{\varphi}_o\cos(\varphi_o - \psi_{Eu}) - \cos\bar{\varphi}_o\sin(\varphi_o - \psi_{Eu})\cos\vartheta_{Eu}] + \\ & + \cos\vartheta_o\sin\vartheta_{Eu}\cos\bar{\varphi}_o\} - \\ & - \cos(\varphi_P - \psi_{Eu})\sin\vartheta_{Eu}[\sin\vartheta_o\sin(\varphi_o - \psi_{Eu})\sin\vartheta_{Eu} + \cos\vartheta_o\cos\vartheta_{Eu}] \Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} K_1^2(P_o,P) = \ \cos\vartheta_o \ \Big\{ & \{\sin\vartheta_P[\sin\overline{\varphi}_P\cos(\varphi_P-\psi_{Eu})-\cos\overline{\varphi}_P\sin(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]+ \\ & +\cos\vartheta_P\sin\vartheta_{Eu}\cos\overline{\varphi}_P\}\times \\ & \times[\sin\overline{\varphi}_o\sin(\varphi_o-\psi_{Eu})+\cos\overline{\varphi}_o\cos(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}] \ - \\ & -\cos(\varphi_o-\psi_{Eu})\sin\vartheta_{Eu}[\sin\vartheta_P\sin(\varphi_P-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_P\cos\vartheta_{Eu}] \Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} K_2^2(P_o,P) = \ & \{\sin\vartheta_P[\sin\overline{\varphi}_P\cos(\varphi_P-\psi_{Eu})-\cos\overline{\varphi}_P\sin(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]+ \\ & +\cos\vartheta_P\sin\vartheta_{Eu}\cos\overline{\varphi}_P\}\times \\ & \times\{\sin\vartheta_o[\sin\overline{\varphi}_o\cos(\varphi_o-\psi_{Eu})-\cos\overline{\varphi}_o\sin(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}]+ \\ & +\cos\vartheta_o\sin\vartheta_{Eu}\cos\overline{\varphi}_o\} \ + \\ & + \ [\sin\vartheta_P\sin(\varphi_P-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_P\cos\vartheta_{Eu}]\times \\ & \times[\sin\vartheta_o\sin(\varphi_o-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_o\cos\vartheta_{Eu}] \end{aligned}$$

It should be noted that these operators are indeed functions of the points P_o and P, i.e. of the coordinates (φ_o,ϑ_o) and (φ_P,ϑ_P) only. For ψ_{Eu} and ϑ_{Eu}, this is evident from (36) and (37), while, for $\overline{\varphi}_o$ and $\overline{\varphi}_P$, the following relations can easily be established

$$\left.\begin{aligned} \cos\overline{\varphi}_o &= \cos\vartheta_o\cos(\varphi_o-\psi_{Eu}) \\ \cos\overline{\varphi}_P &= \cos\vartheta_P\cos(\varphi_P-\psi_{Eu}) \end{aligned}\right\}, \tag{39}$$

and the above statement again holds (we remember that $\overline{\vartheta}_P=\overline{\vartheta}_o=0$).

The fact that the operators (38) are obtained by using a *heuristic* procedure – and not by solving the homogeneous system (11), i.e. the system

$$\frac{dV^\alpha}{ds} \ + \ \Gamma^\alpha_{\beta\gamma} \ V^\beta\frac{du^\gamma}{ds} \ = \ 0 \tag{40}$$

for an *arbitrary* vector **V** (where (40) represents the condition of its parallel transport along a curve, as well a geodesics) – should not be surprising, since the existence of a fundamental solution (it *does* exist for the system (40) along a *given* curve) does not, implicitly, mean that it is easy to find; on the other hand, this approach could cause the concern regarding the correctness of operators so obtained.

In order to allay this concern, let us look for the fundamental solution of the system (40) when geographical coordinates are in question ($u^1\equiv\varphi,\ u^2\equiv\vartheta$). In this case (when only the three coordinates of Christoffel symbols of the second kind are non-zero: $\Gamma^1_{12}=\Gamma^1_{21}=-\mathrm{tg}\,\vartheta,\ \Gamma^2_{11}=\sin\vartheta\cos\vartheta$), it reduces to

$$\left.\begin{aligned} \frac{dV^1}{ds} \ -\mathrm{tg}\,\vartheta\frac{d\vartheta}{ds}V^1-\mathrm{tg}\,\vartheta\frac{d\varphi}{ds}V^2 \ &= \ 0 \\ \frac{dV^2}{ds} \ + \ \sin\vartheta\cos\vartheta \ \frac{d\varphi}{ds}V^1 \ &= \ 0 \end{aligned}\right\}. \tag{41}$$

Some special cases of parallel transport of a vector along curves on a spherical surface will now be considered.

Let us start with propagation along the equator. In this case, since $\vartheta = 0$, (41) reduces to

$$\left.\begin{aligned} dV^1/ds &= 0 \\ dV^2/ds &= 0 \end{aligned}\right\}; \tag{42}$$

it is obvious that the following two solutions of this system

$$\begin{Bmatrix} V^1_{(1)} \\ V^2_{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} V^1_{(2)} \\ V^2_{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{43}$$

form the fundamental system for (42), because of

$$Det \begin{Bmatrix} V^1_{(1)} & V^1_{(2)} \\ V^2_{(1)} & V^2_{(2)} \end{Bmatrix} = Det \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \neq 0 \tag{44}$$

(s. e.g. p. 73 in [9]). However, when parallel transport along the equatorial circle is in question ($\vartheta_P = \vartheta_o = 0, \quad \varphi_P \neq \varphi_o$), then the operators K^α_β reduce to

$$\left\{K^\alpha_\beta\right\} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}; \tag{45}$$

so, the matrix of these coefficients is obviously fundamental.

In the next case, parallel transport on a spherical surface is still in question, but now along a meridian. Then $\varphi = const.$, and the system (41) reduces to

$$\left.\begin{aligned} dV^1/V^1 &= \operatorname{tg}\vartheta \, d\vartheta \\ dV^2/ds &= 0 \end{aligned}\right\}; \tag{46}$$

the following two solutions of this system

$$\begin{Bmatrix} V^1_{(1)} \\ V^2_{(1)} \end{Bmatrix} = \begin{Bmatrix} \cos\vartheta_o / \cos\vartheta \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} V^1_{(2)} \\ V^2_{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \tag{47}$$

form its fundamental system due to

$$Det \begin{Bmatrix} V^1_{(1)} & V^1_{(2)} \\ V^2_{(1)} & V^2_{(2)} \end{Bmatrix} = Det \begin{Bmatrix} \dfrac{\cos\vartheta_o}{\cos\vartheta} & 0 \\ 0 & 1 \end{Bmatrix} \neq 0 \, . \tag{48}$$

On the other hand, bearing in mind that parallel transport along a meridian is in question ($\varphi_P = \varphi_o, \quad \vartheta_P \neq \vartheta_o$), for the operators K^α_β we obtain

$$\left\{K^\alpha_\beta\right\} = \begin{Bmatrix} \dfrac{\cos\vartheta_o}{\cos\vartheta_P} & 0 \\ 0 & 1 \end{Bmatrix}. \tag{49}$$

Therefore, it is obvious that the matrix of these coefficients is fundamental in this case, too (we suppose the point P to be variable, i.e. that $\vartheta_P \equiv \vartheta$).

Remark 1. It now seems to be the right moment to compare the results (38), obtained for the operators $\left\{K^{\alpha}_{\beta}\right\}$, with the expressions for "Euclidean shifters" $\left\{g^{\cdot i}_{j}\right\}$ in spherical polar coordinates (s. e.g. p. 146 in [10] and p. 401 in [14])

$$\left\{\begin{matrix} \begin{matrix}\cos\vartheta_P\cos\vartheta_o\cos(\varphi_P-\varphi_o)+\\ +\sin\vartheta_P\sin\vartheta_o\end{matrix} & -r_o\cos\vartheta_P\cos\vartheta_o\sin(\varphi_P-\varphi_o) & \begin{matrix}-r_o\cos\vartheta_P\sin\vartheta_o\cos(\varphi_P-\varphi_o)+\\ +r_o\sin\vartheta_P\cos\vartheta_o\end{matrix} \\ -\dfrac{\cos\vartheta_o}{r_P\cos\vartheta_P}\sin(\varphi_P-\varphi_o) & \dfrac{r_o\cos\vartheta_o}{r_P\cos\vartheta_P}\cos(\varphi_P-\varphi_o) & \dfrac{r_o\sin\vartheta_o}{r_P\cos\vartheta_P}\sin(\varphi_P-\varphi_o) \\ \begin{matrix}-\dfrac{1}{r_P}\sin\vartheta_P\cos\vartheta_o\cos(\varphi_P-\varphi_o)+\\ +\dfrac{1}{r_P}\cos\vartheta_P\sin\vartheta_o\end{matrix} & -\dfrac{r_o}{r_P}\sin\vartheta_P\cos\vartheta_o\sin(\varphi_P-\varphi_o) & \begin{matrix}\dfrac{r_o}{r_P}\sin\vartheta_P\sin\vartheta_o\cos(\varphi_P-\varphi_o)+\\ +\dfrac{r_o}{r_P}\cos\vartheta_P\cos\vartheta_o\end{matrix} \end{matrix}\right\},$$

i.e. with the corresponding submatrix relating to a spherical surface (r_o=r_P)

$$\left\{\begin{matrix} \dfrac{\cos\vartheta_o}{\cos\vartheta_P}\cos(\varphi_P-\varphi_o) & \dfrac{\sin\vartheta_o}{\cos\vartheta_P}\sin(\varphi_P-\varphi_o) \\ -\sin\vartheta_P\cos\vartheta_o\sin(\varphi_P-\varphi_o) & \sin\vartheta_P\sin\vartheta_o\cos(\varphi_P-\varphi_o)+\cos\vartheta_P\cos\vartheta_o \end{matrix}\right\}. \tag{50}$$

At first glance, we notice that (38) differs from (50). This, however, may not seem immediately obvious to a more inquisitive reader (because of the complexity of expression (38), which can probably be further simplified), so we can consider two special cases. First, let the points P_o and P lie on the equator ($\vartheta_P = \vartheta_o = 0,\ \ \varphi_P \neq \varphi_o$); then (50) reduces to

$$\left\{\begin{matrix} \cos(\varphi_P-\varphi_o) & 0 \\ 0 & 1 \end{matrix}\right\}, \tag{51}$$

and this differs obviously from the matrix (45) corresponding to the operators K^{α}_{β} in that case. But if the points P_o and P lie on a meridian ($\varphi_P = \varphi_o,\ \ \vartheta_P \neq \vartheta_o$), then (50) reduces to

$$\left\{\begin{matrix} \dfrac{\cos\vartheta_o}{\cos\vartheta_P} & 0 \\ 0 & \cos(\vartheta_P-\vartheta_o) \end{matrix}\right\}, \tag{52}$$

and this also differs from the matrix (49) now corresponding to the operators K^{α}_{β}. Therefore, we can indeed say that the operators of parallel transport with respect to a surface (and, generally, in a Riemannian space) differ in principle from the "Euclidean shifters" for the corresponding enveloping Euclidean space, more precisely from their "surface" part. This just confirms the above mentioned note in [15], p. 130

Remark 2. Now, when we have obtained the *analytical* expressions for the operators of parallel transport K^{α}_{β} along the great circles on a spherical surface, the covariant coordinates of a vector shifted on this surface from point P_o to point P (along the arc of the great circle connecting them) would be calculated according to the formula

$$v^{\alpha}(P) = K_{\beta}^{\alpha}(P_o, P)\, v^{\beta}(P_o) \tag{53}$$

(where $v^1 \equiv v^{\varphi}$, $v^2 \equiv v^{\vartheta}$), and we can then determine the Cartesian coordinates of this vector at the point P in the usual way

$$v^i(P) = \left.\frac{\partial z^i}{\partial \varphi}\right|_P v^{\varphi}(P) + \left.\frac{\partial z^i}{\partial \vartheta}\right|_P v^{\vartheta}(P) \tag{54}$$

(but now $v^1 \equiv v^x \equiv v^{z^1}$, $v^2 \equiv v^y \equiv v^{z^2}$, $v^3 \equiv v^z \equiv v^{z^3}$).

This procedure is used to calculate the Cartesian coordinates of a given unit vector **v** after its parallel transport on a spherical surface (with the radius r) from point P_o to point P along the great circle; these points are given by their geographical coordinates $\{\varphi_o, \vartheta_o\}$ and $\{\varphi_p, \vartheta_p\}$, where the angle α_o between this unit vector and the geographic parallel was prescribed at point P_o too. The results for a few arbitrarily selected pairs of points on the spherical surface are quoted in the Table 1. In this table, the Cartesian coordinates of the vector **v** obtained directly (without introducing the notion of the operator of parallel transport with respect to a surface) from the condition that a vector shifted along a geodesic line must close a constant angle with this curve at every its point, are also quoted (s. p. 143 in [6]). This was performed by a special software tool, used to generate the Figure 1, as well.

Table 1

P_o	P
$\varphi_o = 3°$	$\varphi_P = 76°$
$\vartheta_o = 15°$	$\vartheta_P = 79°$
$\alpha_o = 23°$	

	Cartesian coordinates of a given unit vector **v** after parallel transport with respect to a spherical surface ($r = 5$) from point P_o to point P along the great circle	
$\mathbf{v}_P$	(*analytical approach*)	(*numerical approach*)
v_P^x:	–0.5609105726399270	–0.5609105726399270
v_P^y:	0.8179382961038478	0.8179382961038477
v_P^z:	–0.1278916465899297	–0.1278916465899297

P_o	P
$\varphi_o = 10°$	$\varphi_P = 80°$
$\vartheta_o = 15°$	$\vartheta_P = 85°$
$\alpha_o = 60°$	

	Cartesian coordinates of a given unit vector **v** after parallel transport with respect to a spherical surface ($r = 10$) from point P_o to point P along the great circle	
$\mathbf{v}_P$	(*analytical approach*)	(*numerical approach*)
v_P^x:	–0.9592179801699705	–0.9592179801699705
v_P^y:	0.2824986141850212	0.2824986141850212
v_P^z:	–9.7672668738299610E-3	–9.7672668738299595E-3

P_o	P
$\varphi_o = 17°$	$\varphi_P = 66°$
$\vartheta_o = 10°$	$\vartheta_P = 77°$
$\alpha_o = 30°$	

	Cartesian coordinates of a given unit vector **v** after parallel transport with respect to a spherical surface ($r = 10$) from point P_o to point P along the great circle	
$\mathbf{v}_P$	(*analytical approach*)	(*numerical approach*)
v_P^x:	–0.8188552843021616	–0.8188552843021616
v_P^y:	0.5723252631531620	0.5723252631531620
v_P^z:	–4.3815710961823556E-2	–4.3815710961823552E-2

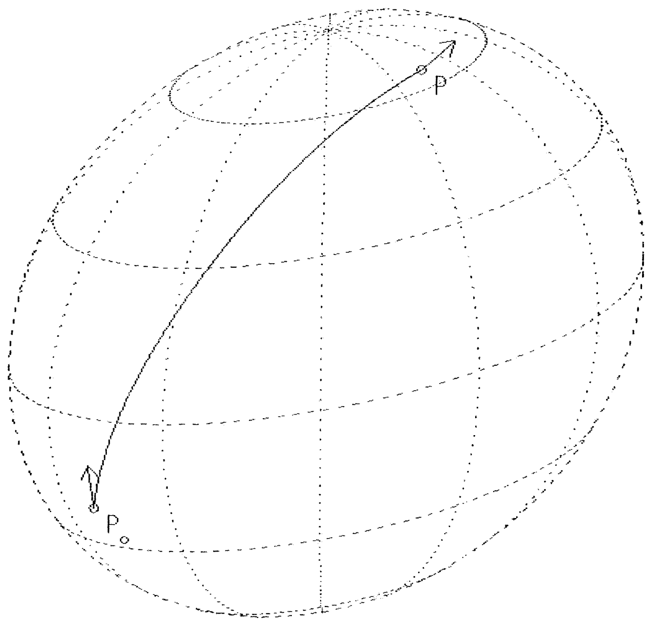

Fig. 1.

The conformance of these two groups of results represents a *numerical* confirmation of the correctness of the previously obtained expression for shifting operators on a spherical surface; we consider this examination to be a very advisable one – on the one hand, because of the fact that these expressions, as well as the approach to their derivation, are new (at least judging from the available literature) and, on the other hand, because the complexity[13] of these operators indisputably increases the possibility of an error.

At the end of this section we conclude the following: even though the former efforts to determine the shifting operators might resemble a "search for the Holy Grail", we have nevertheless managed to obtain, for a particular example, a closed form of these operators, but along a *known* geodesic line.

However, the question from the above mentioned discussion – does the reduction of the order of the differential equations of geodesics make their solving possible? – is not resolved in this manner. Since, on the one hand, it was pointed out (p. 40 in [15]) that the further integration of the equations of the first order (14) "*is not solved generally*", and, on the other hand, we are more and more convinced that the reduction of the order of the equations (9) was performed at the price of introducing the unknown functions[14] K_{β}^{α} to the equations (14) – we dare say that further integration of these equations is not possible, either, because of the existence of a

[13] Which can probably be reduced by using a software tool for symbolic transformation, differentiation etc.
[14] More precisely, it is known that these coefficients are shifting operators, but along an unknown curve!

CIRCULUS VICIOSUS OF ABSOLUTE INTEGRATION OF DIFFERENTIAL EQUATIONS OF GEODESICS:

To reduce the order of the differential equations of a geodesic line (9) and obtain its first order equations (14), one should know the operators of parallel transport along this still unknown geodesic line on the surface under consideration. On the other hand, to determine these operators as a fundamental solution for the system (11), one must know the geodesic line along which this system is to be satisfied!

In this situation, we can do nothing but wonder: "What next?". Even the most well-intentioned researcher would point out to the correctness, checked innumerable times, of the dictum "Back to school!", meaning – since we do not notice any possibility of cutting the above vicious circle – an attempt to find the origins of this circulus vitiosus. Therefore let us remember that "the concept of absolute derivative is made to depend on the concept of parallel displacement of a given vector at one point on a curve C to other points on C" ([12], p. 178); namely, the introduction of the notion of absolute and covariant derivatives implies a certain concept of parallel transport; however, the subsequent introduction of the notion of parallel transport in non-Euclidean space (s. e.g. [6], p. 142) includes a condition in which the covariant derivative arises, and this is a sort of circulus vitiosus as well! In view of the fact that the operation of absolute integration is introduced as an inverse to the one of absolute differentiation, we logically reach the

CONCLUSION

that the above mentioned vicious circle is only the *consequence* of a situation inherent to the existing approach to covariant differentiation in non-Euclidean spaces. In other words, the impossibility of using the concept of absolute integration for an effective determination of geodesics in non-Euclidean space is not the deficiency of this concept itself – it is *impossible in principle* within the theory based on the usual procedure of covariant differentiation in these spaces.

Therefore, the dilemma arising from the discussion mentioned in the title of this paper is substituted with the following one: whether, and how, to attempt to introduce another definition of the operation of covariant differentiation in non-Euclidean spaces (generalizing some characteristics common to both Euclidean and non-Euclidean spaces), without causing the mentioned *circulus vitiosus* ? However, this should be the subject of a future communication, because too much heretical ideas have already been presented in this paper.

Acknowledgement. *The author is especially indebted to Professor Veljko Vujičić (Mathematical Institute, Belgrade, Yugoslavia), not only for a critical review of this paper, but also for a concerned, unobtrusive and unselfish introduction to scientific work three decades ago*.

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PRILOG RASPRAVI O APSOLUTNOM INTEGRALJENJU DIFERENCIJALNIH JEDNAČINA GEODEZIJSKIH LINIJA U NEEUKLIDSKOM PROSTORU

Zoran Drašković

U radu je: a) još jednom istaknuto da postoje operatori paralelnog pomeranja po površi duž krive zadate na toj površi; b) dobijen je zatvoreni oblik tih operatora za slučaj paralelnog pomeranja po sfernoj površi, a duž velikih krugova; c) iznet je stav da apsolutno integraljenje diferencijalnih jednačina geodezijskih linija u neeuklidskim prostorima principski nije moguće zbog samog postupka kojim je u tim prostorima uvedeno kovarijantno diferenciranje.