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DERIVATION OF GENERALIZED VARIATIONAL PRINCIPLES WITHOUT USING LAGRANGE MULTIPLIERS PART III: APPLICATIONS TO THERMOPIEZOELECTRICITY

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Abstract. *By using the semi-inverse method proposed by He, some new variational principles are established for thermopiezoelectricity. The present theory provides a more complete theoretical basis for the finite element applications, and the other modern numerical techniques such as meshfree particle methods.*

1. INTRODUCTION

Recent interest in piezoelectric materials stems from their potential applications in intelligent structural systems. In 1988, Chandrasekharaiah [2] proposed a generalized linear thermoelasticity theory for piezoelectric media. Based on his theory, Ashida & Tauchert [1] studied an inverse problem for determination of transient surface temperature from piezoelectric sensor measurement. There exist many other applications, detailed discussion can be found in Refs. [1~5] and references cited thereby. The rapid development of computer science and the finite element applications reveals the importance of searching for a classical variational principle for the thermopiezoelectricity, which is the theoretical basis of the finite element methods. Though it is easy to establish a Gurtin-type functional (involving convolutions), it is very difficult to construct a classical variational model due to the strongly coupled constitutive relations and the terms of the first-order derivatives with respect to time involving in the heat conduction equation, which leads to non-self adjoint of the system of the equations. The variational model (not Gurtin-type) for the system is unknown at this time to the best of author's knowledge. Therefore, it is urgent to establish a classical variational representation for the system, in our approach we will apply the semi-inverse method [6~9], which appears to be one of the best and most convenient ways to establish variational principles for the physical problems, to arrive at our aim. To eliminate the convolutions in the functional, a

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new dynamic (continuous) differencing transformation technique [6] will be applied.

2. Mathematical formulation of thermopiezoelectricity

The basic equations in a generalized linear thermoelasticity theory for piezoelectric media can be written down as follows [2]

a) Equilibrium equations

$$\sigma_{ij,j} + f_i = \rho u_{i,tt}\,, \tag{2.1}$$

in which σ_{ij} is the symmetric stress tensor, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$, $u_{i,tt} = \partial^2 u_i/\partial t^2$, f_i represents the mechanical body force.

b) Constitutive equations

$$\sigma_{ij} = a_{ijkl}\ \gamma_{kl} - e_{mij}E_m - b_{ij}\theta\,, \tag{2.2}$$

$$\rho S = c\theta + b_{ij}\gamma_{ij} + c_i E_i\,, \tag{2.3}$$

$$D_m = \varepsilon_{mj}E_j + e_{mij}\gamma_{ij} + c_m\theta\,, \tag{2.4}$$

in which γ_{ij} is the symmetric strain tensor D_i is the vector of the electric displacement, E_i is the vector of the electric field, S is the entropy. The elastic moduli a_{ijkl} measured at constant (zero) electric field, and the piezoelectric moduli e_{mij} , and the dielectric permittivity ε_{ij} have the following symmetry properties , respectively

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}\,,$$
$$e_{mij} = e_{mji}\,, \text{ and } \varepsilon_{ij} = \varepsilon_{ji}\,.$$

c) Strain-displacement relations

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,, \tag{2.5}$$

where u_i is the vector of the elastic displacement.

d) Maxwell's equations for piezoelectric materials

$$D_{i,i} = 0 \ \text{ or } \ \nabla\cdot\mathbf{D} = 0\,, \tag{2.6}$$

$$E_i = \Phi_{,i} \ \text{ or } \ \mathbf{E} = \nabla\Phi\,, \tag{2.7}$$

where Φ is the electric potential.

e) Fourier's law for heat conduction

$$c\theta_0\frac{\partial\theta}{\partial\ t} + b_{ij}\frac{\partial\gamma_{ij}}{\partial\ t} + c_i\frac{\partial E_i}{\partial\ t} = q_{i,i} + \rho Q\,, \tag{2.8}$$

$$\tau\frac{\partial q_i}{\partial t} + q_i = -k_{ij}\theta_{,j}\,, \tag{2.9a}$$

or

$$\theta_{,i} = -K_{ij}\left(\tau\frac{\partial q_i}{\partial t} + q_i\right), \tag{2.9b}$$

where $\theta = T - \theta_0$, T is the temperature and θ_0 is the initial temperature , Q is the strength of the internal heat source, K_{ij} is the inverse of k_{ij}.

Our aim of this paper is to establish a generalized variational principle for the above discussed problem, whose stationary conditions should satisfy all the field equations and boundary/initial conditions. The present paper deals in facts with the very difficult inverse problem of the calculus of variations, the non-self adjoint of the system makes the problem more difficult. In order to make the system self-adjoint, and avoid the Gurtin-type variational principle, the time-derivative term in the coupled heat conduction equations (2.8) and (2.9) should be expressed as follows[13,14]:

$$\frac{\partial \Psi(t')}{\partial t} = \frac{\Psi(t) - \Psi(t^{(n-1)})}{t - t^{(n-1)}} = \frac{\Psi(t') - \Psi(0)}{t'}$$

where Ψ is an arbitrary function, $t' = t - t^{(n-1)}$.

The equations (2.8) and (2.9b), therefore, can be re-written in the forms

$$c\theta_0\theta + b_{ij}\gamma_{ij} + c_i E_i - t' q_{i,i} = \alpha\,, \tag{2.11}$$

$$t'\theta_{,i} + (K_{ij}\tau + t')q_i = \beta \tag{2.12}$$

where $\alpha = c\theta_0\theta^{(n-1)} + b_{ij}\gamma_{ij}^{(n-1)} + c_i E_i^{(n-1)} + t'\rho Q$ and $\beta = K_{ij}\tau q_i^{(n-1)}$.

3. Generalized variational principles

We have illustrated the basic idea of the semi-inverse method in the Part I and Part II of this series of paper. An energy-like trial-functional with 8 kinds of independent variations ($\sigma_{ij}, \gamma_{ij}, u_i, \theta, q_i, D_i, E_i$ and Φ) can be constructed as follows

$$J(\sigma_{ij}, \gamma_{ij}, u_i, \theta, q_i, D_i, E_i, \Phi) = \int_{t^{(n-1)}}^{t^{(n)}} \int L \,\mathrm{d}V \,\mathrm{d}t + IB, \tag{3.1a}$$

where

$$L = u_i(\sigma_{ij,j} + f_i) + F\,, \tag{3.1b}$$

in which F is an unknown function, L is a trial-Lagrangian.

Now we will identify the unknowns step by step.

Step 1

Making the above trial-functional (3.1a) stationary with respect to σ_{ij} results in the following trial-Euler equation:

$$-\frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{\delta F}{\delta\sigma_{ij}} = 0\,. \tag{3.2}$$

We search for such F so that the above trial-Euler equation(3.2) satisfies one of its field equations, saying, the equation (2.5). Accordingly we can set

$$F = \sigma_{ij}\gamma_{ij} + F_1 \tag{3.3}$$

where F_1 is a newly introduced unknown function, which should be free from σ_{ij}.

The trial-Lagrangian, therefore, can be renewed as follows

$$L = u_i(\sigma_{ij,j} + f_i) + \sigma_{ij}\gamma_{ij} + F_1 . \tag{3.4}$$

Step 2

The stationary condition with respect to γ_{ij} reads

$$\sigma_{ij} + \frac{\delta F_1}{\delta \gamma_{ij}} = 0 . \tag{3.5}$$

We set

$$F_1 = \gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl} + e_{mij}E_m + b_{ij}\theta) + F_2 , \tag{3.6}$$

so that the trial-Euler equation (3.6) satisfies the field equation (2.2). The trial-Lagrangrian (3.4) can be further renewed as

$$L = u_i(\sigma_{ij,j} + f_i) + \sigma_{ij}\gamma_{ij} + \gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl} + e_{mij}E_m + b_{ij}\theta) + F_2 \tag{3.7}$$

where F_2 is free from γ_{ij}.

Step 3

The trial-Euler equation with respect to u_i can be expressed as

$$\sigma_{ij,j} + f_i + \frac{\delta F_2}{\delta u_i} = 0 . \tag{3.8}$$

Supposing that the above equation is the field equation (2.1), we can identify the unknown as follows

$$F_2 = \tfrac{1}{2}\rho u_{i,t}u_{i,t} + F_3 , \tag{3.9}$$

where F_3 is free from u_i.

Step 4

By the same manipulation, the trial-Euler equation for $\delta\theta$ reads

$$b_{ij}\gamma_{ij} + \frac{\delta F_3}{\delta \theta} = 0 . \tag{3.10}$$

We set

$$\frac{\delta F_3}{\delta \theta} = c\theta_0\theta + c_i E_i - t' q_{i,i} - \alpha , \tag{3.11}$$

so that the trial-Euler equation (3.10) satisfies the field equation (2.11). From the equation (3.11) the unknown F_3 can be identified as

$$F_3 = \theta(\tfrac{1}{2}c\theta_0\theta + c_iE_i - mt'q_{i,i} - \alpha) + nt'\theta_{,i}q_i + F_4, \tag{3.12}$$

where m and n are constants, and satisfy the identity $m + n = 1$, F_4 is free from θ.

The trial-Lagrangian, therefore, can be rewritten in the form

$$\begin{aligned} L = {} & u_i(\sigma_{ij,j} + f_i) + \sigma_{ij}\gamma_{ij} + \gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl} + e_{mij}E_m + b_{ij}\theta) \\ & + \theta(\tfrac{1}{2}c\theta_0\theta + c_iE_i - mt'q_{i,i} - \alpha) + nt'\theta_{,i}q_i + F_4 . \end{aligned} \tag{3.13}$$

Step 5

Now the stationary condition for δq_i can be expressed as

$$(m+n)t'\theta_{,i} + \frac{\delta F_4}{\delta q_i} = 0. \tag{3.14}$$

In view of the field equation (2.12), the unknown F_4 can be determined as follows

$$F_4 = \tfrac{1}{2}(K_{ij}\tau + t')q_iq_j - \beta q_i + F_5 . \tag{3.15}$$

Substituting the equation (3.15) into the equation (3.13) yields the following modified trial-Lagrangrian

$$\begin{aligned} L = {} & u_i(\sigma_{ij,j} + f_i) + \sigma_{ij}\gamma_{ij} + \gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl} + e_{mij}E_m + b_{ij}\theta) \\ & + \theta(\tfrac{1}{2}c\theta_0\theta + c_iE_i - mt'q_{i,i} - \alpha) + nt'\theta_{,i}q_i + \tfrac{1}{2}\rho u_{i,t}u_{i,t} \\ & + \tfrac{1}{2}(K_{ij}\tau + t')q_iq_j - \beta q_i + F_5 . \end{aligned} \tag{3.16}$$

Step 6

Processing as the previous steps, the trial-Euler equation for δE_m reads

$$e_{mij}\gamma_{ij} + c_m\theta + \frac{\delta F_5}{\delta E_m} = 0 . \tag{3.17}$$

In view of the equation(2.4), the unknown F_5 can be determined as follows

$$F_5 = -E_iD_i + \tfrac{1}{2}E_i\varepsilon_{ij}E_j + F_6 . \tag{3.18}$$

Step 7

The stationary condition for δD_i reads

$$-E_i + \frac{\delta F_6}{\delta D_i} = 0 . \tag{3.19}$$

We set

$$F_6 = \xi D_i\Phi_{,i} - \varsigma D_{i,i}\Phi + F_7 \tag{3.20}$$

with $\xi + \zeta = 1$, where ξ and ζ are constants, so that the trial-Euler equation (3.19) satisfies the field equation (2.7).

Step 8

The trial-Euler equation for $\delta\Phi$ reads

$$-(\xi+\varsigma)D_{i,i}+\frac{\delta F_7}{\delta\Phi}=0\,. \tag{3.21}$$

We set

$$F_7=0\,, \tag{3.22}$$

so that the trial-Euler equation (3.21) satisfies the field equation (2.6). The Lagrangrian, therefore, has the following final form

$$\begin{aligned}L=&u_i(\sigma_{ij,j}+f_i)+\sigma_{ij}\gamma_{ij}+\gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl}+e_{mij}E_m+b_{ij}\theta)\\&+\theta(\tfrac{1}{2}c\theta_0\theta+c_iE_i-mt'q_{i,i}-\alpha)+nt'\theta_{,i}q_i+\tfrac{1}{2}\rho u_{i,t}u_{i,t}\\&+\tfrac{1}{2}(K_{ij}\tau+t')q_iq_j-\beta q_i-E_iD_i+\tfrac{1}{2}E_i\varepsilon_{ij}E_j+\xi D_i\Phi_{,i}-\varsigma D_{i,i}\Phi\,.\end{aligned} \tag{3.23}$$

Note: If the electrical and heat effects are not taken into consideration, then we obtain the Hellinger-Reissner principle in elasticity.

Now if we introduce an energy function A and its complementary $B,$ which are defined respectively as

$$A=-\gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl}+e_{mij}E_m+b_{ij}\theta)-\tfrac{1}{2}E_i\varepsilon_{ij}E_j+c_i\theta E_i\,, \tag{3.24}$$

From the above definitions, we have the following relations

$$\frac{\partial A}{\partial\gamma_{ij}}=a_{ijkl}\gamma_{kl}-e_{mij}E_m-b_{ij}\theta=\sigma_{ij} \tag{3.25}$$

By constraining the obtained generalized variational functional , we can obtain the following important functional:

$$J_1(u_i,\Phi)=\int_{t^{(n-1)}}^{t^{(n)}}\int(A-f_iu_i)dVdt+IB\,, \tag{3.26}$$

4. Conclusion

In the paper, we have succeeded in obtaining a generalized variational principle with some arbitrary constants, from which various variational principles can be obtained by constraining the functional by selectively field equations. Details have been discussed in Refs. [7,9].

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IZVOĐENJE UOPŠTENIH VARIJACIONIH PRINCIPA BEZ KORIŠĆENJA LAGRANGE-OVIH MNOŽILACA DEO III: PRIMENA NA TERMOPIEZOELEKTRIČNOST

Ji-Huan He

Korišćenjem polu-inverzne metode predložene od strane He-ija, nekoliko novih varijacionih principa je ustanovljeno za termopiezoelektričnost. Sadašnja teorija obezbeđuje mnogo potpuniju teoretsku osnovu za primene konačnih elemenata kao i za druge moderne numeričke tehnike kao što su bezmrežne metode čestice.