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VARIATIONAL THEORY FOR NONLINEAR PIEZOELECTRICITY

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Abstract. *The present paper concentrates on the construction of a generalized variational principle (non Gurtin-type and not involving convolutions) for nonlinear piezoelectricity. By the semi-inverse method proposed by He, a family of variational principles is established directly from the field equations and boundary conditions. Present theory provides a more complete theoretical basis for the finite element applications, variational-based meshless method(element-free method), and the other direct variational methods such as Ritz's , Trefftz's and Kantorovitch's methods.*

1. INTRODUCTION

The phenomenon of piezoelectricity was first discovered by the Curie brothers, Pierre and Jaques in 1880 when Pierre Curie was only 21 years old. The brothers [6] discovered that a crystal of sufficiently low symmetry develops an electric polarization under the influence of an external mechanical force, and about one year later the inverse effect was predicted, i.e. deformation of a crystal experiences an electric field. Piezoelectricity is one of the basic properties of crystals, ceramics, polymers, liquid crystals and some biological tissues (e.g. bone and tendon). Recent interest in piezoelectric materials stems from their potential applications in intelligent structural systems, and piezoelectricity is currently enjoying a greatest resurgence in both fundamental research and technical applications. Much achievement has been made in recent years [1~3, 5, 19,20,13,17].

The rapid development of computer science and the finite element applications reveals the importance of searching for a classical variational principle for the piezoelectricity, which is the theoretical basis of the finite element methods. Recent research also reveals that variational theory is also a powerful tool for meshless particle method or element-free method [9]. The present author has successfully established variational theory for linear piezoelectricity [17] and linear thermopiezo-electricity [13],

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and in this paper we will establish a variational functional for nonlinear piezoelectricity by the semi-inverse method [7-18].

2. Mathematical formulation

The basic equations for nonlinear piezoelectric media can be written in the forms [5, 13, 17]

a) Equilibrium equations

$$\sigma_{ij,j} + f_i = 0 , \tag{1}$$

in which $\sigma_{i,j}$ is the symmetric stress tensor, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j, f_i$ represents the mechanical body force.

b) Constitutive equations

$$\sigma_{ij} = a_{ijkl} \; r_{kl} - e_{mij} E_m + \frac{1}{2} C_{ijklpq} r_{kl} r_{pq} - g_{mijkl} E_m r_{kl} - \frac{1}{2} Q_{ijmn} E_m E_n , \tag{2}$$

$$D_m = \varepsilon_{mj} E_j + e_{mij} r_{ij} + \frac{1}{2} g_{mijkl} r_{ij} r_{kl} + \frac{1}{2} \varepsilon_{mnl} E_n E_l + Q_{mnij} E_n r_{ij} , \tag{3}$$

in which r_{ij} is the symmetric strain tensor D_i is the vector of the electric displacement, E_i is the vector of the electric field. a_{ijkl} and C_{ijklpq} – second and third-order moduli of elasticity are respectively fourth and sixth-rank tensors being measured on condition that electric field is constant(zero); ε_{mn} and ε_{mnl} – second and third-order dielectric penetrabilities which are second and third-rank tensors being measured on condition that deformation is constant(zero); e_{mij} are the piezoelectric moduli which are third-rank tensor components; g_{mijkl} are elastoelectric coefficients which are fifth-rank tensor components; Q_{ijmn} are electrostriction coefficients which are fourth-rank tensor components.

For linear piezoelectric media it follows

$$C_{ijklpq} = 0, g_{mijkl} = 0, \quad Q_{ijmn} = 0 , \quad \text{and} \quad \varepsilon_{mnl} = 0 .$$

c) Strain-displacement relations

$$r_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) , \tag{4}$$

where u_i is the vector of the elastic displacement.

d) Maxwell's equations for piezoelectric materials

$$D_{i,i} = 0 \;\text{ or }\; \nabla \cdot \boldsymbol{D} = 0 , \tag{5}$$

$$E_i = \Phi_{,i} \;\text{ or }\; \boldsymbol{E} = \nabla\Phi , \tag{6}$$

where Φ is the electric potential.

e) Boundary conditions

On A_1 surface displacement is prescribed

$$u_i = \overline{u}_i \qquad (\text{on} \quad A_1) \tag{7}$$

and on the complementary part A_2 the traction is given

$$\sigma_{ij} n_j = \overline{p}_i \qquad (\text{on} \quad A_2) \tag{8}$$

where $A_1 + A_2 = A$ covers the total boundary surface.

Suppose that on A_3, the electric potential and on A_4, the surface charges are given as

$$\Phi = \overline{\Phi} \qquad (\text{on} \quad A_3) \tag{9}$$

$$D_i n_i = \overline{D}_n \qquad (\text{on} \quad A_4) \tag{10}$$

where $A_3 + A_4 = A$ covers the total boundary surface.

Our aim of this paper is to establish a generalized variational principle for the above discussed problem, whose stationary conditions should satisfy all the field equations and boundary conditions. The present paper deals in fact with the very difficult inverse problem of the calculus of variations, and the semi-inverse method of establishing generalized variational principle will be applied hereby.

3. Generalized variational principle

The traditional approach to establishing generalized variational principles is the well-known Lagrange multiplier method. By such method the constraints of a known functional can be eliminated. In sometime, however, the variational crisis (some multipliers become zero) might occur during the derivation of generalized variational principles [4,11,12]. On the other hand, the Lagrange multiplier method is not valid to deduce a variational representation directly from the field equations and boundary conditions for the present problem.

We will use the semi-inverse method [7-18] to search a variational functional for the present study. The basic idea of the semi-inverse method is to construct an energy-like integrate with some an unknown function as a trial-functional. There are various methods to construct trial-functionals for practical problems. Details can be found in Ref. [7].

If we want to establish a functional with 6 kinds of independent variations (σ_{ij}, r_{ij}, u_i, D_i, E_i and Φ) , we can construct an energy-like trial-functional as follows

$$J(\sigma_{ij}, r_{ij}, u_i, D_i, E_i, \Phi) = \iiint L \,\mathrm{d}V + IB, \tag{11a}$$

where

$$L = \sigma_{ij} r_{ij} + F \ , \tag{11b}$$

$$IB = \sum_{k=1}^{4} \iint_{A_k} G_k \,\mathrm{d}A \ , \tag{11c}$$

in which F and G_i (i = 1–4) are unknowns, L is a trial-Lagrangian.

Now we will identify the unknowns step by step.

Step 1

Calculating variation of functional (11) with respect to σ_{ij} results in the following trial-Euler equation:

$$r_{ij} + \frac{\delta F}{\delta \sigma_{ij}} = 0\,, \tag{12}$$

where $\delta F / \delta \Psi$ is call variational derivative, which is defined in xyz coordinates as

$$\frac{\delta F}{\delta \Psi} = \frac{\partial F}{\partial \Psi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \Psi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \Psi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial \Psi_z}\right).$$

We search for such an F that the above trial-Euler equation(12) satisfies one of its field equations, saying, the equation (4).

If

$$\frac{\delta F}{\delta \sigma_{ij}} = -\frac{1}{2}(u_{i,j} + u_{j,i})\,, \tag{13}$$

then the equation(12) turns out to be (4). From (13) the unknown F can be identified as

$$F = -\frac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + F_1\,, \tag{14}$$

where η and μ are constants and should satisfy the identity $\eta + \mu = 1$, F_1 is a newly introduced unknown function, which should be free from σ_{ij}.

The trial-Lagrangian, therefore, can be renewed as follows

$$L = \sigma_{ij}r_{ij} - \tfrac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + F_1\,. \tag{15}$$

Step 2

Making the renewed trial-functional (11) stationary with respect to u_i yields the following trial-Euler equation

$$(\eta + \mu)\sigma_{ij,j} + \frac{\delta F_1}{\delta u_i} = 0\,. \tag{16}$$

Supposing that the above equation is the field equation (1), we can identify the unknown F_1 as follows

$$F_1 = f_i u_i + F_2\,, \tag{17}$$

where F_2 is free from u_i.

The substitution of the equation (17) into the trial-Lagrangian (15) yields a new one, which reads

$$L = \sigma_{ij}r_{ij} - \tfrac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + f_i u_i + F_2 \tag{18}$$

Step 3

The stationary condition with respect to r_{ij} reads

$$\sigma_{ij} + \frac{\delta F_2}{\delta r_{ij}} = 0\,. \tag{19}$$

We set

$$\frac{\delta F_2}{\delta r_{ij}} = -(a_{ijkl}\ \ r_{kl} - e_{mij}E_m + \frac{1}{2}C_{ijklpq}r_{kl}r_{pq} - g_{mijkl}E_m r_{kl} - \frac{1}{2}Q_{ijmn}E_m E_n) \tag{20}$$

in equation (19), so that the Euler equation (19) becomes (2). From (20), one can readily obtain

$$\begin{gathered} F_2 = -\frac{1}{2}r_{ij}a_{ijkl}\ \ r_{kl} + r_{ij}e_{mij}E_m - \frac{1}{6}r_{ij}C_{ijklpq}r_{kl}r_{pq} + \\ + \frac{1}{2}r_{ij}g_{mijkl}E_m r_{kl} + \frac{1}{2}r_{ij}Q_{ijmn}E_m E_n + F_3 \end{gathered}, \tag{21}$$

The trial-Lagrangrian (18), therefore, can be further renewed as

$$\begin{gathered} L = \sigma_{ij}r_{ij} - \tfrac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + f_i u_i - \\ -\frac{1}{2}r_{ij}a_{ijkl}\ \ r_{kl} + r_{ij}e_{mij}E_m - \frac{1}{6}r_{ij}C_{ijklpq}r_{kl}r_{pq} + \frac{1}{2}r_{ij}g_{mijkl}E_m r_{kl} + \frac{1}{2}r_{ij}Q_{ijmn}E_m E_n + F_3 \end{gathered} \tag{22}$$

where F_3 is free from r_{ij}.

Step 4

Processing as the previous steps, the trial-Euler equation for δE_m reads

$$r_{ij}e_{mij} + \frac{1}{2}r_{ij}g_{mijkl}r_{kl} + r_{ij}Q_{ijmn}E_n + \frac{\delta F_3}{\delta E_m} = 0\,. \tag{23}$$

We set

$$\frac{\delta F_3}{\delta E_m} = -D_m + \varepsilon_{mj}E_j + \frac{1}{2}\varepsilon_{mnl}E_n E_l \tag{24}$$

so that equation (23) satisfies the field equation (3). From (24) we have

$$F_3 = -E_m D_m + \frac{1}{2}E_m\varepsilon_{mj}E_j + \frac{1}{6}E_m\varepsilon_{mnl}E_n E_l + F_4\,. \tag{25}$$

We, therefore, obtain the following renewed trial-Lagrange function

$$\begin{gathered} L = \sigma_{ij}r_{ij} - \tfrac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + f_i u_i - \\ -\frac{1}{2}r_{ij}a_{ijkl}\ \ r_{kl} + r_{ij}e_{mij}E_m - \frac{1}{6}r_{ij}C_{ijklpq}r_{kl}r_{pq} + \frac{1}{2}r_{ij}g_{mijkl}E_m r_{kl} + \frac{1}{2}r_{ij}Q_{ijmn}E_m E_n + \\ -E_m D_m + \frac{1}{2}E_m\varepsilon_{mj}E_j + \frac{1}{6}E_m\varepsilon_{mnl}E_n E_l + F_4\,, \end{gathered} \tag{26}$$

where F_4 is free from E_i.

Step 5

The stationary condition for δD_i and $\delta\Phi$ read

$$-E_i + \frac{\delta F_4}{\delta D_i} = 0\,, \tag{27}$$

$$\frac{\delta F_4}{\delta \Phi} = 0\,. \tag{28}$$

The above two equations should satisfy the equations (6) and (5) respectively. So we can determine the unknown F_4 as

$$F_4 = \xi D_i \Phi_{,i} - \varsigma D_{i,i}\Phi \tag{29}$$

with $\xi + \varsigma = 1$, where ξ and ς are constants. The Lagrangrian, therefore, has the following final form

$$\begin{aligned} L = {} & \sigma_{ij} r_{ij} - \frac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i + f_i u_i - \\ & - \frac{1}{2} r_{ij} a_{ijkl}\ \ r_{kl} + r_{ij} e_{mij} E_m - \frac{1}{6} r_{ij} C_{ijklpq} r_{kl} r_{pq} + \frac{1}{2} r_{ij} g_{mijkl} E_m r_{kl} + \frac{1}{2} r_{ij} Q_{ijmn} E_m E_n + \\ & - E_m D_m + \frac{1}{2} E_m \varepsilon_{mj} E_j + \frac{1}{6} E_m \varepsilon_{mnl} E_n E_l + \xi D_i \Phi_{,i} - \varsigma D_{i,i}\Phi\,, \end{aligned} \tag{30}$$

Applying the Green's theory, on the boundaries, we obtain the following trial-Euler equations:

$$\delta u_i :\ -\eta\sigma_{ij} n_j + \frac{\partial G_k}{\partial u_i} = 0 \quad (k = 1\text{–}2), \tag{31}$$

$$\delta\sigma_{ij} :\ \mu u_i n_j + \frac{\partial G_k}{\partial \sigma_{ij}} = 0 \quad (k = 1\text{–}2), \tag{32}$$

$$\delta\Phi :\ \xi D_i n_i + \frac{\partial G_k}{\partial \Phi} = 0 \quad (k = 3\text{–}4), \tag{33}$$

$$\delta D_i :\ -\varsigma\Phi n_i + \frac{\partial G_k}{\partial D_i} = 0 \quad (k = 3\text{–}4). \tag{34}$$

On A_1, the above trial-Euler equations should satisfy the boundary condition (7) or an identity, so the unknown G_1 can be identified as follows

$$G_1 = \eta\sigma_{ij} n_j (u_i - \overline{u}_i) - \mu\sigma_{ij} n_j \overline{u}_i\,. \tag{35}$$

It is easy to prove that the stationary conditions for δu_i and $\delta\sigma_{ij}$ on the A_1 are an identity and the equation (7) respectively.

By the same manipulation, we have

$$G_2 = \eta\overline{p}_i u_i - \mu u_i(\sigma_{ij} n_j - \overline{p}_i)\,, \tag{36}$$

$$G_3 = -\xi(\Phi - \overline{\Phi}) D_i n_i + \varsigma\overline{\Phi} D_i n_i\,, \tag{37}$$

$$G_4 = -\xi\Phi\overline{D}_n + \varsigma\Phi(D_i n_i - \overline{D}_n)\,, \tag{38}$$

Note: If the electrical and heat effects are not taken into consideration, then we have

$$J(\sigma_{ij}, r_{ij}, u_i) = \int L' \,\mathrm{d}V\,\mathrm{d}t + IB' \tag{39a}$$

where

$$\begin{aligned} L' &= \sigma_{ij}\gamma_{ij} - \frac{1}{2}\eta\sigma_{ij}(u_{i,j} + u_{j,i}) + \mu\sigma_{ij,j}u_i \\ &\quad - \frac{1}{2}\gamma_{ij}a_{ijkl}\gamma_{kl} + f_i u_i + \frac{1}{2}\rho u_{i,t}u_{i,t} \end{aligned} \tag{39b}$$

and

$$\begin{aligned} IB' &= \iint_{A_1} \{\eta\sigma_{ij}n_j(u_i - \bar{u}_i) - \mu\sigma_{ij}n_j\bar{u}_i\}dA \\ &\quad + \iint_{A_2} \{\eta\bar{p}_i u_i - \mu u_i(\sigma_{ij}n_j - \bar{p}_i)\}dA \end{aligned} \tag{39c}$$

Setting $\eta = 1$, $\mu = 0$ in equation (39) results in the well-known Hu-Washizu principle of elasticity[21]. So we obtain Hu-Washizu-like principle for nonlinear piezoelectricity by setting $\eta = 1$, $\mu = 0$, $\xi = 1$, $\varsigma = 0$. The Lagrangian of Hu-Washizu-like principle can be expressed as

$$\begin{aligned} L_1 &= \sigma_{ij}[r_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})] + f_i u_i - \\ &- \frac{1}{2}r_{ij}a_{ijkl}\ r_{kl} + r_{ij}e_{mij}E_m - \frac{1}{6}r_{ij}C_{ijklpq}r_{kl}r_{pq} + \frac{1}{2}r_{ij}g_{mijkl}E_m r_{kl} + \frac{1}{2}r_{ij}Q_{ijmn}E_m E_n + \\ &- D_m(E_m - \Phi_{,m}) + \frac{1}{2}E_m\varepsilon_{mj}E_j + \frac{1}{6}E_m\varepsilon_{mnl}E_n E_l\,. \end{aligned} \tag{40}$$

For simplicity, we introduce a generalized strain-piezoelectric energy density defined as

$$\begin{aligned} \mathrm{A} &= \frac{1}{2}r_{ij}a_{ijkl}\ r_{kl} - r_{ij}e_{mij}E_m + \frac{1}{6}r_{ij}C_{ijklpq}r_{kl}r_{pq} - \frac{1}{2}r_{ij}g_{mijkl}E_m r_{kl} - \frac{1}{2}r_{ij}Q_{ijmn}E_m E_n + \\ &- \frac{1}{2}E_m\varepsilon_{mj}E_j - \frac{1}{6}E_m\varepsilon_{mnl}E_n E_l\,. \end{aligned} \tag{41}$$

From (41), we have the following relations

$$\frac{\partial \mathrm{A}}{\partial r_{ij}} = a_{ijkl}\ r_{kl} - e_{mij}E_m + \frac{1}{2}C_{ijklpq}r_{kl}r_{pq} - g_{mijkl}E_m r_{kl} - \frac{1}{2}Q_{ijmn}E_m E_n = \sigma_{ij}\,, \tag{42}$$

$$\frac{\partial \mathrm{A}}{\partial E_m} = -r_{ij}e_{mij} - \frac{1}{2}r_{ij}g_{mijkl}r_{kl} - r_{ij}Q_{ijmn}E_n - \varepsilon_{mj}E_j - \frac{1}{2}\varepsilon_{mnl}E_n E_l = -D_m\,, \tag{43}$$

Constraining the functional by selectively field equations(1),(6) and boundary conditions(7) and (9), we can obtain

$$J(u_i, \Phi) = \iiint(\mathrm{A} - f_i u_i)dV - \iint_{A_2}\bar{p}_i u_i dA + \iint_{A_2}\Phi\bar{D}_n dA \tag{44}$$

which is the minimum principle for nonlinear piezoelectricity.

4. CONCLUSION

In the paper, we have succeeded in obtaining a generalized variational principle with some arbitrary constants, from which various variational principles (including the minimum principle)can be obtained by constraining the functional by selectively field equations or boundary conditions.

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REFERENCES

1. Ashida F, Tauchert TR. *An inverse problem for determination of transient surface temperature from piezoelectric sensor measurement*, ASME J. App. Mech.,1998, **65:** 367-373
2. Chandrasekharaiah D S. *A generalized linear thermo-elasticity theory for piezoelectric media*. ACTA Mechanica, 1998, **71:** 39-49
3. Chen TY. *Further correspondences between plane piezoelectricity and generalized plane strain in elasticity*, Proc. R. Soc. Lond., 1971, **A454:** 873-884
4. Chien W Z. *Method of high-order Lagrange multiplier and generalized variational principles of elasticity with more general forms of functionals*, Applied Math. & Mech. ,1983, **4**(2): 137-150
5. Chizhikov, S.I. , Sorokin, N.G. and Petrakov, V.S. *The elastoelectric effect in the non-centrosymmetric crystals, in: Piezoelectricity*, eds G. W. Taylor et al, Gordon & Breach Science Publishers, New York, 1985, 75-91
6. Curie, J, Curie, P. *Development par compression de l'etricite polaire das les cristaux hemledres a faces inclinees*, Bulletin No.4 de la Societee Mineralogique de France, **3**(1880)
7. He JH. *Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics*. Int.. J. Turbo & Jet-Engines, 1997, **14**(1): 23-28
8. He J H. *A variational theory for one-dimensional unsteady compressible flow: an image plane approach*. Applied Math. Modelling, 1998, **22:** 395-403
9. He J H. *Treatment shocks in transonic aerodynamics in meshless method: Part I Lagrange multiplier approach*, Int. J. Turbo & Jet-Engines, 1999, **16**(1): 19-26
10. He J H. *Hybrid problems of determining unknown shape of bladings in compressible S_2-flow in mixed-flow turbomachinery via variational technique*, Aircraft Engineering and Aerospace Technology, 1999, **71**(2): 154-159
11. He JH. *On variational crisis and generalized variational principle of elasticity*. J. University of Shanghai for Science & Technology (in Chinese), 1999, **21**(2): 127-130
12. He J H. *An overview of variational crises and its recent developments*. J. University of Shanghai for Sciences and technology(in Chinese), 1999, **21**(1): 29-35
13. He J H. *A variational principle for thermopiezoelectricity based on Chandrasekharaiah's generalized linear theory*, J. University of Shanghai for Science and Technology, 1999, 21(4): 356-365
14. He J H. *Inverse problems of determining the unknown shape of oscillating airfoils in compressible 2D unsteady flow via variational technique*, Aircraft Engineering and Aerospace Technology, 2000, 72(1): 18-24
15. He J H. *A classical variational model for micropolar elastodynamics*, International Journal of Nonlinear Sciences and Numerical Simulation, 2000, 1(2): 133-139
16. He J H. *A variational model for micropolar fluids in lubrication journal bearing*, International Journal of Nonlinear Sciences and Numerical Simulation, 2000, 1(2):139-142
17. He J H. *Coupled variational principles of piezoelectricity*, Int. J. Engineering Science, (accepted)
18. He J H. *Generalized Hellinger-Reissner principle*, ASME Journal of Applied Mechanics, (accepted)
19. Maugin GA. *The Mechanical Behavior of Electromagnetic Solid Continua*, North-Holland, 1984
20. Maugin G A. *Continuum Mechanics of Electromagnetic Solids*, North-Holland-Amsterdam, 1988
21. Washizu K. *Variational Methods in Elasticity and Plasticity*, Pergamon Press, Oxford, 1982

VARIJACIONA TEORIJA ZA NELINEARNU PIEZOELEKTRIKU

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Rad je usresređen na konstruisanje generalizovano varijacionog principa (koji nije Gurtin-ovog tipa i koji ne obuhvata konvolucije) za nelinearnu piezoelektriku. Pomoću semi-inverzne metode koju je predložio He, ustanovljena je familija varijacionih principa direktno iz jednačina polja i graničnih uslova. Ova teorija obezbedjuje kompletniju teorijsku osnovu za primenu konačnih elemenata, meshless metoda (metoda bez elementa) zasnovanog na varijaciji i drugih direktnih varijacionih metoda kao što su Ritz-ova, Trefftу-ova i Kantorovitch-ov metoda.