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ACTION OF FORCE – FORMALITY OR ESSENCE

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Abstract. *It is proved that the author's term "action of force" is not mathematical formality, but it has essential physic significance. This is an expression with whom the action of force can be determined in time. The relation between force and action is established, similar as between force and power. Important examples in analytical, celestuial and quantum mechanics are mentioned.*

1. Introduction. The concept of *Action* is not uniformly and unambiguously determined in the classical mechanics. Therefore, for the sake of a clearer understanding of this paper, it is necessary to clarify this statement and then to proceed to generalizing and defining the concept of *Action*.

The first concept of the *action* can be found in the work by Leibnitz (**1669**) [3; pp. 782 - 789], as the *actio formalis*, whose dimension is product of mass, velocity and path. In forth Newton's definition, (**1687**), it is written that: "Vis impressa est *actio* in corpus exercita, ad mutandum eius statum vel movendi uniformiter in directum" ("An impress'd force is an *actio* exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line").[1] Accordingly, the force is defined by means of the concept of *action* that has not been previously defined and thus it is assumed to be clear and known. The concept of actio was later used by Christian Wolff (**1726**). Wolff wrote: "*actio* consist of

[1]Newton Is., Mathematical Principles of natural Philosophy (tren. into English by Robert Thorp, MA,London, 1969.

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mass, velocity and space" [3]. Maupertuis P. L. wrote (**1746**): "The quantity of action is a product of the bodies' masses, their velocities and the distance they are traversing" [3; pp. 53, 881]. Euler [3; pp. 70-75, 791] found (**1748**) that the sum of all *momentary actions* has the form

$$\int dt\Big(\int Vdv + V'dv' + V''dv'' + \dots\Big), \tag{*}$$

where V, V', V'' are forces expressed as functions of distances v, v', v'', ([3], pp. 76, 882). Let's also quote Hamilton who wrote (**1834**)" that integral

$$A_1 = \int \sum (X'dx + Y'dy + Z'dz) = \int 2E_k dt$$

represents collected living force (vis viva) often termed as *action of system* from its initial to its final position" [3]. In the Van Nostrand's Scientific Encyclopedia, Second Edition (New York, 1947) it is formulated in the following way: "ACTION. In certain discussion of dynamics there is need of an expression for the product of twice the mean total kinetic energy of a system of particles, during a specified interval of time, by the duration of the interval. This product is called *action*. Mathematically, it is expressed by

$$A_1 = 2\int_{t_0}^{t_1} E_k dt \tag{1}$$

in which E_k is kinetic energy and t_0, t_1 are the times of beginning and ending of the interval". The significance of formula (1) is also stressed by the fact that it is used to formulate principle of least action that has been formulated and elaborated by the most distinguised and deserving theorists of analytical Mechanics. K. Jacobi, [3; p. 797], even wrote that the principle of least action is the "mother" of the entire analytical mechanics.

In further development of Analytical mechanics the concept of *action* was accepted in the form of the functional

$$A_2 = \int_{t_0}^{t_1} L dt \tag{2}$$

where $L(q, \dot{q}, t)$ is kinetic potential, often called Lagrange's function.

Physical dimension of *action* is ML^2T^{-1} (dim mass = M, dim length = L, dim time = T).

For our purpose here let us also quote a few sentences from A. Sommerfeld's book [4; pp.255,328] " ... just as power is defined as *energy magnitude* : *time*, so *action* is defined as *energy×time*". ... "When we speak about cause and action, we imply that action is a consequence or result". ... "However, afterwards, if the term action is sanctioned by Helmholtz and Planck, each attempt to substitute it by some other term will be without perspective". ... "As an example, *the elementary quantum of Planck's action* can be used".

$$\int pdq = h = 6,6424 \cdot 10^{-34} Js. \tag{3}$$

where p iz generalized impulse and q is generalized independent coordinate. Let's quote M. Planck. "Characteristically, the principle of the least action, being fully adapted in the mechanics, by Lagrange, didn't have any focal and practical influence in the science. It has been observed almost as the mathematical curiosity, as something interesting but, superficial with respect to the Newton's law of motion. Alrready in 1837, Poisson called it "only useless rule". It was not before Thomson and Tait, G.Kirchhoff, K. Neumann, L. Boltzman and other's who showed it very useful tool in solving several problems in hydrodynamics and elasticity, that the principle began to be appreciable"[2]. Discussing the "element of energy $E = a'\nu$ of the oscillator", where a' is constant, independent of the nature of oscillator, Planch wrote: "I marked the constant a' by the letter h and called it, because it has the dimension of energy and time product, the *elementary quantum of action* or element of action, with difference to the element of energy $h\nu$". Using measured meaning of constants a and C, $(a' = aC)$, Planch obtained $h = 6,55 \cdot 10^{-27} erg \cdot sec$.

According to the afore mentioned the following therms can be underlined: Leibnitz's "actio formalis", Newton's words "Vis impressa est actio..."; Maupertuius' "quantite d'action", Euler's "action des forces"; "action of systems" (1), "Hamilton's action" (2); Planch's "elementary quantum of action". Perhaps this is the reason for the term of "action" to be less present in physics, then term "power", which is uniquely defined. There are some other reasons as well. In paper [6] it is shown that functional (2) is not invariant with respect to Lagrange's generalized coordinates $q = (q^1, \ldots, q^n)^T \in M^n$ and Hamilton's $p, q \in T^*M^n$ canonical variables for rheonomic systems. Namely, to put it more clearly, for mechanical systems with time-independent constraints, functional (1) and (2) can be written with respect to Descartes' coordinates y, curvilinear coordinates x, generalized independent coordinates q as well as with respect to Hamilton's coordinates p, q in invariant form

$$A_1 = \int_{t_0}^{t_1} 2E_k dt = \int_{t_0}^{t_1} p_i dq^i = \int_{t_0}^{t_1} a_{ij}\dot{q}^i dq^j = \int_{t_0}^{t_1} p_i \dot{q}^i dt. \tag{4}$$

$$A_2 = \int_{t_0}^{t} L dt = \int_{t_0}^{t_1} (E_k - E_p) dt = \int_{t_0}^{t_1} (p_i dq^i - H dt); \quad H := E_k + E_p. \tag{5}$$

[2]M.Planch, Das Princip der kleinsten Wirkung", "Die Kultur der Gegenwart", T.1, Physik, 195

However, if the constraints of the system depend on time, relation (4) and (5) lose both mathematical and physical sence, since it is

$$A_1 = \int_{t0}^{t_1} 2E_k dt \neq \int_{t_0}^{t_1} p_i dq^i \neq \int_{t_0}^{t_1} a_{ij}\dot{q}^j dq^i \neq \int_{t_0}^{t_1} p_i \dot{q}^i dt.$$

$$A_2 == \int_{t_0}^{t_1} L dt \neq \int_{t_0}^{t_1} (p_i \dot{q}^i - H) dt, \quad H \neq E_k + E_p.$$

For the above, it was necessary to introduce a general formula of action, which will keep positive mathematical and physical characteristics of defined formulas (1), (2) and (3).

2. Definition of the term Action of force.

Here we will retain the accepted dimensions of action, but we will uniformly define the term *action of force.*

Definition 1. *The Action $A(\boldsymbol{F})$ of a force $\boldsymbol{F}$ during an interval of time $t_1 - t_0$ is determined by integral*

$$A(\boldsymbol{F}) := \int_{t_0}^{t_1} W(\boldsymbol{F}) dt = \int_{t_0}^{t_1} \Big(\int_S \boldsymbol{F} d\boldsymbol{r} \Big) dt \tag{6}$$

where $W(\boldsymbol{F})$ is the work of all resulting forces $\boldsymbol{F}_\nu$ on the ν-th material or dynamical point M_ν on the path S_ν $(M_\nu \in S_\nu); d\boldsymbol{s}_\nu = d\boldsymbol{r}_\nu$. As it is well known (see, for example: [6]) *the work of the forces, in general case, is curvilinear integral*

$$W(\boldsymbol{F}) := \int_S \boldsymbol{F} d\boldsymbol{s} = \int_{S_\nu} \sum_{\nu=1}^{N} \boldsymbol{F}_\nu d\boldsymbol{r}_\nu.$$

This definition - just like Euler's action of force (*) – has more important both mathematical and physical differences with respect to action (2) or (1). The action (6) is invariant with respect to all the above-mentioned transformations, that is

$$A = \int_{t_0}^{t} W(\boldsymbol{F}) dt = \int_{t_0}^{t} W(Y) dt = \int_{t_0}^{t} W(X) dt = \int_{t_0}^{t} W(Q) dt =$$
$$\int_{t_0}^{t} \Big(\int_S \boldsymbol{F} d\boldsymbol{r} \Big) dt = \int_{t_0}^{t} \Big(\int_S Y dy \Big) dt = \cdots = \int_{t_0}^{t} \Big(\int_S Q dq \Big) dt$$

both for scleronomic and rheonomic mechanical systems.

For material point of constant mass m, moving with the velocity $\boldsymbol{v}$; $v \in [0, v]$, it will be

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} \Big(\int_S -m\frac{d\boldsymbol{v}}{dt}d\boldsymbol{r} \Big) dt = -\int_{t_0}^{t_1} E_k dt. \tag{7}$$

Although integrals (1) and (2) seem to be similar in form to the integral (7), there is a basic difference between them. Namely, for a constant velocity motion ($v = const.$) integral (7) is always zero,

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} \Big(\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} -m\dot{\boldsymbol{v}}d\boldsymbol{r} \Big) dt = -m \int_{t_0}^{t_1} \Big(\int_{\boldsymbol{v}_0}^{\boldsymbol{v}_1} \boldsymbol{v}d\boldsymbol{v} \Big) dt = -\int_{t_0}^{t_1} \Big(E_k(v_1) - E_k(v_0) \Big) dt = 0,$$

while integral (1) and (2) may attain arbitrarily large values. In additional, integral (1) is the action by definition, and in relation (7) it appears as the consequence of action of inertia force (7).

A very simple **example 1** of a weighty point motion of mass m at constant velocity $v = c$ along an ideally smooth horizontal straight line very clearly points aut to a quantitative difference between action (1) and action of forces (6). Actions (1) and (2), n this case are

$$A_1 = 2\int_{t_0}^{t_1} \Big(\frac{1}{2}mv^2 \Big) dt = mc^2(t_1 - t_0), \tag{8}$$

$$A_2 = \int_{t_0}^{t_1} L dt = mc^2 \frac{t_1 - t_0}{2}. \tag{9}$$

However, action (7) is

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} (-m\frac{d\boldsymbol{v}}{dt}) dt = 0, \tag{10}$$

since in this example $\frac{d\boldsymbol{v}}{dt} = 0$. It is enough to give some thought to result (9) in order to reach the conclusion that such an action does not exist in nature while it is senseless in theory. Classic terms od action (1) and (2), differ from the new introduced ones, (6) and (7), not only by quantity, but also logically, which is seen from the following text. Action of a force should be taken into account in considering the motio of a mechanical system. If the trajectory is given by parametrized finite equations of the form $\boldsymbol{r} = \boldsymbol{r}(\lambda, t)$ or by differential equations of motion $m\dot{\boldsymbol{v}} = \boldsymbol{F}$, then the curvilinear integral of work $W(\boldsymbol{F})$ can be reduced to a definite integral

$$W = \int_S \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{r} = \int_{t_0}^{t} \boldsymbol{F}(\boldsymbol{r}(t), \boldsymbol{v}(t), t)\boldsymbol{v}(t) dt = \int_{t_0}^{t} \mathcal{F}(t) dt.$$

That corresponds to known relations

$$\frac{dW}{dt} = \mathcal{F} = \boldsymbol{F}\frac{d\boldsymbol{r}}{dt} \longrightarrow dW = \boldsymbol{F}d\boldsymbol{r}.$$

Example 2. Material point of the mass m moves along the spiral: $x = a\cos\omega t, y = b\sin\omega t, z = ct$. Let us find action of the force in the interval $[t_0 = 0, t_1 = 2\pi]$. Since $\ddot{x} = -m\omega^2 a\cos\omega t = -m\omega^2 x$; $\ddot{y} = -m\omega^2 b\sin\omega t = -m\omega^2 y$; $\ddot{z} = 0$, we have $F_x = -m\omega^2 x$, $F_y = -m\omega^2 y$, $F_z = 0$. So:

$$W(\boldsymbol{F}) = \int_S \boldsymbol{F}d\boldsymbol{r} = -m\omega^2\Big(a^2\int_0^{2\pi}\cos\omega t d\cos\omega t + b^2\int_0^{2\pi} sin\omega t d\sin\omega t\Big) = 0;$$

and then

$$A(\boldsymbol{F}) = 0.$$

Example 3. The force $\boldsymbol{F} = -\mu\boldsymbol{v}$ acting on a material point of masse m. Action of force $\boldsymbol{F}$ is

$$A(\boldsymbol{F}) = \int_{t_0}^{t_1}\Big(\int_S(-\mu\boldsymbol{v}d\boldsymbol{r}\Big)dt.$$

Since from the differential equation of motion $m\dot{\boldsymbol{v}} = -\mu\boldsymbol{v}$ follows that $\boldsymbol{v} = \boldsymbol{v}_0 - \frac{\mu}{m}(\boldsymbol{r} - \boldsymbol{r}_0)$, we get

$$A(\boldsymbol{F}) = -\int_{t_0}^{t}\Big(\mu\int_{\boldsymbol{r}0}^{\boldsymbol{r}}[\boldsymbol{v}_0 - \frac{\mu}{m}(\boldsymbol{r} - \boldsymbol{r}_0)]d\boldsymbol{r}\Big)dt. \qquad (11)$$

3. Planch's elementary quantum of action

Although, Quantum of Physics is based on the standpoint (see for example [9]) that using the classical Mechanics it is not possible to explain the base of Theory of quantum, "The hypothesis of elementary quantum of action is a good step in explaining physical phenomena, as it assumes finite domains, instead of infinitisimal once:

$$\int\int dqdp = h. \qquad (12)$$

where q is a generalized independent coordinate of Physical systems and p corresponding impuls (momentum); quantity h is *the elementary quantum of action* and presented by universal constant with dimension of energy × time" (M. Planch, [2; pp. 286-287]). Relation (12), using the Green's theorem can be written as:

$$A = \int\int dqdp = \int pdq = \int_{t_0}^{t_1} 2E_k dt = h. \qquad (13)$$

In the further text on model of atom and physical data let's us Berkely Physics cource – Quantum Physics (IV) by Wikhmann [9;, Sec. 2.29]. "Our discussion is therefore very defective, but it nevertheless gives us some kind of picture of the nature of heavy atoms. It follows from this picture that transitions in thestate of motion of the autermast or optical electrons will involve energies of the other of an electron volt, wich roughly corresponds to wavelengths of the emitted photons in the optical region,...". Electron mass $m_e = (9.10908 \pm 0,00013) \cdot 10^{-31}$kg; velocity of light $c = (997925 \pm 0,000001) \cdot 10^8 \text{ms}^{-1}$. With numerical values of mass, velocity and time, or impuls and wave lenght, Planch's elementari quantum of action can be calculated. I will presume that the carreer of action is the part of electron of mass $\Delta m = 1.3 \cdot 10^{-35}$kg, as error of measuring of the electon mass. Let's call it a *"crumb"*. At direct calculation of the quantum of action with formula (1) and (2) the problem of limits of the interval $t_1 - t_0$ appears. However, the integral of action of force [6] removes that problem, as W is a work of force which is expeling a "crumb" from atom. This is in correspondance with Einstein's explanation of fotoeffect [10; Sec.42]. If a particle expelts from a body, with velicity $\boldsymbol{u}$ as it is well known, the reactive force $\boldsymbol{F} = \dot{m}\boldsymbol{u}$ is appearing. At the expelling crumb of mas Δm, the reactive force $\dot{m}u$ is acting on particle, while the force $-\dot{m}u$ is acting on atom, in correspondence of the principle of acting and counteraction. The time of duration of action of force is the time necessary for a crumb to accelerate from zero velocity to the velocity c. If, as example, we assume that the wave length of mass emition $\Delta m = 0,00013 \cdot 10^{-31}$kg is equal to $\lambda = 1,7 \cdot 10^{-7}$m, then the interval of time of crumb expeling, on light speed $c = 2,997926 \cdot 10^8 \text{ms}^{-1}$ is equal

$$\tau = \frac{\lambda}{c} = 0,567^{-15} = 5,67^{-16} s.$$

Thus, the action of this crumb is

$$\begin{aligned} A &= \int_0^\tau \Big(\int_m^{m-\Delta m} u^2 d(\Delta m) \Big) dt = \int_0^\tau \Big(\Delta m u^2 \Big) dt = \Delta m u^2 \tau \\ &= 1,3 \cdot 10^{-35} \cdot 2,997926^2 \cdot 10^{16} \cdot 5,67 \cdot 10^{-16} = 6.6247 \cdot 10^{-34} Js, \end{aligned} \tag{14}$$

which corresponds to the Planch's elementary action.

4 De Broglie's relation At the similar but more general level, de Broglie's relation [9; Sec 5.23, p 196]

$$h = \lambda p, \tag{15}$$

where p is the momentum of the particle, and λ its de Broglie wave length cam be derived. Let us show this again on the classical way. If a particle of mass m makes a road of c meters for 1 second, then the wave of wave lengte λ will make it for time $\tau = \frac{\lambda}{c}$. With substitution in the relation (3), relation

$$h = \int_0^\tau 2E_k dt = mc^2\tau = mc^2\frac{\lambda}{c} = p\lambda \tag{16}$$

is dotained, wich is de Broglie's relation (15).

"Both p and λ are independently measurable quantities, and by measuring a pair of corresponding variables (p, λ) we can determine Planck's constant h. It is remarkable empirical fact that we always get the same value for h, irresoctive of which kind of partucle we are observing, and that this is so is not a trivialy".[10;, p. 200].

5. Classical mechanical approach to the basic principle of quantum phisics

Let's quote the first Berkely Physics cource–Quantum Physics ([9] Sec. 1.46). "As we will learn later the relation (46a; $\frac{E}{\nu} = X_2$) expresses a very basic principle of quantum phisics, namely that energy and frequency are *universally* related by $E = h\nu$. Such a relationship is entirely foreign to classical physics, and the mysterious constant $X_2 (= h) in (46a)$ is a manifestation of secrets of nature unsuspected at the time."

Solving the problem in general form with canonical variables, $p := (p_1, \dots p_n)$, $q := (q^1, \dots q^n)$ Hamilton wrote the action (2) in the forme:

$$A_2 = \int_{t_0}^{t_1} L(q, \dot{q}) dt = \int_{t_0}^{t-1} [2E_k(q,p) - E(p,q)] dt. \tag{17}$$

or, regarding relation (1) or (13) and (6),

$$A_2 = A_1 - \int_0^{\tau} E dt. \tag{18}$$

For conservative system, which kinetical potential $L = E_k - E_p = 0$, it follows

$$A_2 = h - E\tau = 0, \tag{19}$$

or

$$E = h\tau^{-1}. \tag{20}$$

For an oscillatory process, in which the frequency $\nu = \tau^{-1}$, the basic relation of quantum physic is:

$$E = h\nu. \tag{21}$$

Example 4. Harmonic oscillator.

Let's defini the action (3) for harmonic oscilator.

$$\ddot{x} + \omega^2 x = 0, \omega^2 = \frac{c}{m}, c > 0, m > 0, x(t_0) = x_0, \dot{x}(0) = \dot{x}_0.$$

Finally equation of motion is

$$x(t) = x_0 \cos(\omega t) + \frac{\dot{x}}{\omega} \sin(\omega t).$$

As the $p(t) = m\dot{x}$, introducing it to the relation (3) it follows

$$h = \int_0^\tau m\dot{x}dx = \int_0^\tau m\dot{x}^2 dt = \int_0^\tau \frac{m}{2}(\dot{x}_0^2 + \omega x_0^2)dt = \int_0^\tau (E_k + E_p)dt = E\tau, \quad (22)$$

because

$$E_k = \frac{m}{2}\dot{x}^2, \quad E_p = \frac{c}{2}x^2; \quad E_k + E_p = E = const.$$

Further, as

$$\tau = \nu^{-1} = \frac{2\pi}{\omega}$$

we obtain:

$$E = h\nu = \hbar\omega$$

where:

$$\hbar = \frac{h}{2\pi} = 1.05449 \cdot 10^{-34} Js.$$

We can observe that from the previous equality it follows

$$\int_0^\tau m\dot{x}dx - (E_k + E_p)dt = \int_0^\tau Ldt = 0.$$

The qualitative analysis of this relation shows that separate phase trajectories is correspond to the zero kinetical potential $L = 0$. This also corresponds to the crumb expeling from the atom after the cyclus ν is finished.

6. Two bodies model

In scientific literatur it is accepted that the gravitation force betweem two bodies of the mass m_1 and m_2 is defined by formula

$$F = -\varkappa \frac{m_1 m_2}{\rho^2} \quad (23)$$

Using the variational Maupertuis–Lagrange's principle Ji-Huan He obtained a more general result[3]

$$F = -\chi \frac{m_1 m_2}{\rho}. \quad (24)$$

[3] See his manuscript "A varitional approach to problem of two bodies"

Let"s check this result by using our general principle of action on the problem of two bodies. In the paper [8] and monography [6], the principle of action is formulated by the relation:

$$\delta \int_{t_0}^{t_1} [W(\boldsymbol{F}) - W(\boldsymbol{I})]dt = 0 \tag{25}$$

where $W(\boldsymbol{F})$ is work of force $\boldsymbol{F}$ and $W(\boldsymbol{I})$ is work od inerctia force $\boldsymbol{I}$. Distance $\rho(t)$ between bodies is changes in time and thus it can be written as

$$f_1 = \Big[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\Big]^{\frac{1}{2}} - \rho(t) = 0. \tag{26}$$

Thus it is only possible to claim, that beside geometrical equation (26), forces of inertia also egsist. As the negative work of inertia force is equal to kinetic energy [6], relation (25) by using the boundary conditions $\delta x(t_0) = \delta x(t_1) = \ldots \delta z(t_1) = 0$, is reduced to:

$$-\int_{t_0}^{t_1} [m_1(\ddot{x}_1\delta x_1 + \ddot{y}_1\delta y_1 + \ddot{z}_1\delta z_1) + m_2(\ddot{x}_2\delta x_2 + \cdots + \ddot{z}_2\delta z_2)] = 0. \tag{27}$$

With this, the equation

$$\delta f_1 = (x_2 - x_1)\delta(x_2 - x_1) + \cdots + (z_2 - z_1)\delta(z_2 - z_1) = 0. \tag{28}$$

shuld be satisfied. If the eqation is integrated in the interval $(t_1 - t_0)$, multiplied by Lagrange's multiplier λ, and then summed with equation (27), it is obtained

$$\lambda_1 \int_{t_0}^{t_1} [(x_2 - x_1)\delta(x_2 - x_1) + \cdots + (z_2 - z_1)\delta(z_2 - z_1)]dt$$
$$- \int_{t_0}^{t_1} [m_1\ddot{x}_1\delta x_1 + \cdots + m_2\ddot{x}_2\delta x_2)]dt = 0.$$

This variational relation is equivalent to the following system of differential equations of motion of two bodies

$$\begin{aligned} m_1\ddot{x}_1 &= -\lambda(x_2 - x_1), \\ m_1\ddot{y}_1 &= -\lambda(y_2 - y_1), \\ m_1\ddot{z}_1 &= -\lambda(z_2 - z_1); \end{aligned}$$

$$\begin{aligned} m_2\ddot{x}_2 &= \lambda(x_2 - x_1), \\ m_2\ddot{y}_2 &= \lambda(y_2 - y_1), \\ m_2\ddot{z}_2 &= \lambda(z_2 - z_1). \end{aligned}$$

Using these equations and conditions of acceleration the multiplier λ can bi defined as:

$$\lambda = \chi \frac{m_1 m_2}{\rho^2}, \qquad ()$$

where:

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2},$$

and

$$v_{or}^2 := (\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2 + +(\dot{z}_2 - \dot{z}_1)^2.$$

Thus, the gravitation force of the two bodies of mass m_1 and m_2 is

$$F = \chi \frac{m_1 m_2}{\rho}.$$

The same result is obtained with precondition that the motion of body has to sastisfy the second Kepler's law, we can write in the form

$$[\boldsymbol{\rho}, \boldsymbol{v}_{or}] = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \dot{x}_2 - \dot{x}_1 & \dot{y}_2 - \dot{y}_1 & \dot{z}_2 - \dot{z}_1 \end{vmatrix} = \boldsymbol{c} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3 = 0.$$

That means that beside equations (26) there are additional three equations

$$f_2 = (y_2 - y_1)(\dot{z}_2 - \dot{z}_1) - (\dot{y}_2 - \dot{y}_1)(z_2 - z_1) = c_1,$$
$$f_3 = (z_2 - z_1)(\dot{x}_2 - \dot{x}_1) - (\dot{z}_2 - \dot{z}_1)(x_2 - x_1) = c_2,$$
$$f_4 = (x_2 - x_1)(\dot{z}_2 - \dot{z}_1) - (\dot{z}_2 - \dot{x}_1)(y_2 - y_1) = c_3.$$

Integrating the equation of variation

$$\delta f_2 = \dot{z}\delta y + y\delta\dot{z} - z\delta\dot{y} - \dot{y}\delta z = 0,$$

where $x := x_2 - x_1, \quad y := y_2 - y_1, \quad z := z_2 - z_1$, we obtein:

$$\int_{t_0}^{t_1} \delta f_2 dt = 2\int_{t_0}^{t_1} (\dot{z}\delta y - \dot{y}\delta z)dt = 0,$$

and then

$$\int_{t_0}^{t_1} \delta f_3 dt = 2\int_{t_0}^{t_1} (\dot{x}\delta z - \dot{z}\delta x)dt = 0$$

$$\int_{t_0}^{t_1} \delta f_4 dt = 2\int_{t_0}^{t_1} (\dot{y}\delta x - \dot{x}\delta y)dt = 0.$$

With succesive multiplying of these relations by multipliers β_2, β_3 and summing with eqation (27), it follows

$$\lambda \int_{t_0}^{t_1} (x\delta x + y\delta y + z\delta z)dt + \beta_1 \int_{t_0}^{t_1} (\dot{x}\delta y - \dot{y}\delta z)dt$$
$$+ \beta_2 \int_{t_0}^{t_1} (\dot{y}\delta z - \dot{z}\delta x)dt + \beta_3 \int_{t_0}^{t_1} (\dot{z}\delta x - \dot{x}\delta y)dt$$
$$- \int_{t_0}^{t_1} [m_1(\ddot{x}_1\delta x_1 + \ddot{y}_1\delta y_1 + \ddot{z}_1\delta z_1) + m_2(\ddot{x}_2\delta x_2 + \ddot{y}_2\delta y_2 + \ddot{z}_2\delta z_2)]dt = 0.$$

Remembering that $x := x_2 - x_1, \ldots, z := z_2 - z_1$ follows 6 eqations of motion follow:

$$\begin{aligned} m_1\ddot{x}_1 &= -\lambda x + \beta_2\dot{z} - \beta_3\dot{y}, \\ m_1\ddot{y}_1 &= -\lambda y - \beta_1\dot{z} + \beta_3\dot{x}, \\ m_1\ddot{z}_1 &= -\lambda z + \beta_1\dot{y} - \beta_2\dot{x}; \\ m_2\ddot{x}_2 &= \lambda x - \beta_2\dot{z} + \beta_3\dot{y}, \\ m_2\ddot{y}_2 &= \lambda y - \beta_3\dot{x} + \beta_1\dot{z}, \\ m_2\ddot{z}_2 &= \lambda z - \beta_1\dot{y} + \beta_2\dot{x}. \end{aligned} \tag{29}$$

Substitung accelerations $\ddot{x}, \ddot{y}, \ddot{z}$ from eqations (29) into the condition of accelerations

$$\begin{aligned} \ddot{f}_1 &= x(\ddot{x}_2 - \ddot{x}_1) + y(\ddot{y}_2 - \ddot{y}_2) + z(\ddot{z}_2 - \ddot{z}_1) + v_{or}^2 - \dot{\rho}^2 - \rho\ddot{\rho} = 0, \\ \dot{f}_2 &= y(\ddot{z}_2 - \ddot{z}_1) - z(\ddot{y}_2 - \ddot{y}_1) = 0, \\ \dot{f}_3 &= z(\ddot{x}_2 - \ddot{x}_1) - x(\ddot{z}_2 - \ddot{z}_1) = 0, \\ \dot{f}_4 &= x(\ddot{y}_2 - \ddot{y}_1) - y(\ddot{x}_2 - \ddot{x}_1) = 0, \end{aligned}$$

it is obtained:

$$\beta_2 = \beta_3 = \beta_4 = 0;$$
$$\lambda = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho^2},$$

where:

$$v_{or}^2 := \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$

This corresponds to Vujičić's rezult, obtained by principle of eqillibrium [6],[7].

Finally, let's get back to the title: Action od force – formality or essence. On the example of inertial motion, a difference between action (1) and action of inercia force (7) was clearly observed. In the action (2), the integral

$$\int_{t_0}^{t_1} E_p dt \tag{30}$$

appears, where E_p is potential energy. In many books (see for example [1]) in mechanics and physics, it is learnt that the gravitational potential is

$$E_p = -G\frac{m_1 m_2}{\rho}$$

Assuming that the satelite of mass m_2 is moving on the circular orbit at the distance $\rho = R = const.$ from the centre of inertia of the body of mass m_1 it follows that the action (30) is

$$\int_{t_0}^{t_1} G\frac{m_1 m_2}{R}(t_1 - t_0) = G\frac{m_1 m_2}{R}(t_1 - t_0).$$

As it is seen, it will increase, which has no physical sence. On the other hand, for the same bodies and motion at the gravitation force

$$F = \chi\frac{m_1 m_2}{\rho}$$

the action of force (6) will be equal to zero.

Conclusion. It is proved that the auttor's term "action of force" is not mathematical formality, but it has esential physical significance. This is an expression with withwhom the action of force can be determined in time. The relation between force and action is established, similar as between force and the power. Important examples in analytical, celestuial and quantum mecanics are mintioned.

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DEJSTVO SILE-FORMALNOST ILI SUŠTINA

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Pojam "Dejstvo" nije jedinstveno i jednoznačno definisan u klasičnoj mehanici. Postoje razlike i u matematičkom i u fizičkom smislu izmedju pojedinih definicija dejstva. Zato se ponekad postavlja pitanje o fomalnosti ili suštini ovog pojma u analitičkoj mehanici. Reč "Dejstvo" (lat. Actio) u fizici prvo nalazimo kod Leibnitz-a kao "Actio formalis" (1669). kasnije (1687) Is. Newton je, definišući sile napisao: "Vis impresa est actio in corpus axercita",... (Nametnuta ili pokretačka sila je dejstvo...)." O razvoju pojma dejstva dovoljno i iscrpno vidi u antologiji "Varijacioni principi mehaniki" (na ruskom jeziku) (1959) od Polaka. Uočiće se i pojmovi: Mopertuis-ova količina dejstva, Euler-ovo dejstvo sila, "Dejstvo po Lagrange-u", Hamilton-ovo dejstvo,... Planck-ovo elementarno dejstvo. Možda je ta raznovrsnost razlog što pojam dejstva nije usvojen u fizici i mehanici , kao što je to pojam snage.

Ovde je redefinisan pojam "dejstvo sile" sa pretenzijom da postane jedinstven u fizici i mehanici. Kao i u većini dosadašnjih definicija usvojena je fizička dimenzija dejstva ML^2T^{-1}, *a "dejstvo sile" je definisano kao integral*

$$\mathbf{A} = \int_{t_0}^{t} \mathbf{W}(\mathbf{F})dt = \int_{t_0}^{t} \mathbf{W}(Y)dt = \int_{t_0}^{t} \mathbf{W}(\mathbf{x})dt = \int_{t_0}^{t} \mathbf{W}(\mathbf{Q})dt = = \int_{t_0}^{t}(\int_S \mathbf{F}d\mathbf{r})dt = \int_{t_0}^{t}(\int_S \mathbf{F}d\mathbf{r})dt = \int_{t_0}^{t}(\int_S Ydy)dt = \int_{t_0}^{t}(\int_S Qdq)dt \quad (1)$$

gde su x, y, q u istom redosledu Descartes-ove koordinate, krivolinijske i generalisane koordinate; $\mathbf{W}(\mathbf{F})$ *je rad sile* $\mathbf{F}$. *Polazeći od definicije dejstva sile i principa dejstva i protivdejstva, odredjuje se na klasičan način elementarno Planck-ovo dejstvo:*

$$\mathbf{A}(\mathbf{F}) = -\mathbf{A}(\mathbf{I}) = h = 6.62 \cdot 10^{-34}\, Js \quad (2)$$

gde je h Planck-ovo elementarno dejstvo. Za klasični harmonijski oscilator lako se dobija osnovna relacija kvantne teorije:

$$\mathbf{E} = h\nu \quad (3)$$

Sve to kao i neke numeričke vrednosti u navedenim primerima, pokazuje da nova formula dejstva sila nije matematički formalizam, nego ima suštinski značaj, ne manji od pojma snage. Na osnovu te definicije formulisan je generalisani integralni varijacioni princip dejstva relacijom:

$$\delta\int_{t_0}^{t}[\mathbf{W}(\mathbf{Q}) - \mathbf{W}(\mathrm{I})]dt = 0$$

gde je $\mathbf{W}(\mathrm{I})$ *rad generalisane sile inercije* I.