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SOME ELEMENTS OF THE INTERDEPENDENCE BETWEEN THE PRODUCTIVITY FUNCTION AND THE PRODUCTION FUNCTION OF TWO VARIABLES

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Abstract. *Proceeding from the fact that labour and capital are two basic production factors and that their investment is materialized through the elements of production process, the contemporary analysis of productivity and production functions treats the concept of aggregate (integral) productivity as being as relevant as the concepts of productivity of living labour and productivity of capital. The paper deals with some economic and methodological aspects of aggregate productivity and its interdependence with productivity function of two variables.*

1. INTRODUCTION

Productivity has a central place in the analysis of economic efficiency both at the level of an enterprise and economy at large. An appropriate analysis of productivity depends on what production factors, apart from labour, are taken into consideration. Also, in the analysis of the determinants and interdependence between the product and the production factors we can use both the model of one-variable production function and the model of production function of several variables, which includes labour, capital and productivity as the factors of scope and dynamics of the product.

Since we have already considered some aspects of the interdependence between the one-variable productivity function and production function in our previous papers [5], we will on this occasion analyze some aspects of the interdependence having in mind the concept of aggregate productivity (productivity of several variables). Taking into consideration the complexity of the problem, this paper we'll point out only some aspects of the problem. In that sense, we will determine the elements of the concept of productivity of two variables (1) and, then, some aspects of interdependence between the

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productivity function of two factors and the production function of two variables (2).

2. THE ELEMENTS OF PRODUCTIVITY OF TWO FACTORS

The concept of productivity of one production factor always means the productivity of living human labour. In that sense, the expression of the productivity of labour is $p_u = q/u$, where p_u is the productivity of labour, q is the physical scope of a product Q and u is the labour (precisely, the investment quantity of labour force). If the product is expressed as a function of invested labour: $q = f(u)$, it follows the expression of one-factor productivity function: $p = f(u)/u$. This is the expression of productivity of human labour, as one of the production factors. This concept of productivity assumes only the productivity of labour since the other production factors (instruments and subjects of labour) are factors of the productivity of labour.

However, according to the concept of productivity of several factors, all production factors have a productivity. Therefore, in addition to the productivity of labour, there is the productivity of capital (v), as an economic expression of invested instruments and subjects of labour. The expression of this productivity, for the set of discrete values, is: $p_v = q/v$. If the product is expressed as a function of invested capital: $q = f(v)$, then the functional expression of the productivity of capital will be: $p_v = f(v)/v$.

Besides the labour and the capital, in the contemporary analysis of production factors enterpreneurship and innovation are often taken as additional production factors. If this concept of production factors is accepted, the expressions of partial (one-factor) productivity have to be converted. However, since enterpreneurial and innovative labour are considered as individual expressions in the scope of total labour as one of the basic production factors, the analysis has to be done in the context of partial effects of the productivity of living labour. This is, probably, an explanation why the contemporary economic analysis of productivity and production function has concerned only labour and capital, that is labour, capital and productivity (often in the form of effects of the technological progress).

Taking into consideration the fact that labour (u) and capital (v) are the basic production factors and that their investment is materialized through the elements of both production and technological process, the economic analysis treats the concept of productivity of living labour (1), the concept of productivity of capital (2) and the concept of aggregate (integral) productivity (3) as equally relevant concepts. In addition, it has to be emphasized that a rate of productivity expresses both the capability and efficiency of joint investment and exploitation of labour and capital (in the above-mentioned sense). Therefore, the expression of aggregate productivity (p_a) will be

$$p_a = \frac{q}{u + v} . \tag{1.1}$$

The relation (1.1) shows an obvious interdependence between the product (q) and the productivity of labour and capital (p_a) and it is necessary to consider the interdependence for the set of discrete values and some relations of functional interdependence between those elements.

3. THE ELEMENTS OF INTERDEPENDENCE BETWEEN THE PRODUCTIVITY OF TWO FACTORS AND THE PRODUCT FOR THE SET OF DISCRETE VALUES

Starting from the above-mentioned elements of the aggregate productivity in the expression (1.1), it follows a statement that they are given in an absolute expression as well as that they are valid for a certain period of time (precisely, as a state at the end of a certain period). Therefore, in the analysis of the interdependence between the productivity and the product certain relations relevant for a basic period (a) and for a successive period (b) have to be considered.

The expression of productivity which in its denominator has both labour and capital has a feature of aggregate productivity, since it includes both the influence of labour and capital on a realized scope of production. The scope of product (q) for the basic period of time is a direct function of the investment quantity of labour (u) and capital (v) and their aggregate productivity (p_a), that is

$$q = p_a (u + v). \tag{2.1}$$

From the relation (2.1) we can find the value of labour:

$$u = \frac{q - p_a v}{p_a}$$

and the value of capital:

$$v = \frac{q - p_a u}{p_a} \ .$$

However, in order to apply the model (2.1) as a measure of the aggregate productivity we need to solve the problem of expressing labour and capital, since labour is expressed in units of labour and capital is in monetary units. First, this is a crucial problem with significant economic implications on applying the concept of aggregate productivity. Second, in the majority of works in the domain of macroeconomic analysis this problem has not been stated in the right sense and its solution has been taken for granted. Third, the differences in economic meaning of the expressions of labour and capital investment are not based on the differences already existing between these two production factors.

In the domain of methodological problems of this question, it seems possible and acceptable to apply the solution based on the conditional units of labour, capital or another element, but in the same form of investment. Namely, the units of labour are a form of investment quantities, since the units of capital are a form of committed resources (capital) in the reproduction. In that sense, starting from the units of labour (u) and the fact that their cost expression is in cost of labour, based on the market earnings or labour cost (C_{lo}) as well as that the average interest (d_i) is the cost expression of capital exploitation in the market economy ($d_i = v \cdot K_{ro}$), a satisfied solution could be found in expressing the invested capital in the units of labour, so

$$L_v = \frac{v \cdot K_{ro}}{C_{lo}}, \tag{2.2}$$

where L_v is the invested capital expressed in the equivalent of labour, and K_{ro} is the average rate of interest.

From the relations (1.1) and (2.2), the expression of aggregate productivity [1] is as follows:

$$p_a = \frac{q}{u+v} = \frac{q}{u+L_v} = \frac{q}{L_{uv}}, \tag{2.3}$$

where L_{uv} is the aggregate expression of the invested labour and capital in the units of labour (the objective value of labour).

From the formula (2.3) we can conclude that there is a direct dependence between the productivity and the scope of product, so the product is a function of the investment of labour and capital and the level of aggregate productivity, that is

$$q = p_a \cdot L_{uv}, \quad \text{and} \quad L_{uv} = q / p_a\,.$$ [2]

Applying the relation (2.3) as an adequate methodological solution for measuring the aggregate productivity in the current period, the relative relation between the labour and the capital is a basis or a measure for determining their contribution to the realized scope of product and, consequently, the contribution of labour ($p_{a(u)}$) and capital ($p_{a(v)}$) to the realized productivity, so $p_{a(u)} : p_{a(v)} = u : L_v$. It is possible, in addition, to make a direct estimate as follows:

$$p_{a(u)} = \frac{u}{L_{uv}} \qquad \text{and} \qquad p_{a(v)} = \frac{L_v}{L_{uv}}\,.$$

However, it is especially relevant to explore and determine the problems of interdependence between the productivity and the product in two (or more) comparable periods.

On the basis of the relation (2.3), the aggregate productivity in period i is $p_a^{(i)} = q^{(i)} / L_{uv}^{(i)}$, $i = 1,2,...,n$, and productivity in period i+1 is

$$p_a^{(i+1)} = \frac{q^{(i+1)}}{L_{uv}^{(i+1)}} = \frac{q^{(i)} \pm \Delta q^{(i)}}{L_{uv}^{(i)} \pm \Delta L_{uv}^{(i)}} = \frac{q^{(i)}(1 \pm \dfrac{\Delta q^{(i)}}{q^{(i)}})}{L_{uv}^{(i)}(1 \pm \dfrac{\Delta L_{uv}^{(i)}}{L_{uv}^{(i)}})} \qquad (i = 1,2,...,n).$$

If we introduce the labels:

$$K_q^{(i)} = \frac{\Delta q^{(i)}}{q^{(i)}}, \qquad K_{Luv}^{(i)} = \frac{\Delta L_{uv}^{(i)}}{L_{uv}^{(i)}} \qquad (i = 1,2,...,n),$$

than *the coefficient of change in aggregate productivity* of period i will be:

[1] Taking into consideration a specific character of the problems, an estimate on a hypothetical example is presented in addition of the paper.

[2] The concept of integral (global) productivity has been the subject of works by many authors. They have mostly emphasized the importance of applying *the index of global factor productivity* [3, p.76], that can be calculated as the average product of ponderous investment quantities of labour and capital (with suitable prices or factor participations in functional distribution of the product, as the ponders).

$$K_{p_a}^{(i)} = \frac{\Delta p_a^{(i)}}{p_a^{(i)}} = \frac{p_a^{(i+1)} - P_a^{(i)}}{P_a^{(i)}} = \frac{p_a^{(i+1)}}{P_a^{(i)}} - 1 = \frac{1 \pm K_q^{(i)}}{1 \pm K_{Luv}^{(i)}} - 1, \quad (i = 1,2,...,n), \tag{2.4}$$

where $K_q{}^{(i)}$ is the coefficient of product change in period *i*, and $K_{Luv}^{(i)}$ is the coefficient of change in investment quantity of labour and capital (in the equivalent of labour), also in period *i*.

From the formula (2.4) we can conclude that there is a direct but not proportional dependence between the change in productivity and the change in product. So, if the coefficient of change in aggregate productivity is determined (for a future or a past period), it is possible to determine the objective or realized coefficient of change in product, that is

$$K_q^{(i)} = (K_{p_a}^{(i)} + 1)(1 \pm K_{Luv}^{(i)}) - 1 \quad (i = 1,2,...,n).$$

Since there is a possibility of simultaneous changes in the scope of product, labour and capital in the successive periods, the change in aggregate productivity is a function of the changes in productivity based on product dynamics and the changes in productivity based on labour and capital dynamics. Namely, the formula (2.4) could be written as

$$K_{p_a}^{(i)} = \frac{\pm K_q^{(i)}}{1 \pm K_{Luv}^{(i)}} - \frac{\pm K_{Luv}^{(i)}}{1 \pm K_{Luv}^{(i)}} \quad (i = 1,2,..., n),$$

and, introducing the label:

$$\frac{\pm K_q^{(i)}}{1 \pm K_{Luv}^{(i)}} = K_{p_a(q)}^{(i)} \quad (i = 1,2,..., n), \tag{2.5}$$

where $K_{p_a(q)}^{(i)}$ is the coefficient of change in productivity of period *i* based on product dynamics, and, at the same time, the label

$$\frac{\pm K_{Luv}^{(i)}}{1 \pm K_{Luv}^{(i)}} = K_{p_a(Luv)}^{(i)} \quad (i = 1,2,..., n), \tag{2.6}$$

where $K_{p_a(Luv)}^{(i)}$ is the coefficient of change in productivity of period *i* based on labour and capital dynamics, then, on the basis of (2.5) and (2.6), the formula (2.4) could be written as

$$K_{p_a}^{(i)} = K_{P_a(q)}^{(i)} - K_{p_a(Luv)}^{(i)} \, (i = 1,2,...,n). \tag{2.7}$$

According to (2.3), we obtain that $L_{uv}=u+L_v$, so the coefficient of change in investment quantity of labour and capital of period *i*, $K_{Luv}^{(i)}$, could be expressed as

$$K_{Luv}^{(i)} = \frac{\Delta L_{uv}^{(i)}}{L_{uv}^{(i)}} = \frac{\pm \Delta u^{(i)}}{L_{uv}^{(i)}} + \frac{\pm \Delta L_v^{(i)}}{L_{uv}^{(i)}} \quad (i = 1,2,...,n),$$

and, introducing the labels:

$$\frac{\pm \Delta u^{(i)}}{L_{uv}^{(i)}} = K_{Luvu}^{(i)} \qquad (i = 1,2,...,n), \tag{2.8}$$

$$\frac{\pm \Delta L_v^{(i)}}{L_{uv}^{(i)}} = K_{Luvv}^{(i)} \qquad (i = 1,2,...,n), \tag{2.9}$$

we obtain

$$K_{Luv}^{(i)} = K_{Luvu}^{(i)} + K_{Luvv}^{(i)} \qquad (i = 1,2,...,n). \tag{2.10}$$

Now, it is possible to determine the coefficient $K_{p_a(Luv)}^{(i)}$ expressing the change in productivity based on the changes in labour and capital in period *i*, which have already been expressed in the formula (2.7).

In the case when there are no changes in the scope of product *q* in the successive periods, according to the relation (2.3), we obtain

$$K_{p_a(Luv)}^{(i)} = \frac{\Delta p_a^{(i)}}{P_a^{(i)}} = \frac{\dfrac{q^{(i)}}{L_{uv}^{(i)} \pm \Delta L_{uv}^{(i)}} - \dfrac{q^{(i)}}{L_{uv}^{(i)}}}{\dfrac{q^{(i)}}{L_{uv}^{(i)}}} \qquad (i = 1,2,..., n),$$

that is, after composing and applying the relation $K_{Luv}^{(i)} = \Delta L_{uv}^{(i)} / L_{uv}^{(i)}$, we obtain

$$K_{p_a(Luv)}^{(i)} = \frac{\mp K_{Luv}^{(i)}}{1 \pm K_{Luv}^{(i)}} \qquad (i = 1,2,...,n),$$

and then, according to (2.10), it follows:

$$K_{p_a(Luv)}^{(i)} = \frac{\mp K_{Luvu}^{(i)}}{1 \pm K_{Luv}^{(i)}} + \frac{\mp K_{Luvv}^{(i)}}{1 \pm K_{Luv}^{(i)}} .$$

From the previous relation, if we introduce the labels:

$$\frac{\mp K_{Luvu}^{(i)}}{1 \pm K_{Luv}^{(i)}} = K_{p_a(Lu)}^{(i)} \qquad (i = 1,2,...,n), \tag{2.11}$$

$$\frac{\mp K_{Luvv}^{(i)}}{1 \pm K_{Luv}^{(i)}} = K_{p_a(Lv)}^{(i)} \qquad (i = 1,2,...,n), \tag{2.12}$$

we obtain

$$K_{p_a(Luv)}^{(i)} = K_{p_a(Lu)}^{(i)} + K_{p_a(Lv)}^{(i)} \qquad (i = 1,2,...,n) \tag{2.13}$$

and, according to the starting assumption that the product is not changed, it is equal to the coefficient of change in aggregate productivity.

In practice, however, there are almost always some simultaneous changes in scope of product ($K_q^{(i)}$) as well as the changes in investment of labour ($K_u^{(i)}$) and investment of capital ($K_{Lv}^{(i)}$), that is the investment of labour and capital ($K_{Luv}^{(i)}$). It is very often the case that both the product and the investment of production factors are simultaneously increased, with a sharper increase in the product ($K_q^{(i)} > K_{Luv}^{(i)}$).

In that case, from the formula (2.7) and (2.13) it follows that

$$K_{p_a}^{(i)} = K_{p_a(q)}^{(i)} - (K_{p_a(Lu)}^{(i)} + K_{p_a(Lv)}^{(i)}) \quad (i = 1,2,...,n). \text{ [3]} \tag{2.14}$$

The relation between the coefficient of change in aggregate productivity and the coefficients of individual changes in factors determines the coefficients of partial elasticity of productivity. If there is a change only in labour or in capital, the coefficient of partial elasticity of productivity based on labour (that is, on capital), labelled as ε_u (that is ε_v) is determined according to the formula

$$\varepsilon_u = \frac{K_{p_a}^{(i)}}{K_u^{(i)}} \left(\varepsilon_{Lv} = \frac{K_{p_a}^{(i)}}{K_{Lv}^{(i)}} \right) \quad (i = 1,2,...,n),$$

where $K_{p_a}^{(i)}$ is the coefficient of change in aggregate productivity given in the formula (2.4), and $K_u^{(i)} = \Delta u^{(i)} / u^{(i)}$ is the coefficient of change in labour (that is $K_{Lv}^{(i)} = \Delta L_v^{(i)} / L_v^{(i)}$ is the coefficient of change in capital).

Since labour and capital are two different production factors, it is necessary to determine an individual objective change in labour ($K_u^{(i)}$) and an individual objective change in capital ($K_{Lv}^{(i)}$) for a certain change in aggregate productivity ($K_{p_a}^{(i)}$) and for a certain change in product ($K_q^{(i)}$). Since, from the relation (2.8), $\Delta u^{(i)} = u^{(i)} K_u^{(i)}$, it follows

$$K_{Luvu}^{(i)} = \frac{\Delta u^{(i)}}{L_{uv}^{(i)}} = \frac{u^{(i)} \cdot K_u^{(i)}}{L_{uv}^{(i)}} \quad (i = 1,2,...,n)$$

and from the previous relation we obtain

$$K_u^{(i)} = \frac{L_{uv}^{(i)} \cdot K_{Luvu}^{(i)}}{u^{(i)}} \quad (i = 1,2,...,n). \tag{2.15}$$

From the relations (2.15), (2.10) and (2.4), we obtain

$$K_u^{(i)} = \left[\left(\frac{1 \pm K_q^{(i)}}{1 + K_{P_a}^{(i)}} - 1 \right) - K_{Luvv}^{(i)} \right] \frac{L_{uv}^{(i)}}{u^{(i)}} \quad (i = 1,2,...,n). \tag{2.16}$$

From the relation (2.9), $\Delta L_v^{(i)} = L_v^{(i)} \cdot K_{Lv}^{(i)}$, it follows

$$K_{Luvv}^{(i)} = \frac{L_v^{(i)} \cdot K_{Lv}^{(i)}}{L_{uv}^{(i)}} \quad (i = 1,2,...,n),$$

[3] The formula (2.14) is an analog to the mentioned index of global factor productivity [3, p. 76].

and, from the previous relation, we obtain

$$K_{Lv}^{(i)} = \frac{L_{uv}^{(i)} \cdot K_{Luvv}^{(i)}}{L_v^{(i)}} \qquad (i = 1,2,...,n). \tag{2.17}$$

From the relations (2.17), (2.10) and (2.4), it follows

$$K_{Lv}^{(i)} = \left[\left(\frac{1 \pm K_q^{(i)}}{1 + K_{P_a}^{(i)}} - 1\right) - K_{Luvu}^{(i)}\right]\frac{L_{uv}^{(i)}}{L_v^{(i)}} \qquad (i = 1,2,...,n). \tag{2.18}$$

We can give an illustration of the previous models through a hypothetical example. It is necessary to mention that the solutions in the example were found by using the models from the text.

Table 1. The contribution of labour and capital to the aggregate productivity (the set of discrete values)

Or. No.	Elements	Mark	Period I	Period II	Δ	K	K
1	2	3	4	5	6	7	8
1.	Product	q	10	14	4	0,40	K_q
2.	Labour	u	10	12	2	0,20	K_u
3.	Capital	v	2000	2600	600	0,30	K_v
4.	Earnings (price of labour)	C_{lo}	10	10			
5.	Rate of interest	K_{ro}	0,10	0,10			
6.	Interest	d_i	200	260	60	0,30	K_{di}
7.	Capital equivalent	L_v	20	26	6	0,30	K_{Lv}
8.	Labour and capital	L_{uv}	30	38	8	0,267	K_{Luv}
9.	Coefficient of labour investment	K_{Luvu}				0,067	K_{Luvu}
10.	Coefficient of capital investment	K_{Luvv}				0,200	K_{Luvv}
11.	Productivity of labour	p_u	1,00	1,167	0,167	0,167	Kp_u
12.	Productivity of capital	p_v	0,50	0,5385	0,0385	0,077	Kp_v
13.	Aggregate productivity	p_a	0,333	0,3684	0,0354	0,1063	Kp_a
14.	Partial elasticity of labour	ε_u				0,5315	ε_u
15.	Partial elasticity of capital	ε_v				0,3540	ε_v

4. THE PRODUCTIVITY FUNCTIONS OF TWO FACTORS

In this part of the paper we will analyze a functional dependence between productivity and production factors. Namely, the subject will be the functions of partial productivity of two factors, that is the productivity function of labour, the productivity function of capital and the integral factor productivity function. We will form these functions proceeding from the production function of two factors:

$$q = \Phi(u,v), \tag{3.1}$$

where q is the scope of product, u is the labour (the investment of labour force), v is the investment of capital, and all the other investments are assumed to be fixed on the given conditions. We also start from the assumption that the function (3.1) is defined only for nonnegative values of the arguments u and v as well as it is two times differentiable with the infinite partial derivatives, so the first infinite derivatives on u and v are positive ($\Phi_u > 0$ and $\Phi_v > 0$), and the second direct partial derivatives are negative ($\Phi_{uu} < 0$ and $\Phi_{vv} < 0$). It is also assumed that (3.1) is a linear homogeneous function with constant returns on the scope of production, i.e. the sum of the production partial elasticities on the arguments u and v is equal to one ($E_{q,u} + E_{q,v} = 1$).

4.1. THE PARTIAL PRODUCTIVITY FUNCTIONS

The *productivity of labour* function of two factors, labelled as p_u, we obtain if the production function (3.1) is divided with u, i.e.

$$p_u = \frac{q}{u} = \frac{\Phi(u,v)}{u} = \varphi_1(u,v), \tag{3.2}$$

and the *productivity of capital* function of two factors, labelled as p_v, we obtain as a quotient of the production function and the capital v, that is

$$p_v = \frac{q}{v} = \frac{\Phi(u,v)}{v} = \varphi_2(u,v). \tag{3.3}$$

The function (3.2) is also called the function of *average product of labour*, and the function (3.3) - the function of *average product of capital* [2, p. 58].

If the production function (3.1) is defined in the domain $D = [0,a] \times [0,b] \subset R^2$ where a and b are positive real numbers, then the partial productivity function (3.2) is defined in the domain $D_1 = (0,a] \times [0,b]$, and the function (3.3) in the domain $D_2 = [0,a] \times (0,b]$. Let us assume that for the production function (3.1) is valid:

$$\Phi(0,v) = \Phi(a,v) = \Phi(u,0) = \Phi(u,b) = 0,$$

then, taking into consideration the domain of definition for the productivity function (3.2), is valid:

$$\varphi_1(u,0) = \varphi_1(u,b) = \varphi_1(a,v) = 0,$$

and, for the function (3.3):

$$\varphi_2(0,v) = \varphi_2(a,v) = \varphi_2(u,b) = 0.$$

We will examine some characteristics of the partial productivity function (3.2). To determine the point $(u_0,v_0) \in D_1$ where the productivity function (3.2) has a maximum (where a maximal productivity of labour force is achieved), it is necessary to determine the partial derivatives of this function.

Differentiating the formula (3.2) on u, we can obtain the partial derivative on the variable u (the marginal productivity on the investment quantity of labour force u):

$$\frac{\partial p_u}{\partial u} = \frac{1}{u}\left(\Phi_u(u,v) - \frac{\Phi(u,v)}{u}\right) = \frac{1}{u}(\Phi_u(u,v) - p_u) \tag{3.4}$$

where $\Phi_u(u,v)$ expresses the marginal product of labour and p_u is the average product of investment of labour force [2, p. 55].

Differentiating the formula (3.2) on the variable v, we obtain the partial derivative of this function on v (marginal productivity on the quantity of capital v):

$$\frac{\partial p_u}{\partial v} = \frac{1}{u}\Phi_v(u,v), \tag{3.5}$$

where $\Phi_v(u,v)$ expresses the marginal product of capital investment.

A necessary condition for achieving the extreme values of the productivity function (3.2) is that the first partial derivatives are zero, i.e.

$$\Phi_u(u,v) - p_u = 0 \wedge \Phi_v(u,v) = 0 \Leftrightarrow \Phi_u(u,v) = p_u \wedge \Phi_v(u,v) = 0. \tag{3.6}$$

If (u_o,v_o), where $u_o > 0$, $v_o > 0$, is a solution of the previous system of two equations with two unknowns, then the necessary condition for achieving the extreme values of the productivity function (3.2) is at the level of the investment of labour force u_o and capital v_o when the marginal product of investment of labour force is equal to the average product of that investment and the marginal product of investment of capital is zero.

Defining a sufficient condition for achieving the maximum of the function (3.2) requires determining the second partial derivatives:

$$\frac{\partial^2 p_u}{\partial u^2} = \frac{1}{u}\Phi_{uu}(u,v) - \frac{2}{u^2}\left(\Phi_u(u,v) - \frac{\Phi(u,v)}{u}\right), \tag{3.7}$$

$$\frac{\partial^2 p_u}{\partial u \partial v} = \frac{1}{u^2}(u\Phi_{uv}(u,v) - \Phi_v(u,v)), \tag{3.8}$$

$$\frac{\partial^2 p_u}{\partial v^2} = \frac{1}{u}\Phi_{vv}(u,v). \tag{3.9}$$

If we, one after another, change (u_o,v_o) in (3.7), (3.8) and (3.9), considering that (u_o,v_o) is the solution of the equations (3.6), we obtain

$$A = \frac{1}{u_o}\Phi_{uu}(u_o,v_o)\,,\ B = \frac{1}{u_o}\Phi_{uv}(u_o,v_o)\,,\ C = \frac{1}{u_o}\Phi_{vv}(u_o,v_o).$$

On the basis of the assumptions for the production function (3.1), we conclude that $A < 0$ and $C < 0$. The function of partial productivity (3.2) at the point (u_o,v_o) has the maximum $p_{\max} = \varphi_1(u_o,v_o)$ if $B^2 - AC < 0$ and $A < 0$ $(C < 0)$ [4, p.148]. The point $M(u_o,v_o, \varphi_1(u_o,v_o))$ in the three-dimensional Euclidean space is the maximum point of the function of productivity of labour $p_u = \varphi_1(u,v)$. The graph of this function is an area in the three-dimensional space, shown in Fig. 1.

If the area shown in Fig. 1 is cut by the plane π which contains the maximum point M and is parallel with the coordinate plane uOp, we obtain the curved line k which contains the maximum point M. If we make an ortogonal projection of the curved line k on the coordinate plane uOp, we obtain the curved line k', which is the graph of the one-variable productivity function (this curved line has been shown in Fig. 1, in the paper [5]).

In a similar way we can analyze the partial productivity function (3.3) and show it on a graph.

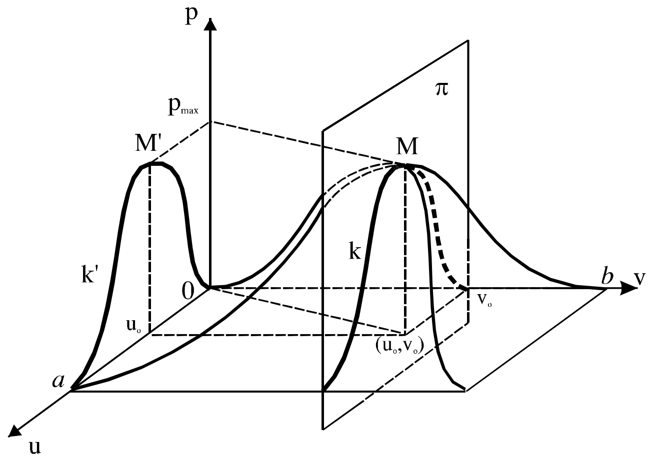

Fig. 1.

4.2. THE INTEGRAL FACTOR PRODUCTIVITY

The integral factor productivity is also referred to as: the total factor productivity or the global factor productivity in literature [1, p. 76]. The starting point in setting a formula for the determination of integral factor productivity is a dynamic form of the production function of two factors

$$q = A\Phi(u,v), \tag{3.10}$$

where $A = A(t)$, $A'(t) > 0$ is "a measure of effects of moving the production function in time" [1, p. 77 (81)], and the factors of labour $u = u(t)$ and capital $v = v(t)$ are also a function of the time t. The production function of two factors $\Phi(u,v)$ has the same assumptions as the function (3.1) as well as an additional assumption that the technological progress is neutral in Hicks sense [1, p. 73]. If we determine the total differential for the function (3.10), we obtain

$$dq = A'(t)\Phi(u,v)dt + A\frac{\partial\Phi(u,v)}{\partial u}du + A\frac{\partial\Phi(u,v)}{\partial v}dv,$$

that is

$$dq = \Phi(u,v)dA + Au\Phi_u(u,v)\frac{du}{u} + Av\Phi_v(u,v)\frac{dv}{v},$$

where $\Phi_u(u,v)$ $\Phi_v(u,v)$ is the partial derivative of the function $\Phi(u,v)$ on labour u (on capital v). After dividing the right and the left side of the previous relation with q, we obtain the equation of growth rate

$$\frac{dq}{q} = \frac{dA}{A} + \frac{u\Phi_u(u,v)}{\Phi(u,v)}\frac{du}{u} + \frac{v\Phi_v(u,v)}{\Phi(u,v)}\frac{dv}{v}. \tag{3.11}$$

Since the ratios

$$\frac{u\Phi_u(u,v)}{\Phi(u,v)} = \varphi_u \ , \ \frac{v\Phi_u(u,v)}{\Phi(u,v)} = \varphi_v \ , \tag{3.12}$$

express, one after another, a part of labour and capital in the distribution of realized production and since $\varphi_u + \varphi_v = 1$ [2, p. 75], the equation of the rate of growth (3.11) is expressed as

$$\frac{d\mathrm{A}}{\mathrm{A}} = \frac{dq}{q} - \left(\varphi_u \frac{du}{u} + \varphi_v \frac{dv}{v}\right). \tag{3.13}$$

In the conditions of perfect competition (when the factor prices are equal to their marginal products) [2, p. 101], where the part of labour and capital in (3.12) are equal to the appropriate production partial elasticities, i.e. $\varphi_u = \mathrm{E}_{q,u}$, $\varphi_v = \mathrm{E}_{q,v}$ the relation (3.13) could be expressed as

$$\frac{d\mathrm{A}}{\mathrm{A}} = \frac{dq}{q} - \mathrm{E}_{q,u}\frac{du}{u} - \mathrm{E}_{q,v}\frac{dv}{v}. \tag{3.14}$$

The formulas (3.13) and (3.14) are used for determining some integral factor productivity measures. If the factor part of labour φ_u and capital φ_v are constant, from (3.13) we can derive the Solow's measure of technological progress (the geometric index of global productivity) [1, p. 77-79].

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ELEMENTI MEĐUZAVISNOSTI IZMEĐU DVOFAKTORSKE FUNKCIJE PRODUKTIVNOSTI I PROIZVODNE FUNKCIJE

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Polazeći od činjenice da su rad i kapital dva osnovna proizvodna faktora i da se njihovo ulaganje materijalizuje preko elemenata procesa proizvodnje, u savremenoj analizi produktivnosti i proizvodnih funkcija je, pored koncepata produktivnosti živog rada i produktivnosti kapitala, relevantan i koncept agregatne (integralne) produktivnosti. U radu se analiziraju neki ekonomski i metodološki aspekti agregatne produktivnosti i njena međuzavisnost sa dvofaktorskom proizvodnom funkcijom.