FACTA UNIVERSITATIS
Series: **Economics and Organization** Vol. 8, N° 2, 2011, pp. 121 - 129

MATHEMATICAL MODELS OF MONOPOLISTIC COMPETITION: THEORETICAL PRINCIPLES AND APPLICATIONS

UDC 51-7:339.137.2

Milan Božinović[1], Vladica Stojanović[2]

[1]Faculty of Economics, University of Priština (in K. Mitrovica), Serbia
milan_bozinovic@yahoo.com
[2]Faculty of Sciences and Mathematics, University of Priština (in K. Mitrovica), Serbia
vlast70@gmail.com

Abstract. *In this paper we will consider the problem of monopolistic competition of two or more firms that offer the same products. Special attention will be put on the construction of a general mathematical model of oligopolistic competition. This model was applied in practical research, i.e. it was implemented in the production optimization of three monopolistic firms.*

Key Words: *monopolistic competition, models of functional dependence, optimization of economics variables.*

1. INTRODUCTION

The continuously growing antagonism between sellers and buyers, motivated by maximalistic goals of achieving the biggest difference between the cost of production and sale of goods on the market, represents an everlasting economic overpowering that we could call the competition. The main generator of competition is certainly the mechanism of setting the market prices, established on the basis of economic law – the law of supply and demand.

Monopolistic competition as a state on the market is completely opposite to the perfect competition. Regardless of how it is created, the main characteristic of monopolistic competition is the presence of monopoly in either the supply or demand area. This means that a manufacturer is confronted with high number of consumers and vice versa, the consumers are able to choose the goods from many producers of similar products.

Therefore, the amount of supply (or demand) is such, that monopoly has the ability to set the price of the products regardless of the quantity of the production. In oligopolistic markets almost whole production is performed by several companies only. The products

Received May 25, 2011

on this market may not necessarily be differentiated. Therefore, oligopolistic company determines the price and the amount of production is based on the strategic assessment of behavior of the competition, and on the other hand, the competitors' decisions are highly dependent on the strategic planning of the leading company. Although it might seem abstract at the first glance, the monopolistic behavior is completely logical and it provides the basis for determining *the equilibrium conditions* in the oligopolistic market.

In fact, in real situations in the market, there is a question of how to manage the price and the level of production so that they could be ideally balanced, along with the question of its feasibility. Certainly, in this case the market price is not an outcome of the law of supply and demand because there is no possibility of increasing the number of bidders, so the products offered on the market have no close substitutes.

However, the purchasing power of consumers to pay a certain price represents a limiting factor for the price growth, and an individual problem which occurs is the formation of an appropriate model in order to formally describe the behavior of oligopolistic companies on the market. Guided by this idea, and beginning from the standard functional models of economic variables (see, for example, Henderson and Quandt [3]), we will present different ways of mathematical modeling of monopolistic competition and determination of the optimal, equilibrium level of price and demand in which the monopolistic companies achieve maximum profit.

2. The Model of Oligopolistic Competition

Suppose that the oligopolistic competition (of order n) occurs when several companies (firms) $F_1,\ldots, F_n$ appear on the market offering the same product X. If $q_1,\ldots, q_n$ denotes their demand and production volume of each company, respectively, the total demand for X is equal to the sum of individual demands, i.e.

$$q = q_1 + \ldots + q_n, \tag{1}$$

while the law of demand in general, inverse form is given by

$$p = f(q), \tag{2}$$

where p is the price of product X.

Further, $T_1(q)$, $T_2(q)$,...,$T_n(q)$ denotes the total costs of product X in the companies F_1, F_2,..., F_n, respectively, and $C_1(q)$, $C_2(q)$,..., $C_n(q)$ their total income. Then, total income is generally equal to the product of demands and price

$$C_i(q) = q_i p = q_i f(q_1 + \ldots + q_n), \tag{3}$$

while the difference between total revenue and total costs presents the profit of company, given by the functions

$$P_i(q) = C_i(q) - T_i(q), \quad i = 1,\ldots, n. \tag{4}$$

We can put equation (4) in a developed form, taking the function of the total revenue (3). The profit functions of company F_i, i = 1, 2, ..., n, then as follows

$$P_i(q) = q_i f(q_1 + ... + q_n) - T_i(q). \tag{5}$$

In a further analysis of the model of oligopolistic competition we will omit the possibility that in case of changes in production in *i*-th company other monopolists are indifferent to the changes. Specifically, certainly all the other participants will react in a similar way, i.e. changes in production volume in the company pulls F_i change in other businesses. Based on this, we conclude that for the demand q there is a corresponding functional dependency, which is displayed as the following

$$q_i = \begin{cases} \xi_{ij}(q_j), & i \neq j \\ q_j, & i = j. \end{cases} \tag{6}$$

Therefore, a functional relationship between the variables q_i and q_j exists only for $i \neq j$, which is consistent with the fact that this is the demand of different companies. Now, substituting (6) into (5) we obtain the following system of equations

$$P_i(q) = q_i f\left(q_i + \sum_{j \neq i} \xi_{ji}(q_i) \right) - T_i(q), \tag{7}$$

where $i = 1, 2, ..., n$ and q is a common demand. Further, by differentiating equations (7) order of variables q_i, $i = 1, 2, ..., n$, and put that received partial derivatives equate to zero, we get a system of differential equations

$$f(q) + q_i \frac{\partial f(q)}{\partial q_i} \cdot \left(1 + \sum_{j \neq i} \xi'_{ji}(q_i) \right) = \frac{\partial T_i(q)}{\partial q_i}, \tag{8}$$

that completely describes the state of the market within a set of monopolistic firms. In economic terms, any solution of (8), i.e. a finite sequence of numbers

$$(q_1, ..., q_n) \in R^n \tag{9}$$

which identically fulfills this system of equations, is *the optimal value of demand*, or *optimal level of production* for which each of monopolistic firms realize maximum profits.

Specifically, the equations of system (8) can be interpreted as the equilibrium conditions of a monopolistic market. They are realized when the marginal revenues are equal to marginal costs, i.e. if

$$C_g(q) = T_g(q), \tag{10}$$

where $C_g(q)$ and $T_g(q)$ are derivates of function of total income $C(q)$ and total costs $T(q)$ expressed through the demand for X.

This concept in economic theory was given by John Nash in 1951 and it is known as *Nash's equilibrium*. Its basic idea is the formal, mathematical description of the principles that each company "doing business as best he could", taking into account "the actions of its competitors". In this way, he provides the basis for the application of game theory to describe the interaction between monopolistic companies, which will be discussed in details later.

3. Model Application

Let us now consider some specific possibilities of applying the model above, especially the system of differential equations (8), in purpose of finding the optimal level of production values of oligopolistic companies. Because of this, we will firstly point out some specific assumptions about functional dependence between these economic variables.

One of typical features of the organization of production in oligopolistic companies exists in their "agreement" on a common market share and in the quantity of goods that each firm will produce. This situation can be displayed through the ratio quantities of manufactured goods, i.e. participation of companies in a given market segmentation. Using these facts, the system of demand equations (6) can be written as functions of direct proportionality

$$q_j = k_{ji} q_i, \quad i, j = 1,2,...,n, \tag{11}$$

where k_{ji} are coefficients of elasticity of demand q_j with respect to q_i. According to the introduced assumptions, it can easily be seen as valid $k_{ij} = k_{ji}^{-1}$, and $k_{ii} = 1$. Additionally, the system of equations (11) can be written in the form of matrix equation

$$\mathbf{q} = K\mathbf{q}, \tag{12}$$

where $\mathbf{q} = (q1,\ldots,q_n)^T \in R^n$ is, so called, *vector of demand*, while

$$K = \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1n} \\ k_{21} & k_{22} & \ldots & k_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ k_{n1} & k_{n2} & \ldots & k_{nn} \end{bmatrix} \tag{13}$$

is the matrix of corresponding coefficients of proportionality demand defined in (11). Based on (12), then it is clear that the demand vector **q** is the eigenvalue vector of matrix K, which may be of practical use in its calculation.

As the corresponding functional model of the cost of production we will use the parabolic (quadratic) functions

$$T_i(q) = a_i q^2 + b_i q + c_i, \quad i = 1,..., n. \tag{14}$$

Then, we will obtain the cost functions using standard approximation procedures, for which there is a lot of extensive literature (see, for example, Božinović and Stojanović [2] or Render and Stair [7]). On the other hand, to the demand function (2) we assume the simplest functional model of dependency the price p and the demand q of the product X. This model can be mathematically expressed by a linear function of demand

$$f(q) = mq + r, \tag{15}$$

which is easily determined from empirical data. Accordingly, the corresponding model is called *linear model* of monopolistic competition.

Using the afore mentioned functional models, we can find the optimal levels of production values, we mark them with $q_i^*, i = 1,2,..., n,$ for which every company achieves maximum profit by selling products *X*. By simply replacing the function (11)–(15) into the system of equations (8) and its resolved by $q_i = q_i^*$, we find

$$q_i^* = \frac{1}{2}(b_i - r)\left(m\sum_{j=1}^{n} k_{ji} - a_i \right)^{-1}, \tag{16}$$

where $i = 1,2,...,n$.

To illustrate the previous procedure, we will describe a typical example of monopolistic production optimization of three firms A, B, C, dealing with the chemical industry. These companies produce a certain kind of sanitary products intended for hotel chains and larger companies, as primary consumers. Using data of Wiens, i.e. Egwald Web Services [8], we take the participation of these firms in a given market segmentation is 1:2:3, respectively. In this way, if we mark with q_1, q_2, q_3 the appropriate volume of daily production of the product (000 units), the matrix coefficients of proportionality crossed demand forms (13) is

$$K = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}. \tag{17}$$

In the next step, as we have already pointed out above, we can determine the appropriate functional dependence of market prices p, income C_i and costs T_i compared to the volume of production q_i and the total demand q. If the price of the product expressed in euros, using the previously described approximations we obtained

- Demand function: p = 10.871-0.04692q
- Income function: $C_i\,(x)$ = 10.871q -0.04692q^2
- Cost functions:

Firm A: $T_1(q)$ = 39.922 +1.484q + 0.0295q^2
Firm B: $T_2(q)$ = 36.013 +1.3q + 0.0298q^2
Firm C: $T_3(q)$ = 45.213 +1.671q + 0.0278q^2.

As we apply the optimization procedure described above, based on solving systems of equations (8), or direct application of equality (16), we find the optimal level of production value (in 10^3 units of product):

$$q_1^* = 15.091; \quad q_2^* = 28.067; \quad q_1^* = 37.829. \tag{18}$$

Note that the obtained values correspond to a somewhat "ideal" proportion of participation of these companies in the market. Based on them, by simply replacing in the previous function, we get the corresponding values of market prices and total revenues, production costs and profits, as shown in Table 1.

Table 1. Estimated values of production parameters of monopoly companies

Optimal estimated values	Firms			Σ
	A	B	C	
Demand /000 units/	15.091	28.067	37.829	80.987
Price /€/	6.623	6.920	7.321	---
Income /000 €/	99.943	194.235	276.951	571.129
Costs /000 €/	69.035	95.977	148.208	313.219
Profit /000 €/	30.908	98.258	128.743	257.910

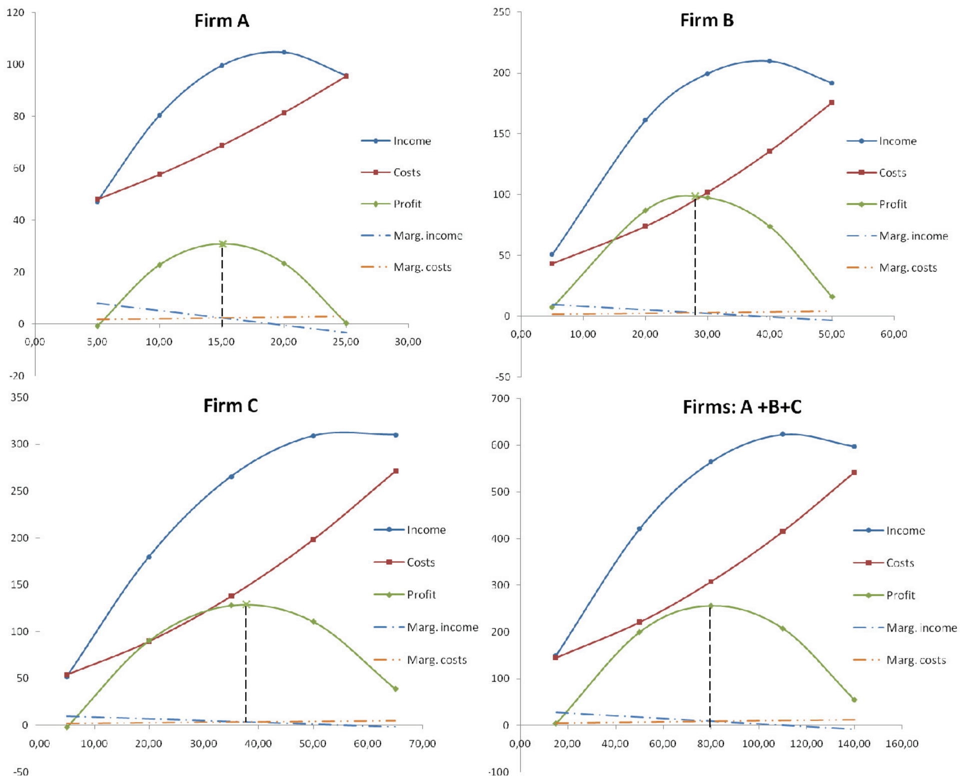

Fig. 1. Charts representing income, costs and profits of the companies together with optimal levels of production

Say, for example, that the market prices of that product for all three companies share a lot of balanced values and they are in a range from 6.62 to 7.32 euros. On the other hand, total profit of all firms, which amounts to 257.910 euros, is the sum of individual maximum profits which are realized as the difference of total revenue and production costs of the product. Functional dependence of income, costs and profits, along with optimal levels of production of all three companies, can be displayed graphically as in Fig. 1.

4. The Model of "Bipolistic" Competition

In the simplest case, when only two competitors perform in oligopolistic market, i.e. when it comes to so-called *bipolistic competition*, every company has only one "competitor" which must be taken into account when making its decisions. Then, both companies have a common goal, i.e. their interest is to "set up" their production of product X so that they achieve maximum profits in the market. In this section we describe the mathematical algorithm of determining the maximum profit of two opposing monopoly enterprises F_1 and F_2. It is clear that all discussion described above is valid if n=2, and we begin from that point of view.

Therefore, demand function for product X in the market be given in the form (2), where the total demand is

$$q = q_1 + q_2. \tag{19}$$

The system of differential equations (8) then becomes

$$\begin{aligned} f(q) + q_1 \frac{\partial f(q)}{\partial q_1} \cdot \left(1 + \xi_{21}'(q_1)\right) &= \frac{\partial T_1(q)}{\partial q_1} \\ f(q) + q_2 \frac{\partial f(q)}{\partial q_2} \cdot \left(1 + \xi_{12}'(q_2)\right) &= \frac{\partial T_2(q)}{\partial q_2}. \end{aligned} \tag{20}$$

Note that functional dependency between demand q_1 and q_2 generally has the form

$$q_1 = \xi_{12}(q_2), \tag{21}$$

whereas the "other" demand function ξ_{21} is the inverse of ξ_{12}, i.e. it is valid

$$q_2 = \xi_{21}(q_1) = \xi_{12}^{-1}(q_2). \tag{22}$$

In this way, each equation of system (20) can be expressed as a function of single production volume q_1 or q_2, because

$$q = q_1 + \xi_{21}(q_1) = \xi_{12}(q_2) + q_2. \tag{23}$$

In the previous review, one of the basic assumptions was that any company reacts to change, i.e. increases the volume of production in another company. This situation in practice is certainly real, because it is natural to assume that each company will react to the increase in production volume of its competitor.

However, in some situations we can conclude that by determining the maximum profit of one of the companies F_i, $i = 1,2$ other company F_j, $j \neq i$ will not change its output and vice versa. In this case joint demand q, similarly to before, can be showen as a variable which depends only on one of the variables q_1, q_2. Moreover, then is $\xi'_{21}(q_1) = \xi'_{12}(q_2) = 0$, so the system of equations (20) becomes

$$f(q) + q_i f'(q) = \frac{\partial T_i(q)}{\partial q_i}, \tag{24}$$

where i=1,2. Finally, if we introduce the labels

$$\varphi_i(q_1, q_2) = f(q) + q_i f'(q) - \frac{\partial T_i(q)}{\partial q_i}, \quad (25)$$

then the system of equations (24) can be written in implicit form as

$$\varphi_i(q_1, q_2) = 0, \quad i = 1,2. \quad (26)$$

Implicit functions given by equations (26) graphically represent the two curves in the coordinate plane q_1Oq_2, which intersect at a point M (Fig. 2, left) whose coordinates (q_1, q_2) represent the optimal level of demand, i.e. the solutions that companies F_1 and F_2 provide maximum profit.

Finally, if the functions of total cost $T_1(q)$ and $T_2(q)$ are equals, then the functions (26) are inverse to each other. As is known, the graphics of functions φ_1 and φ_2 are symmetrical in relation to the line $q_1 = q_2$. Hence the intersection of their graphics lie exactly on that line, which is why we conclude that the optimal level of demand from both companies, is given by equasion

$$q_1 = q_2 = \frac{q}{2}(= q_0). \quad (27)$$

Now, a common demand of both companies we can get from equation (24), i.e.

$$f(q) + \frac{q}{2} f'(q) = T_u'\left(\frac{q}{2}\right), \quad (28)$$

where $T_u(q) = T_1(q) + T_2(q)$ is the function of the total cost of the product X, while the corresponding price p_0 is found from the equation $p_0 = f(q/2)$. Fig. 2 provides a graphical view of the above-described process of finding the optimal level of q_0 and p_0.

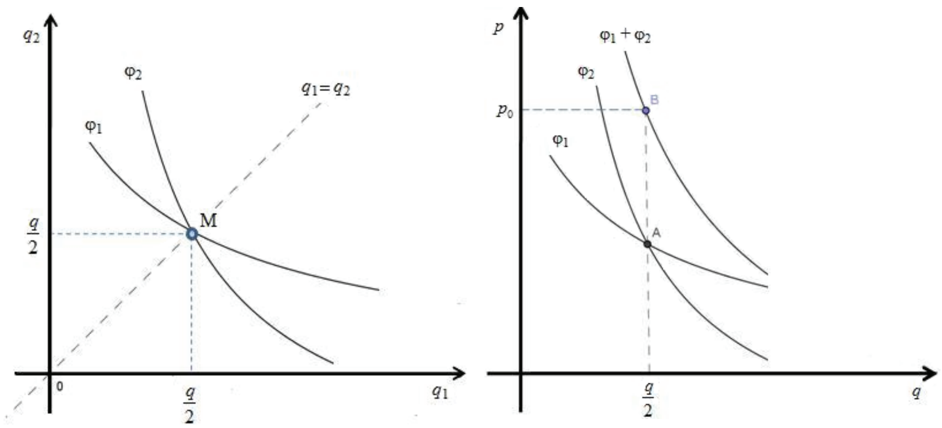

Fig. 2. Determining the optimal level of demand and price in the case of two "bipolistic" competing firms

5. Conclusion

The form of oligopoly as a model of the market structure is quite prevalent, and there are plenty of examples: automotive, steel industry, precious metals industry, oil industry, computer equipment, etc. Looking at individual sectors of the economy where a small numbers of competitors appear, it is clear that each, while making their decisions, must take into account the way in which management actions can affect competition, i.e. how each of them will respond.

Regarding this, the above described procedure to determine the equilibrium level of production may be the basis of the formal description of the joint action of monopolistic firms in the market. As the consequences of monopolistic power, there arise the economic inefficiency and loss of consumers' income in favor of monopolies. Therefore, precise analysis of the behavior of monopolistic enterprises may be the basis for strengthening the free market competition, price controls and the application of other anti-monopolistic measures.

References

1. Behrens, K. and Murata, Y., General equilibrium models of monopolistic competition: A new approach. Journal of Economic Theory, Vol. 136, Issue 1, pp. 776-787, 2007.
2. Božinović, M. and Stojanović, V., Mathematical methods and models in economy of enterprises (in Serbian). Leposavić: High School of Economy, 2006.
3. Henderson, J.M. and Quandt, R.E., Microeconomic Theory: A Mathematical Approach. New York: McGraw-Hill, 1971.
4. Hummels, D. and Levinsohn, J., Monopolistic Competition and International Trade: Reconsidering the Evidence. Oxford Journals: Quarterly Journal of Economics, Vol. 110, Issue3, pp. 799-836, 1995.
5. Lai, H. and Trefler, D., The Gains from Trade with Monopolistic Competition: Specification, Estimation, and Mis-Specification. NBER Working Paper No. 9169, 2002.
6. Petković, D., The impact of market structure on economic efficiency - an empirical analysis of Markets milk in Serbia (in Serbian). Serbian economic forum - Working document, 2008.
7. Render, B. and Stair, R., Quantitative analysis for management. London: Allin and Bacon, 1993.
8. Wiens, E.G. Monopolistic competition. http://www.egwald.ca.

MATEMATIČKI MODELI MONOPOLISTIČKE KONKURENCIJE: TEORIJSKI PRINCIPI I PRIMENE

Milan Božinović, Vladica Stojanović

U ovom radu razmatramo problem monopolističke konkurencije dva ili više preduzeća sa istim proizvodom. Posebna pažnja posvećena je konstrukciji opšteg matematičkog modela oligopolne konkurencije koji je zatim primenjen u optimizaciji proizvodnje triju monopolističkih firmi.

Ključne reči: *monopolistička konkurencija, modeli funkcionalnih zavisnosti, optimizacija ekonomskih veličina.*