FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 22, no. 3, December 2009, 265-284

A Point Source in the Presence of Spherical Material Inhomogenity: Analysis of Two Approximate Closed Form Solutions for Electrical Scalar Potential

Predrag D. Rančić, Miodrag S. Stojanović, Milica P. Rančić, and Nenad N. Cvetković

Abstract: A brief review of derivation of two groups of approximate closed form expressions for the electrical scalar potential (ESP) Green's functions that originate from the current of the point ground electrode (PGE) in the presence of a spherical ground inhomogenity, is presented in this paper.

The PGE is fed by a very low frequency periodic current through a thin isolated conductor. One of approximate solutions is proposed in this paper. Known exact solutions that have parts in a form of infinite series sums are also given in this paper. Here, the exact solution is solely reorganized in order to facilitate comparison to the closed form solutions, and to estimate the error introduced by the approximate solutions. Finally, error estimation is performed comparing the results for the electrical scalar potential obtained applying the approximate expressions and accurate calculations. This is illustrated by numerous numerical experiments.

Keywords: Electrical scalar potential, Green's function, point ground electrode, spherical inhomogenity.

1 Introduction

PROBLEMS of potential fields related to the influence of spherical material inhomogenities have a rather rich history of over 150 years in different fields of mathematical physics. In the fields of electrostatic field, stationary and quasi-stationary current field, and magnetic field of stationary currents, problems of a

Manuscript received on April 25, 2009.

The authors are with Department of Electrical Power Engineering and Theoretical Electrical Engineering, Faculty of Electronic Engineering, University of Niš, 18000 Niš, Serbia (e- mails: [predrag.rancic; miodrag.stojanovic; milica.rancic; nenad.cvetkovic] @elfak.ni.ac.rs).

point source in the presence of a spherical material inhomogenity are gathered in the book by Stratton (1941, [1]) and all later authors that have treated this matter quote this reference as the basic one. The authors of this paper will also consider the results from [1] as referent ones.

The exact solution shown in [1, pp.201-205], related to the point charge in the presence of the dielectric sphere, is obtained solving the Poisson, i.e. Laplace partial differential equation expressed in the spherical coordinate system, using the method of separation of variables. Unknown integration constants are obtained satisfying boundary conditions for the electrical scalar potential continuity and normal component of the electric displacement at the boundary of medium discontinuity, i.e. on the dielectric sphere surface. The obtained general solution for the electrical scalar potential, besides a number of closed form terms, also consists of a part in a form of an infinite series sum that has to be numerically summed.

Among work of other authors that have dealt with this problem, the following will be cited in this paper: Hannakam ([2,3]), Reiß([4]), Lindell et all. ([5–7]) and Veličković ([8,9]). The last cited ones, according to the authors of this paper, gave an approximate closed form solution of the problem. This is also characterized in this paper.

In paper [2], author, through detail analysis manages to express a part of a general solution in a form of infinite sums by a class of integrals whose solutions can not be given in a closed form, i.e. general solution of these integrals have to be obtained numerically. In paper [4], the author considers a problem of this kind with an aim to calculate the force on the point charge in the presence of a dielectric sphere. In order to solve this problem, he uses the Kelvin's inversion factor and introduces a line charge image, which coincides with results from [2].

Starting with the general solution from [1], authors in [5] and [6], using different mathematical procedures, practically obtain the same solutions as in [2], considering separately the case of a point charge outside the sphere ([5]) and the case of a point charge inside the sphere ([6]).

In papers [8] and [9, pp. 97–98], the author deduces the closed form solution for the electrical scalar potential of the point source in the presence of sphere inhomogenity in two steps. In the first one, the author assumes a part of the solution that corresponds to images in the spherical mirror and approximately satisfies boundary conditions on the sphere surface. In the second step, assumed solutions are broadened by infinite sums that approximately correspond to the ones that occur as an exact general solution in [1], i.e. in other words, approximately satisfy the Laplace partial differential equation. Afterwards, unknown constants under the sum symbol are obtained satisfying the boundary condition of continuity of the normal component of total current density on the sphere surface. These solutions enable summing of infinite sums and presenting the general solution in a closed form. Well known mathematical tools from the Legendre polynomial theory were used for the summing procedure.

Finally, the author of this paper has, analysing problems of this kind ([10]), started from the general solution from [1] and, primarily, reorganized certain parts in the following way. A number of terms that correspond to images in the spherical mirror with unknown weight coefficients are singled out. Remaining parts of the general solution are infinite sums whose general, n-th term presents a product of an unknown integration constant, factored function of radial sphere coordinate $r^{-(n+1)}$ and r^n, and the Legendre polynomial of the first kind $P_n(\cos\theta)$. Afterwards, all unknown constants are determined satisfying mentioned boundary conditions, but in such a way that the condition for the electrical scalar potential continuity is completely satisfied, while the condition for the normal component of total current density can be fulfilled approximately. Approximate satisfying of this boundary condition is done in a way to sum a part of the general solution expressed by infinite sums in a closed form. This technique is well known and was very successfully, although under certain assumptions, used by many authors especially in the high frequency domain. For example, one of them is explicitly considered in [11], and one is implicitly given in [12] and [13].

Among five quoted solutions, three will be analyzed in this paper, i.e. the accurate one from [1] as the referent one, approximate one from [8,9] (which will be characterized in detail since this was not done in [8,9]) and the second one, also an approximate model, proposed in this paper ([10]).

In the second section of the paper three groups of cited expressions for the ESP distribution will be given with minimal remarks about their deduction. In this part of the paper, general expressions for the evaluation of the ESP calculation error will be also presented.

A part of numerical experiments whose results justify the use of approximate solutions and also present the error level done along the way, will be presented in the third section of the paper.

Finally, based on the presented theory and performed numerical experiments, corresponding conclusion will be made and a list of used references will be given.

2 Theoretical Background

2.1 Description of the Problem

Spherical inhomogenity of radius r_s is considered. Sphere domain is considered as a linear, isotropic and homogenous semi-conducting medium of known electrical parameters σ_s, ε_s and $\mu_s = \mu_0$ (σ_s - specific conductivity, $\varepsilon_s = \varepsilon_0\varepsilon_{rs}$ - permittivity and $\mu_s = \mu_0$ - permeability). The remaining space is also a linear, isotropic and homogenous semi-conducting medium of known electrical parameters σ_1, $\varepsilon_1 = \varepsilon_0\varepsilon_{r1}$ and $\mu_1 = \mu_0$.

The spherical, i.e. Descartes' coordinate systems with their origins placed in the sphere centre are associated to the problem. At the arbitrary point P', defined by the position vector $\vec{r'} = r'\hat{z}$, a point current source is placed (so-called Point ground electrode - PGE), and is fed through a thin isolated conductor by a periodic current of intensity I_{PGE} and very low angular frequency ω, $\omega = 2\pi f$.

The location of the PGE can be outside the sphere, $r' \geq r_s$, or inside of it, $r' \leq r_s$, which also goes for the observed point P defined by the vector $\vec{r}$, at which the potential and quasi-stationary current and electrical field structure are determined, i.e. for $r \geq r_s$ the point is outside the sphere, and for $r \leq r_s$, inside of it.

In accordance with the last one, the electrical scalar potential $\varphi_{..}(\vec{r})$, total current density vector $\vec{J}_{..}(\vec{r})$ and electrical field vector $\vec{E}_{..}(\vec{r})$, will be denoted by two indices $i, j = 1, s$ where the first one "i" denotes the medium where the quantity is determined, and the other one "j" the medium where the PGE is located. For example: $\varphi_{s1}(\vec{r})$ presents the potential calculated inside the sphere, $r \leq r_s$, when the PGE is located outside the sphere, $r' \geq r_s$. Also, in order to systemize text and ease its reading, the solutions that correspond to references [1], [8,9] and [10], will be denoted in the exponent as follows: S–Stratton, V–Veličković and R-Rančić, respectively. For example: $\varphi_{11}^{S}(\vec{r})$ presents the solution for the potential according to [1]–Stratton outside the sphere, $r \geq r_s$, when the PGE is located at point P' that is also outside the sphere, $r' \geq r_s$.

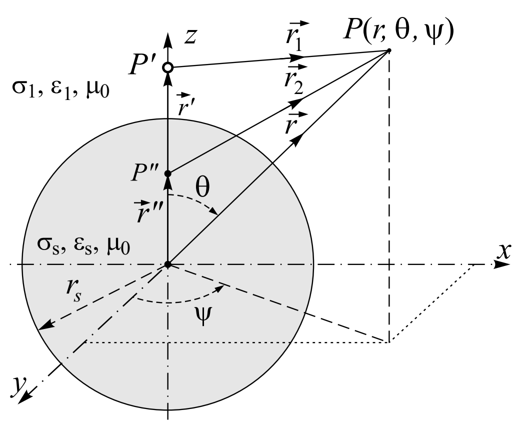

Fig. 1. The PGE outside the sphere.

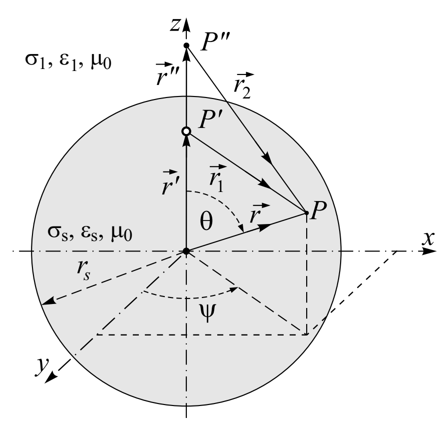

Fig. 2. The PGE inside the sphere.

Problem geometry is illustrated graphically in Figs. 1 and 2, where Fig. 1 corresponds to the case when the PGE is placed outside the sphere, while Fig. 2 refers to its location inside of the sphere. Images in the spherical mirror that are singled out, i.e. points P'' with corresponding position vector $\vec{r}'' = r''\hat{z}$, where $r'' = r_s^2/r'$ is the Kelvin's inversion factor of the spherical mirror, are also given in figures. Distance from the PGE, point P', to the observed point P is denoted by r_1, $r_1 = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$, and distance from the image in the spherical mirror, point P'', to the point P by r_2, $r_2 = \sqrt{r^2 + r''^2 - 2rr''\cos\theta}$.

Finally, the following labels were used in the paper: $\underline{\sigma}_i = \sigma_i + \mathrm{j}\omega\varepsilon_i$ - complex conductivity of the i-th medium, $i = 1, s$; $\underline{\varepsilon}_{ri} = \varepsilon_{ri} - \mathrm{j}\varepsilon_{ii} = \varepsilon_{ri} - \mathrm{j}60\sigma_i\lambda_0$ - complex relative permittivity of the i-th medium, $i = 1, s$ and λ_0 - wave-length in the air; $\underline{\gamma}_i = (\mathrm{j}\omega\mu_0\underline{\sigma}_i)^{1/2}$ - complex propagation constant of the i-th medium, $i = 1, s$; $\underline{n}_{ij} = \underline{\gamma}_i/\underline{\gamma}_j$, - complex refraction index of the i-th and the j-th medium, $i = 1$, s; and R_{1s}, T_{1s}, T_{s1} - quasi-stationary reflection and transmission coefficients defined by the following expression:

$$R_{1s} = \frac{\underline{\sigma}_1 - \underline{\sigma}_s}{\underline{\sigma}_1 + \underline{\sigma}_s} = \frac{\underline{n}_{1s}^2 - 1}{\underline{n}_{1s} + 1} = T_{1s} - 1 = -R_{s1} = 1 - T_{s1}.$$

The time factor $\exp(\mathrm{j}\omega t)$ is omitted in all relations.

2.2 Exact ESP Solution According to [1]

2.2.1 Electrical scalar potential (ESP)

The ESP function for any position of the PGE must satisfy the Poisson, i.e. Laplace partial differential equation, which are, in accordance with introduced labels for the spherical coordinate system, as follows:

- The PGE outside the sphere, $i = 1, s$, $r' \geq r_s$, Fig. 1,

$$\Delta\varphi_{i1} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_{i1}}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi_{i1}}{\partial\theta}\right) =$$

$$= \begin{cases} -\dfrac{I_{PGE}}{2\pi\underline{\sigma}_1}\dfrac{\delta(r-r')\delta(\theta)}{r^2\sin\theta}, & r \geq r_s \\ 0, & r \leq r_s; \end{cases} \tag{1}$$

- The PGE inside the sphere, $i = 1, s$, $r' \leq r_s$, Fig.2,

$$\Delta\varphi_{is} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_{is}}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi_{is}}{\partial\theta}\right) =$$

$$= \begin{cases} 0, & r \geq r_s \\ -\dfrac{I_{PGE}}{2\pi\underline{\sigma}_s}\dfrac{\delta(r-r')\delta(\theta)}{r^2\sin\theta}, & r \leq r_s; \end{cases} \tag{2}$$

where $\delta(r-r')$ and $\delta(\theta)$ are Dirac's δ –functions.

After differential equations, for example (1), are solved applying the method of separation of variables, the unknown integration constants are determined so the obtained solution satisfies the condition for the finite value of the potential at all points $r \in [0,\infty]$, except at $\vec{r}=\vec{r}'$. Remaining integration constants are determined from the electrical scalar potential boundary condition,

$$\varphi_{11}(r=r_s,\theta)=\varphi_{s1}(r=r_s,\theta), \tag{3}$$

and the one for the normal component of the total current density on the discontinuity surface, i.e,

$$\underline{\sigma}_1 \frac{\partial \varphi_{11}(r,\theta)}{\partial r}\Big|_{r=r_s} = \underline{\sigma}_s \frac{\partial \varphi_{s1}(r,\theta)}{\partial r}\Big|_{r=r_s}. \tag{4}$$

Finally, according to [1], the exact solution for the potential distribution, Section 3.23, pp. 204, Eqs. (20)-(21), is for $\vec{r} \geq \vec{r}'$:

$$\varphi_{11}^S(\vec{r}) = \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\left[\frac{1}{r_1}+\sum_{n=0}^{\infty}\frac{n(\underline{\sigma}_1-\underline{\sigma}_s)}{n\underline{\sigma}_s+(n+1)\underline{\sigma}_1}\frac{r_s^{2n+1}}{r'^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}}\right], r \geq r_s, \tag{5a}$$

$$\varphi_{s1}^S(\vec{r}) = \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\sum_{n=0}^{\infty}\frac{(2n+1)\underline{\sigma}_1 r^n}{n\underline{\sigma}_s+(n+1)\underline{\sigma}_1}\frac{P_n(\cos\theta)}{r'^{n+1}}, r \leq r_s, \tag{5b}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of the first kind.

Keeping in mind the duality of electrostatic and quasi-stationary very low frequency current fields, in relation to the solution from [1, Eqs. (20) and (21)]:

- Labels introduced in expressions (5a) and (5b) fit the described geometry and used labels;
- q/ε_2 is substituted by $I_{PGE}/\underline{\sigma}_1$; and
- instead of permittivity, corresponding indexed complex conductivities are used, i.e. $\underline{\sigma}_s$ instead of ε_1, and $\underline{\sigma}_1$ instead of ε_2.

When expressions (5a) and (5b) are reorganized in such a way so they can be compared to approximate expressions, and additionally labelled by S-Stratton, the following exact solution is obtained:

$$\begin{aligned}\varphi_{11}^S(\vec{r}) = V_s\Bigg[&\frac{r_s}{r_1}+R_{1s}\frac{r_s}{r'}\left(\frac{r_s}{r_2}-\frac{r_s}{r}\right)- \\ &-\frac{R_{1s}T_{1s}}{2}\sum_{n=1}^{\infty}\frac{1}{n+T_{1s}/2}\left(\frac{r''}{r}\right)^{n+1}P_n(\cos\theta)\Bigg], r \geq r_s,\end{aligned} \tag{6a}$$

$$\varphi_{s1}^{S}(\vec{r}) = V_s\Big[T_{1s}\frac{r_s}{r_1} - R_{1s}\frac{r_s}{r'} - \\ - \frac{R_{1s}T_{1s}}{2}\frac{r_s}{r'}\sum_{n=1}^{\infty}\frac{1}{n+T_{1s}/2}\left(\frac{r}{r'}\right)^n P_n(\cos\theta)\Big], r \le r_s, \tag{6b}$$

where $V_s = I_{PGE}/(4\pi\underline{\sigma}_1 r_s)$, and R_{1s}, T_{1s} are reflection and transmission coefficients, respectively.

In the same way, final solutions for equations (2), for $r' \le r_s$, that satisfy conditions (3) and (4) are:

$$\varphi_{1s}^{S}(\vec{r}) = V_s\Big[T_{1s}\frac{r_s}{r_1} - R_{1s}\frac{r_s}{r} - \\ - \frac{R_{1s}T_{1s}}{2}\frac{r_s}{r'}\sum_{n=1}^{\infty}\frac{1}{n+T_{1s}/2}\left(\frac{r'}{r}\right)^{n+1} P_n(\cos\theta)\Big], r \ge r_s, \tag{7a}$$

$$\varphi_{ss}^{S}(\vec{r}) = V_s\Big[\frac{T_{1s}}{T_{s1}}\frac{r_s}{r_1} - R_{1s}\frac{T_{1s}}{T_{s1}}\frac{r_s}{r'}\frac{r_s}{r_2} - R_{1s} - \\ - \frac{R_{1s}T_{1s}}{2}\sum_{n=1}^{\infty}\frac{1}{n+T_{1s}/2}\left(\frac{r}{r''}\right)^n P_n(\cos\theta)\Big], r \le r_s. \tag{7b}$$

Comment: The last two expressions are not explicitly given in [1], as (5a) and (5b).

2.2.2 Quasi-stationary electrical and current field structure

Once the potential distributions (6a)–(6b) and (7a)–(7b) are determined, the structure of the quasi-stationary field vectors are:

- Electrical field vector:

$$\vec{E}_{ij} \cong -\mathrm{grad}\varphi_{ij} = -\frac{\partial\varphi_{ij}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\varphi_{ij}}{\partial\theta}\hat{\theta}, \quad i,j = 1,s; \tag{8}$$

- Total current density vector:

$$\vec{J}_{ij}^{\,\mathrm{tot}} = \underline{\sigma}_i\vec{E}_{ij}, \quad i,j = 1,s; \tag{9}$$

- Conduction current density vector:

$$\vec{J}_{ij} = \sigma_i\vec{E}_{ij}, \quad i,j = 1,s. \tag{10}$$

2.3 ESP Solution According to [8] and [9, pp. 97-98]

The ESP solution proposed in [8] considers the following. Firstly, for the case $r' \ge r_s$, solution is proposed in a form:

$$\varphi_{11} \cong \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\left[\frac{1}{r_1} + C_1\frac{1}{r_2} + C_2\frac{1}{r}\right], r \ge r_s, \tag{11a}$$

$$\varphi_{s1} \cong \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\left[C_3\frac{1}{r_1}+C_4\right], r \leq r_s, \tag{11b}$$

where C_1–C_4 are unknown constants that are determined satisfying the condition (3) and the one that the solution (11) is also valid for the case of a sphere with great radius. This solution is identical to the first three terms of the exact solution (6a) and the first two of (6b).

Since solution (11) obtained this way does not satisfy the boundary condition (4), the author broadened solutions (11a) and (11b) with two infinite series of general form:

$$\sum_{n=1}^{\infty} C_{5n}\left(\frac{r_s}{r}\right)^{\pm n} P_n(\cos\theta), \tag{12}$$

where C_{5n} are unknown constants, for "$+n$" in (12) the Eq. (11a) is broadened and Eq. (11b) for "$-n$". Unknown constants C_{5n} are determined using the condition (4), having the ESP final solution:

$$\begin{aligned}\varphi_{11}^{V}(\vec{r}) \cong V_s\Big[&\frac{r_s}{r_1}+R_{1s}\frac{r_s}{r'}\left(\frac{r_s}{r_2}-\frac{r_s}{r}\right)+\\ &+\frac{R_{1s}T_{1s}}{2}\left(\frac{r_s}{r'}\right)\ln\frac{r-r''\cos\theta+r_2}{2r}\Big], r \geq r_s,\end{aligned} \tag{13a}$$

$$\begin{aligned}\varphi_{s1}^{V}(\vec{r}) \cong V_s\Big[&T_{1s}\frac{r_s}{r_1}-R_{1s}\frac{r_s}{r'}+\\ &+\frac{R_{1s}T_{1s}}{2}\left(\frac{r_s}{r'}\right)\ln\frac{r'-r\cos\theta+r_1}{2r'}\Big], r \leq r_s.\end{aligned} \tag{13b}$$

Label V–Veličković in the exponent denotes that solutions (13a) and (13b) correspond to the ones from [8], and V_s is previously introduced constant that appears also in (6a) and (6b).

It should be noted that the introduced extension (12) for "$+n$", approximately satisfies the general solution of the Laplace equation, i.e. Eq. (5a), where r_s/r is factored by $n+1$.

Similarly, the solutions for the potential when, i.e. the PGE is located inside the sphere, are also given in [8]. The solutions are as follows:

$$\begin{aligned}\varphi_{1s}^{V}(\vec{r}) \cong V_s\Big[&T_{1s}\frac{r_s}{r_1}-R_{1s}\frac{r_s}{r}+\\ &+\frac{R_{1s}T_{1s}}{2}\ln\frac{r-r'\cos\theta+r_1}{2r}\Big], r \geq r_s,\end{aligned} \tag{14a}$$

$$\begin{aligned}\varphi_{ss}^{V}(\vec{r}) \cong V_s\Big[&\frac{T_{1s}}{T_{s1}}\frac{r_s}{r_1}-R_{1s}\frac{T_{1s}}{T_{s1}}\frac{r_s}{r'}\frac{r_s}{r_2}-R_{1s}+\\ &+\frac{R_{1s}T_{1s}}{2}\ln\frac{r''-r\cos\theta+r_2}{2r''}\Big], r \leq r_s.\end{aligned} \tag{14b}$$

2.4 ESP Solution Proposed in this Paper [10]

If the general solution from [1] is reorganized under the sum symbol into a form that is for $r' \geq r_s$ given by Eqs. (6a) and (6b), the following is obtained:

$$\varphi_{11}(\vec{r}) = \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\left[\frac{1}{r_1} + C_1\frac{r_s}{r'}\frac{1}{r_2} + B_0\frac{r_s}{r} + \sum_{n=1}^{\infty} B_n\left(\frac{r_s}{r}\right)^{n+1} P_n(\cos\theta)\right], r \geq r_s, \quad (15a)$$

$$\varphi_{s1}(\vec{r}) = \frac{I_{PGE}}{4\pi\underline{\sigma}_1}\left[D_1\frac{1}{r_1} + A_0 + \sum_{n=1}^{\infty} A_n\left(\frac{r}{r_s}\right)^{n} P_n(\cos\theta)\right], r \leq r_s, \quad (15b)$$

where C_1, D_1, B_n and A_n, $n = 0, 1, \ldots$, are unknown constants. Starting from the boundary condition (3) we have $1 + C_1 = D_1$ and $B_n = A_n$, $n = 0, 1, \ldots$. The other boundary condition gives $C_1 = R_{1s}$, so $D_1 = T_{1s}$. If the condition (4) is approximately satisfied, we also have $A_0 = B_0 = -R_{1s}/r'$, and constants B_n, $n = 1, 2, \ldots$, related to (4) are determined from the condition

$$-\frac{R_{1s}T_{1s}}{2r'}\sum_{n=1}^{\infty}\left(\frac{r_s}{r'}\right)^n P_n(\cos\theta) = \sum_{n=1}^{\infty} nB_nP_n(\cos\theta). \quad (16)$$

In Eq.(4) remains a term in a form of a sum, i.e. the error ”e” of satisfying the boundary condition (4) for the radial component of total current density is

$$\begin{aligned} e\{J_{11r}^{\text{tot}}\} &= \frac{I_{PGE}}{4\pi r_s}\sum_{n=1}^{\infty} B_nP_n(\cos\theta) = \\ &= -\underline{\sigma}_1 V_s\frac{R_{1s}T_{1s}}{2r'}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_s}{r'}\right)^n P_n(\cos\theta) = \\ &= \underline{\sigma}_1 V_s\frac{R_{1s}T_{1s}}{2r'}\ln\frac{r' - r_s\cos\theta + r_{1s}}{2r'}, \end{aligned}$$

where r_{1s} is r_1 for $r = r_s$, and B_n, $n = 1, 2, \ldots$, from (16).

If we substitute the solution for, $B_n = A_n$, $n = 0, 1, \ldots$, from (16) into (15a) and (15b) and using known tools from the Legendre polynomial theory, we have:

$$\begin{aligned} \varphi_{11}^{R}(\vec{r}) \cong V_s\Bigg[&\frac{r_s}{r_1} + R_{1s}\frac{r_s}{r'}\left(\frac{r_s}{r_2} - \frac{r_s}{r}\right) + \\ &+ \frac{R_{1s}T_{1s}}{2}\left(\frac{r_s}{r'}\right)\left(\frac{r_s}{r}\right)\ln\frac{r - r''\cos\theta + r_2}{2r}\Bigg], r \geq r_s, \end{aligned} \quad (17a)$$

$$\begin{aligned} \varphi_{s1}^{R}(\vec{r}) \cong V_s\Bigg[&T_{1s}\frac{r_s}{r_1} - R_{1s}\frac{r_s}{r'} + \\ &+ \frac{R_{1s}T_{1s}}{2}\left(\frac{r_s}{r'}\right)\ln\frac{r' - r\cos\theta + r_1}{2r'}\Bigg], r \leq r_s. \end{aligned} \quad (17b)$$

The ESP solution when $r' \leq r_s$ is obtained in a similar way. After obtaining the unknown constants, satisfying the condition (3) and approximately satisfying the condition (4), we have:

$$\begin{aligned}\varphi_{1s}^{R}(\vec{r}) \cong V_s\Big[T_{1s}\frac{r_s}{r_1} - R_{1s}\frac{r_s}{r} + \\ + \frac{R_{1s}T_{1s}}{2}\left(\frac{r_s}{r}\right)\ln\frac{r - r'\cos\theta + r_1}{2r}\Big], r \geq r_s,\end{aligned} \tag{18a}$$

$$\begin{aligned}\varphi_{ss}^{R}(\vec{r}) \cong V_s\Big[\frac{T_{1s}}{T_{s1}}\frac{r_s}{r_1} - R_{1s}\frac{T_{1s}}{T_{s1}}\frac{r_s}{r'}\frac{r_s}{r_2} - R_{1s} + \\ + \frac{R_{1s}T_{1s}}{2}\ln\frac{r'' - r\cos\theta + r_2}{2r''}\Big], r \leq r_s.\end{aligned} \tag{18b}$$

2.5 Analysis of the Presented ESP Solutions

- In contrast to the exact solution [1], both approximate ones have a closed form.
- Obtained approximate solutions (17a) and (17b) are very similar to (13a) and (13b). The only difference is between Eqs. (13a) and (17a) in factor r_s/r that factors the Ln-function. The same respectively goes for solutions (18a) and (18b) when compared to approximate ones (14a) and (14b).
- Solutions (13a) and (13b) satisfy boundary conditions (3) and (4), but do not have a general solution for $r \geq r_s$ that follows from the solution for the Poisson, i.e. Laplace equation [1]. The same goes for (14a) and (14b). A solution to one technical problem of this kind, solved applying this "V" model, is given in [14] and [15].
- Solutions (17a) and (17b) satisfy the general solution for the Poisson, i.e. Laplace partial differential equation, satisfy boundary condition (3), and approximately satisfy boundary condition (4). The same goes for solutions given by (18a) and (18b).
- Expressions (17)–(18) can be also obtained starting from the accurate ones (6)–(7) under a condition $n_{1s} << 1$. In that case, the addend $T_{1s}/2$ in the denominator under the sum symbol is $|T_{1s}/2| = |\underline{n}_{1s}^2/(1+\underline{n}_{1s}^2)| << 1$, so, it can be neglected in relation to the sum index $n \geq 1$. For example, the sum term in (6a) is then approximately

$$-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r''}{r}\right)^{n+1}P_n(\cos\theta) = \frac{r''}{r}\ln\frac{r - r''\cos\theta + r_2}{2r},$$

  and consequently the expression (6a) becomes identical to (17a). Similarly, we obtain remaining expressions (17b), (18a) and (18b). Accordance of the results obtained applying the approximate model "R" and the exact one "S" is better for all values of the refraction coefficient $n_{1s} < 1$, then for the case of $n_{1s} > 1$. This can be easily concluded analysing given expressions "S" and "R". This conclusion is also confirmed by numerical experiments.

2.6 Error Estimation Using the Approximate Expressions for the ESP

All the ESP expressions, accurate ones (6a), (6b), (7a) and (7b) according to [1], approximate ones (13a), (13b), (14a) and (14b) according to [8] and [9] and approximate ones according to R-model proposed in this paper [10, 16, 17], evolved towards the same form so they could be directly compared. Firstly, the terms that associate to spherical mirror imaging are singled out, and they correspond to images with weight coefficients multiplied by quasi-stationary reflection R_{1s}, or transmission T_{1s}, coefficients. Remaining part of the solution is an infinite sum in the case of the exact solution, and in the case of approximate ones, a closed form expressed by Ln-functions.

Error estimation of the ESP calculation is done according to the general expression:

$$\delta_Q = 100\left|\frac{\varphi_{ij}^S(\vec{r}) - \varphi_{ij}^Q(\vec{r})}{\varphi_{ij}^S(\vec{r})}\right|, \quad \text{in } [\%], \tag{19}$$

where $i, j = 1, s$ and $Q = V, R$.

Relative error estimation of satisfying boundary condition (4) can be evaluated according to the following expression:

$$\delta_J = 100\left|\frac{e\{J_{11r}^{\text{tot}}\}}{-\underline{\sigma}_1 \partial\varphi_{11}^s/\partial r}\right|_{r=r_s}, \quad \text{in } [\%]. \tag{20}$$

3 Numerical Results

Based on presented ESP expressions a number of numerical experiments were performed in order to establish the validity of the proposed approximate solutions compared to exact ESP calculations using expressions (6)-(7) according to [1].

The results presented graphically in the figures that follow will be denoted as:

- S-model, Eqs. (6)-(7), ref. [1];
- V-model, Eqs. (13)-(14), ref. [8, 9]; and
- R-model, Eqs. (17)-(18), i.e. the ESP model proposed in this paper.

The first group of numerical results deals with the electrostatic problem of the point charge (PCh) in the presence of the spherical dielectric inhomogenity. In

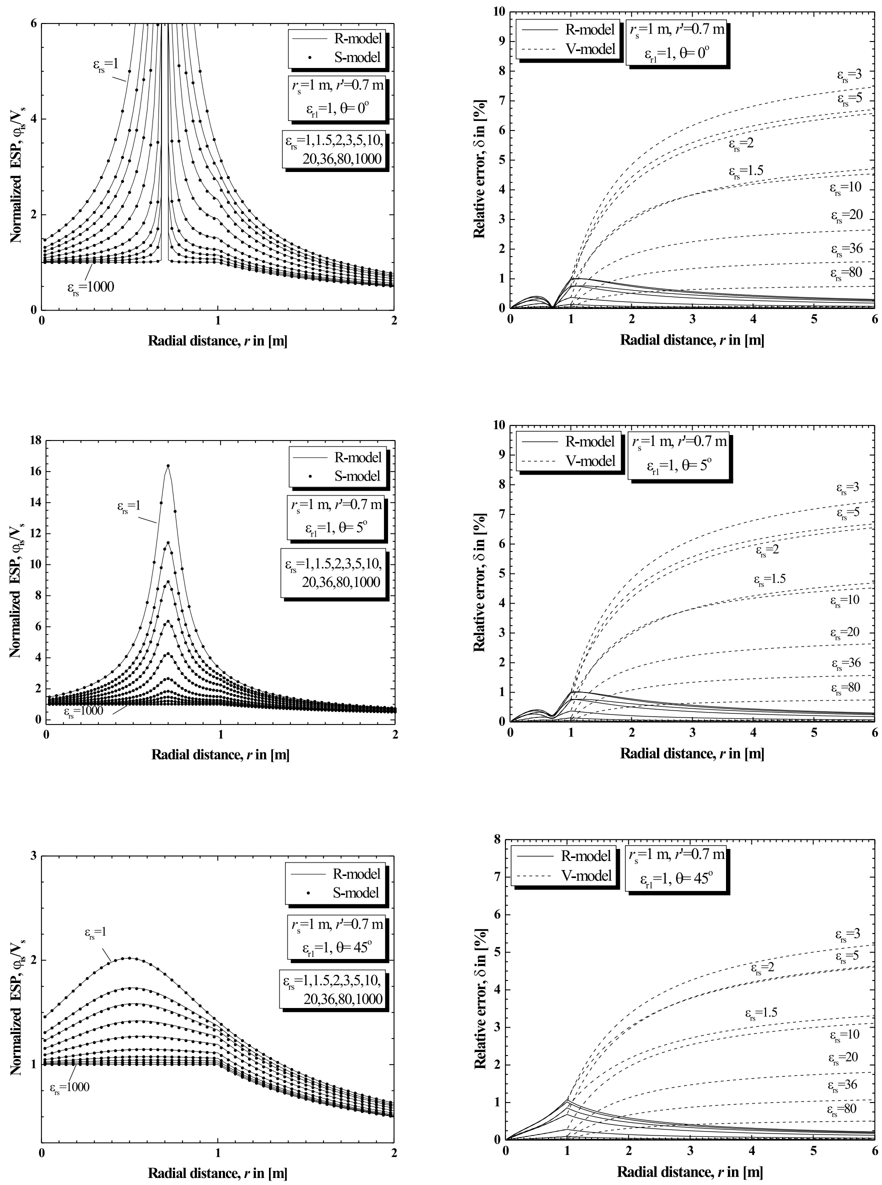

Fig. 3. Point charge inside dielectric sphere. Normalized ESP and corresponding relative error versus radial distance r for different values of angle θ and relative permittivity ε_{rs} taken as parameters.

this case, $\varepsilon_i = \varepsilon_0\varepsilon_{ri}$, $i = 1, s$, should replace $\underline{\sigma}_i$ in all the expressions, having $V_s = Q/(4\pi\varepsilon_1 r_s)$ and $p_{1s} = \varepsilon_1/\varepsilon_s$. Normalized ESP (R–model, solid line) versus radial

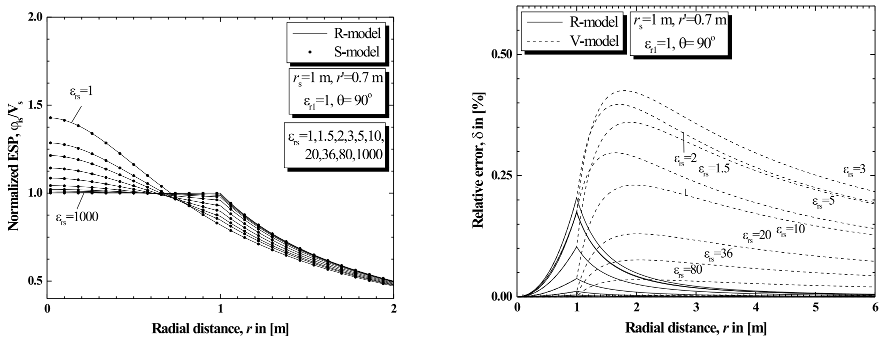

Fig. 3. Continue. Point charge inside dielectric sphere. Normalized ESP and corresponding relative error versus radial distance r for different values of angle θ and relative permittivity ε_{rs} taken as parameters.

distance r for different values of angle $\theta = 0^{\mathrm{o}}, 5^{\mathrm{o}}, 45^{\mathrm{o}}$ and 90^{o}, and different values of relative permittivity $\varepsilon_{rs} = 1, 1.5, 2, 3, 5, 10, 20, 36, 80$ and 1000 as parameters, are given in the left column of the Fig. 3. (the PCh inside the sphere: $r' = 0.7r_s$). For the sake of comparing, the exact normalized ESP values according to S–model

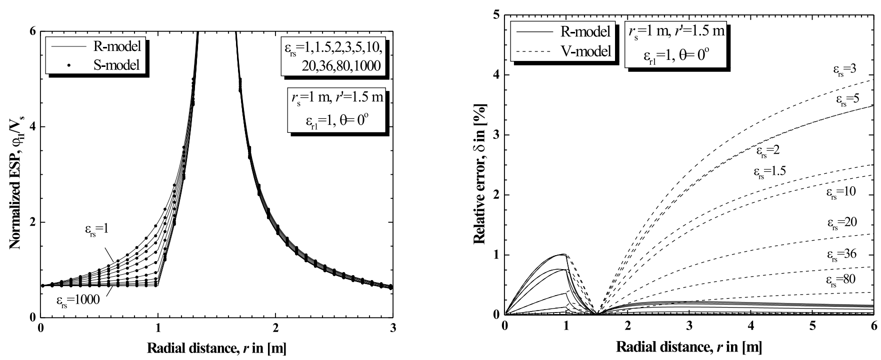

Fig. 4. Point charge outside dielectric sphere. Normalized ESP and corresponding relative error versus radial distance r for different values of angle θ and relative permittivity ε_{rs} taken as parameters.

(solid circle) are presented in the same figures. Corresponding relative error δ in [%], calculated for both R– and V– models is presented in the right column of Fig. 3. Normalized ESP (R- and S-models) and corresponding relative errors (R- and V-models) for the case of $r' = 1.5r_s$ are presented in Fig. 4 (the PCh outside the sphere).

The second group of numerical experiments consider the quasi-stationary field,

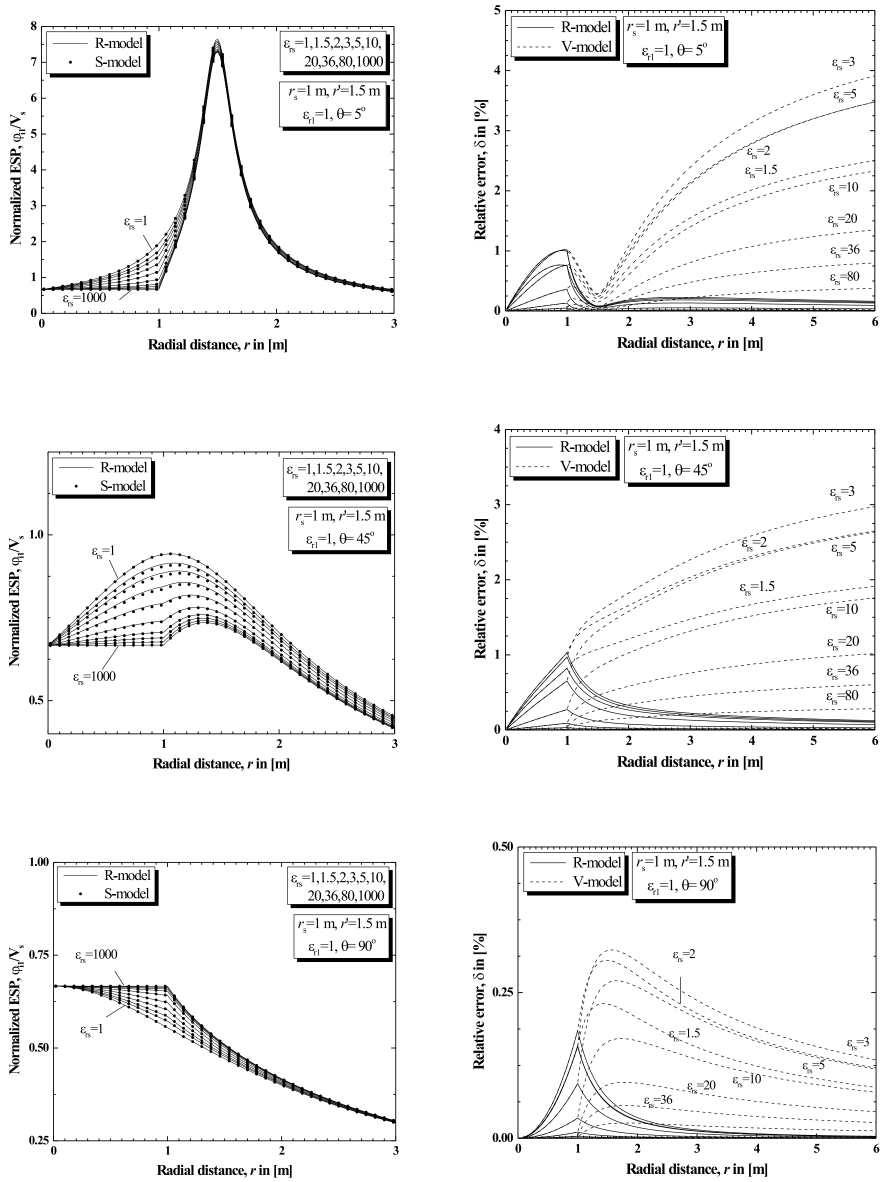

Fig. 4. Continue: Point charge outside dielectric sphere. Normalized ESP and corresponding relative error versus radial distance r for different values of angle θ and relative permittivity ε_{rs} taken as parameters.

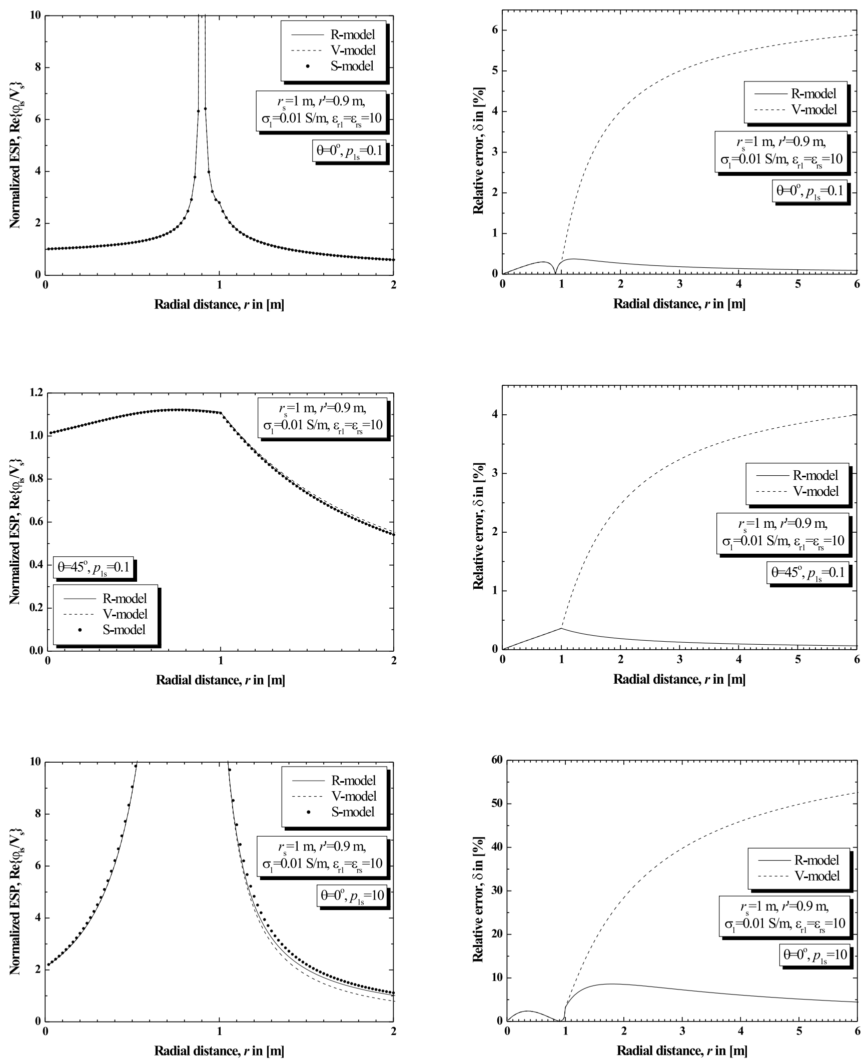

Fig. 5. The PGE inside the sphere. Normalized ESP, φ_{is}/V_s, $i = 1, s$, and corresponding relative errors versus radial distance, for different values of geometry parameters and relation $p_{1s} = \sigma_1/\sigma_s$ taken as parameters.

i.e. the case of the semi-conducting spherical inhomogenity and the PGE fed by the VLF current, $f = 50$Hz. The ESP is calculated as a function of r, and angle $\theta = 0^{\mathrm{o}}, 45^{\mathrm{o}}$ and ratio $p_{1s} = \sigma_1/\sigma_s = 0.1, 10$, are taken as parameters. The rest of system parameters are given in figures. For the sake of comparing, the ESP values

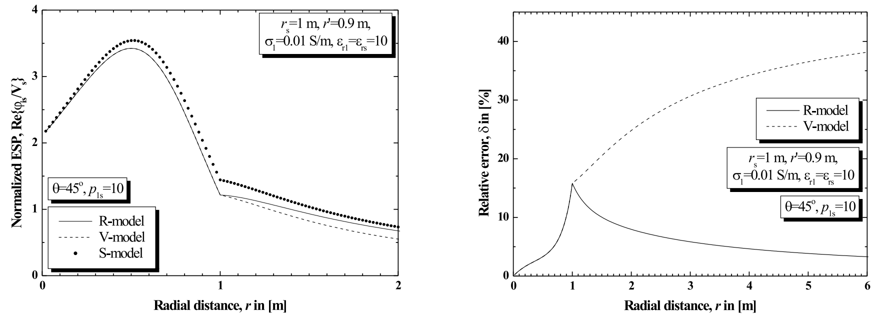

Fig. 5. Continue: The PGE inside the sphere. Normalized ESP, φ_{is}/V_s, $i=1,s$, and corresponding relative errors versus radial distance, for different values of geometry parameters and relation $p_{1s} = \sigma_1/\sigma_s$ taken as parameters.

obtained applying the R–, S– and V–models are presented in the same figures. For each example, relative errors δ in [%], done using the approximate models are also calculated. The results for the case of the PGE placed inside the sphere, $r' = 0.9r_s$, are presented in Fig. 5, and in Fig. 6 the results for the case of the PGE placed outside the sphere, $r' = 1.1r_s$. Based on graphically illustrated results

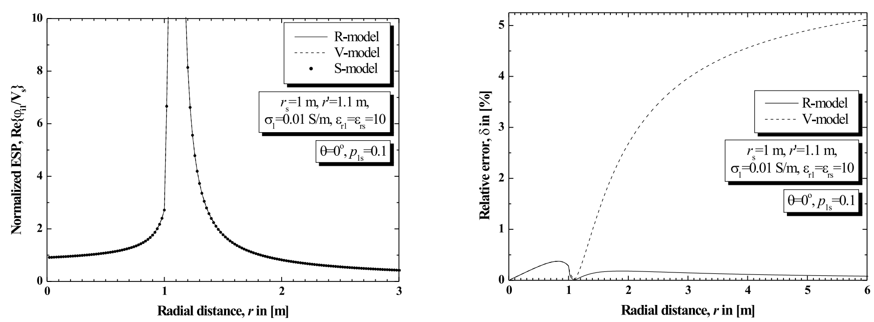

Fig. 6. The PGE outside the sphere. Normalized ESP, φ_{is}/V_s, $i=1,s$, and corresponding relative errors versus radial distance, for different values of geometry parameters and relation $p_{1s} = \sigma_1/\sigma_s$ taken as parameters.

one can conclude that the relative error for the R–model is always $\delta < 1\%$ when the refraction index is $n_{1s} < 1$. For the other case, $n_{1s} > 1$, the maximal error is $\delta < 15\%$, but only for the worst case, i.e. when the field point P is on the sphere surface, $r = r_s$. This conclusion does not apply to the V–model, i.e. the error δ is for certain parameters in a wide range of radial distance r greater than 30% (see Figs. 5 and 6).

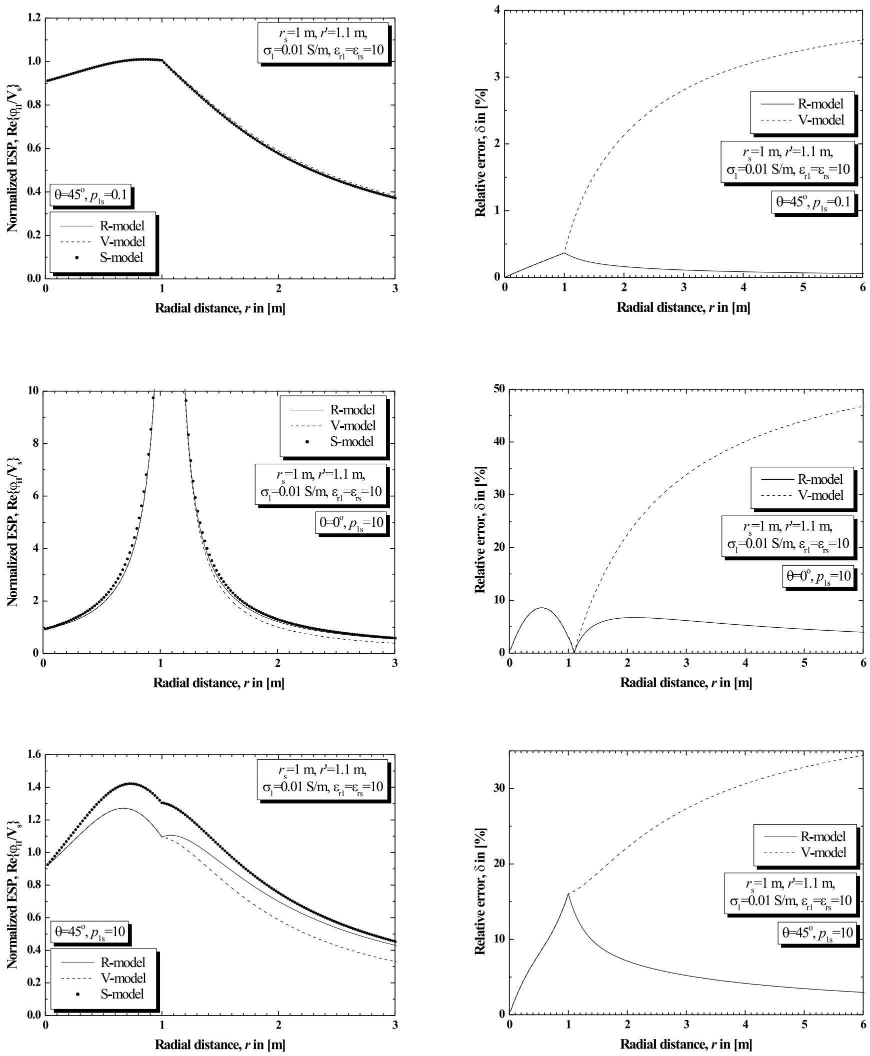

Fig. 6. Continue: The PGE outside the sphere. Normalized ESP, φ_{is}/V_s, $i = 1, s$, and corresponding relative errors versus radial distance, for different values of geometry parameters and relation $p_{1s} = \sigma_1/\sigma_s$ taken as parameters.

Relative error (expressed in %) of the ESP calculation at points on the surface of spherical discontinuity when the PGE is placed outside i.e. inside the sphere, for different ratio $p_{1s} = \sigma_1/\sigma_s$, is presented in Fig. 7.

Large values of relative error correspond to the points where the potential is of small value.

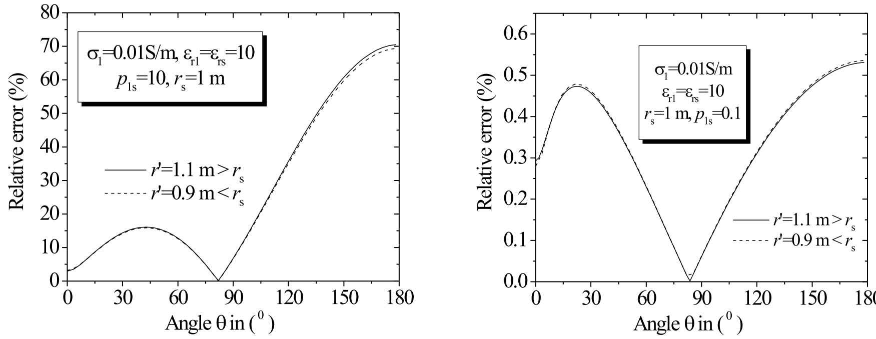

Fig. 7. The relative error of ESP calculation on the sphere surface versus the spherical coordinate θ where $p_{1s} = 10$ and $p_{1s} = 0.1$.

4 Conclusion

A new approximate solution for the Green's function of the ESP that originates from the PGE current in the presence of a spherical ground inhomogenity, when the PGE is fed by a VLF current through a thin isolated ground conductor, was proposed in this paper. The obtained solution is compared to the exact one from [1, pp. 201–205] and also, according to author's opinion, to the approximate solution from [8] and [9, pp. 97-98].

This conclusion (regarding the V-model) is theoretically explained and numerically verified in this paper. Both approximate solutions are in a closed form, which is not the case for the exact one according to [1].

Based on numerical experiments, one can conclude that using the proposed approximate solution, smaller error in the ESP evaluation is then done when the approximate solution from [8] and [9] is used, where the error is estimated in relation to the exact solution from [1]. This is also evident analysing the presented ESP expressions. The error is almost negligible in special cases, e.g. when the refraction coefficient is $n_{1s} < 1$.

Based on everything that was presented, one can conclude that the proposed solution can be successfully used for modelling grounding characteristics in the presence of a spherical and also semi-spherical ground inhomogenity, but also for other problems of this kind.

The proposed approximate R-model can be also applied to derivation of expressions for the Green's function of electrical dipole in the presence of a spherical material inhomogenity and also to other problems of this kind.

Acknowledgment

The authors dedicate the paper to early deceased professor Dragutin M. Veličković (1942-2004).

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