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An Example of Superstable Quadratic Mapping of the Space

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Abstract: It is shown rigorously in this paper that an elementary 3-D quadratic mapping is superstable, i.e. it is superstable for some ranges of its bifurcation parameters. Numerical results that confirm the theory are also given and discussed. These numerical results give a new route to chaos which we call: *the superstable quasi-periodic route to chaos*.

Keywords: 3-D quadratic map, superstability.

1 Introduction

The superstability of a dynamical motion is defined with existence of a minus infinity Lyapunouv exponent, this mean that this motion is attractive. There are several methods for constructing 1-D polynomial mappings with attracting cycles or superstable cycles [1,2] based on Lagrange and Newton interpolations. Superstable phenomena in some 1-D maps embedded in circuits and systems are studied in [3,4], these maps are obtained from the study of nonautonomous piecewise constant circuit and biological models [5–10]. Rich dynamical behaviors can be seen in the presence of superstability [5, 8, 9], especially, the attractivity of the motion that guarantees its stability.

The essential motivation of the present work is to prove rigorously that a family of 3-D quadratic mappings is superstable in the sense that all its behaviors are superstable, i.e. they have a munis infinity Lyapunouv exponent for all bifurcation parameters in a specific region. This property of superstability is probably rare in *n*-D dynamical systems with $n \geq 2$. Also, superstability is a local property in the space of bifurcation parameters, i.e., in general, not all the behaviors of the

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considered system are superstable. For $n = 1$, this property is well investigated analytically and numerically for discrete time systems [1–4].

2 The Superstable 3-D Quadratic Map

The chaotic 3-D quadratic mappings have several potential applications [11–13]. One of the most well known example of this type of mappings is the hyperchaotic generalized Hénon map [14] given by

$$\begin{aligned} x_{k+1} &= a - y_k^2 - bz_k \\ y_{k+1} &= x_k \\ z_{k+1} &= y_k \end{aligned} \tag{1}$$

The map (1) is the simplest 3-D invertible quadratic map. In this paper, we present the simplest 3-D non-invertible quadratic map given by:

$$\begin{aligned} x_{k+1} &= 1 - ay_k^2 \\ y_{k+1} &= x_k + bz_k \\ z_{k+1} &= y_k \end{aligned} \tag{2}$$

First, it is clear that the two maps (1) and (2) are topologically not equivalent because the first is invertible and the second is not for all its bifurcation parameters. Secondly, because we search only bounded superstable states governed by the map (2), we must investigate domains for the parameters $(a, b) \in \mathbb{R}^2$ in which the map (2) has unbounded or bounded orbits. We use the idea of the non-existence of fixed points, because if there is no fixed point, then there is no chaos in the map (2). Indeed, a fixed point (x, y, z) of the map (2) must simultaneously satisfy the following two equalities: $1 - ay^2 = x, x + bz = y, y = z$, hence, if there are no fixed points, then one has that the polynomials $1 - ay^2 - x$ or $x + bz - y$ or $y - z$ are either positive or negative for all $(x, y, z) \in \mathbb{R}^3$.

Assume for example that $1 - ay^2 - x$ is positive, and let $x_0 \geq 0$, then one has for all integer k that $1 - ay_k^2 > x_k$, i.e., $x_{k+1} > x_k > x_{k-1} > .... > x_0 \geq 0$. Let us consider the Euclidean distance $d(x_k, 0) = x_k$ that measures the distance between the first component x_k of the map (2) and the origin 0 on the real line. Then we have $d(x_{k+1}, 0) > d(x_k, 0) > d(x_{k-1}, 0) > .... > d(x_0, 0) \geq 0$. Hence there a real number $\Delta > 0$ such that $d(x_k, 0) = d(x_{k-1}, 0) + \Delta$, which implies that $d(x_k, 0) = d(x_0, 0) + (k+1)\Delta$. Finally, one has $\lim_{k \to +\infty} d(x_k, 0) = +\infty$. If $x_0 < 0$, the same logic applies. However, when there are fixed points, there are domains that contain all bounded orbits, i.e., possibly chaotic attractors. For the map (2) if $a < -(b-1)^2/4$, then all orbits of the map (2) are unbounded. While if $a \geq -(b-1)^2/4$, then the map

(2) has possible bounded orbits. On the other hand, the fixed points of the map (2) are $P_i = \left(1 - ay_i^2, y_i, y_i\right)$, $i = 1,2$, where $y_1 = \left(b - \sqrt{4a + (b-1)^2} - 1\right)/2a$ and $y_2 = \left(b + \sqrt{4a + (b-1)^2} - 1\right)/2a$, and the Jacobian matrix is given by

$$J = \begin{pmatrix} 0 & -2ay & 0 \\ 1 & 0 & b \\ 0 & 1 & 0 \end{pmatrix} \tag{3}$$

and its characteristic polynomial is $\lambda\left(\lambda^2 + 2ay - b\right) = 0$. Hence, the eigenvalues at P_1 are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{\sqrt{4a + (b-1)^2} + 1}$, and at P_2 the eigenvalues are $\omega_1 = 0$, and $\omega_{2,3} = \pm\sqrt{1 - \sqrt{4a - 2b + b^2 + 1}}$, if $a \leq -b(b-2)/4$, and $\omega_{2,3} = \pm i\sqrt{-1 + \sqrt{4a - 2b + b^2 + 1}}$, if $a > -b(b-2)/4$. Some calculations lead to the following results:

1. $|\lambda_1| = 0 < 1, |\lambda_{2,3}| > 1$ for all $a \geq -(b-1)^2/4$, $a \neq 0$, thus P_1 is a saddle fixed point.
2. $|\omega_1| = 0 < 1, |\omega_{2,3}| = \sqrt{1 - \sqrt{4a - 2b + b^2 + 1}} < 1$, if $-(b-1)^2/4 \leq a < -(b+1)(b-3)/4$, thus P_2 is a stable fixed point.
3. $|\omega_1| = 0 < 1, |\omega_{2,3}| = \sqrt{-1 + \sqrt{4a - 2b + b^2 + 1}}$, if $a > -(b+1)(b-3)/4$, thus P_2 is a saddle fixed point.

The goal of this paper is the rigorous proof that the 3-D quadratic map given by equation (2) has minus infinity Lyapunouv exponent for some ranges of $(a,b) \in \mathbb{R}^2$. The method of analysis is the rigorous calculation of the Lyapunouv exponents.

Consider the following 3-D dynamical system:

$$X_{l+1} = g(X_l), X_l \in \mathbb{R}^3, \quad l = 0,1,2,... \tag{4}$$

where the function $g : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field associated with system (4) and $X_l = (x_l, y_l, z_l) \in \mathbb{R}^3$. Let $J(X_l)$ be its Jacobian evaluated at $X_l \in \mathbb{R}^3$, $l = 0,1,2,...$, and define the matrix:

$$T_N(X_0) = J(X_{N-1})\, J(X_{N-2}) ... J(X_1)\, J(X_0) \tag{5}$$

Moreover, let $J_i(X_0, m)$ be the modulus of the i^{th} eigenvalue of the N^{th} matrix $T_N(X_0)$, where $i = 1,2,3$ and $N = 0,1,2,...$

Now, the Lyapunouv exponents for a 3-D discrete-time system are defined by:

$$l_i(X_0) = \ln\left(\lim_{N \to +\infty} J_i(X_0, N)^{\frac{1}{N}}\right), \quad i = 1,2,3 \tag{6}$$

Based on this definition, we will give a rigorous proof that the family of 3-D quadratic maps given by equation (2) is superstable for some ranges of its bifurcation parameters and all initial conditions, i.e. we will prove that the matrix $T_N(X_0)$ has a zero eigenvalue for all $(x_0, y_0, z_0) \in \mathbb{R}^3$. The determinant of Jacobian matrix (3) of the map (2) is $\det J(x,y,z) = 0$, for all $(x,y,z) \in \mathbb{R}^3$, then the matrix $J(x,y,z)$ is singular for all $(x,y,z) \in \mathbb{R}^3$, hence,

$$\det T_N(X_0) = \prod_{i=N-1}^{i=0} \det J(x_i, y_i, z_i) = 0,$$

then and because the determinant of a product is the product the determinants of all matrices, one can deduce that the matrix $T_N(X_0)$ has at least one zero eigenvalue, which means that the map (2) has a munis infinity Lyapunouv exponent, by taking the logarithm of the zero eigenvalue. Finally, the Lyapunouv spectrum of the map (2) is given by $l_{1,2}(X_0)$ are finite numbers and $l_2(X_0) = -\infty$. Note that the map (2) does not display hyperchaotic behaviors due to its smoothness and dissipativity contrary to the situation of the hyperchaotic generalized Hénon map [14].

3 Numerical Computations

In this section, we test numerically the above analytical results, although the different superstable dynamical behaviors of the map (2) are shown in Fig.1 where

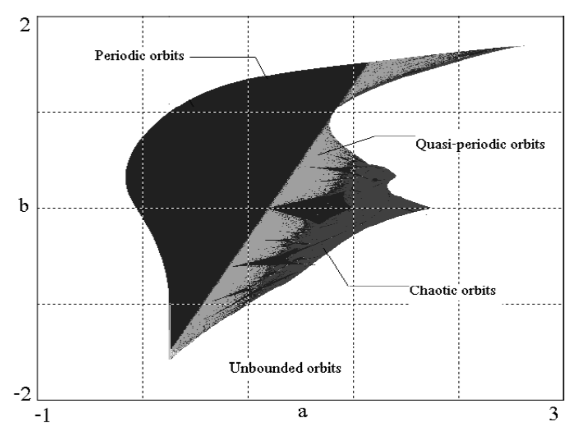

Fig. 1. Regions of dynamical behaviors in the *ab*-plane for the map (2).

regions of unbounded (white), periodic (blue), and chaotic (red) solutions in the ab-plane for the map (2) are obtained using 10^6 iterations for each point. Also, some superstable orbits of the map (2) are shown in Fig.2.

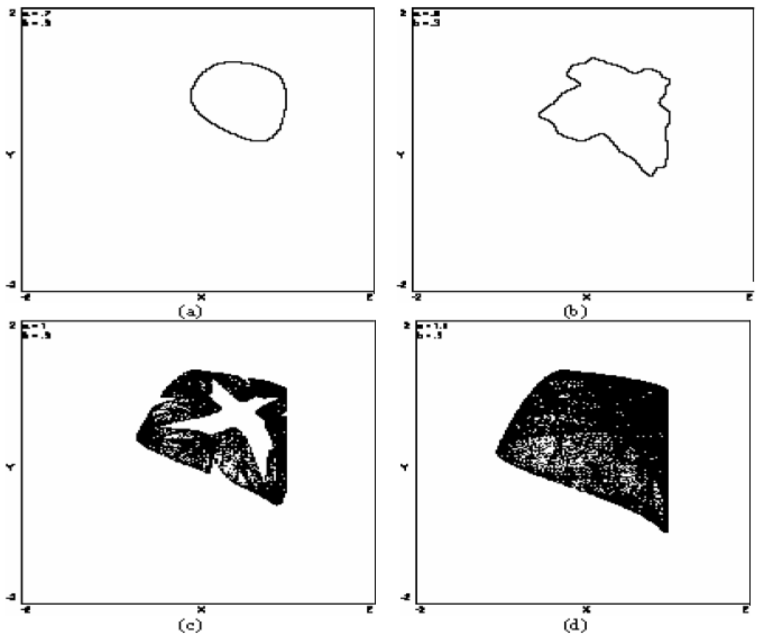

Fig. 2. Superstable attractors obtained from the map (2) with $b = 0.3$, and (a) Periodic orbit for $a = 0.7$. (b) Quasi-periodic orbit for $a = 0.9$. (c) Chaotic orbit for $a = 1.0$ (d) Chaotic orbit for $a = 1.2$.

As shown in Fig.1 the chaotic behaviors of the map (2) results from a quasi-periodic route to chaos, and because all the states of the map (2) are superstable, then we call this route: *the superstable quasi-periodic route to chaos*.

4 Conclusion

It is shown through an elementary example that the superstability phenomenon is possible in 3-D quadratic mappings. Numerical results confirm the theory and give a new route to chaos: *the superstable quasi-periodic route to chaos*.

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