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A Systematic Approach to the Concept of Surface Impedance Boundary Conditions

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Abstract: This paper discusses the general issues, derivation, implementation and applications of Surface Impedance Boundary Conditions (SIBCs) in the time- and frequency-domains. A comprehensive approach based on perturbation methods leads to SIBCs of desired order of approximation as well as systematic implementation within existing formulations for linear and nonlinear media. The approach described here also allows evaluation of errors and appropriateness of SIBCs for specific applications. A suite of SIBCs is proposed, suitable for use in a wide range of practical applications and formulations including FEM, FDTD, FIT and BEM. A general toolbox that can be used for derivation of SIBCs for the users specific formulation and application has been developed and is described here as well.

Keywords: Surface impedance boundary conditions, perturbation methods, skin effect, electromagnetic field formulations.

1 Introduction

THE use of SIBCs in low-penetration problems (either low or high frequencies) can benefit a broad spectrum of applications by reducing the size of the problem to solve and, in some cases, rendering the problem solvable. Although the method has been applied to a large number of problems ranging from electric machines to microwaves, the derivation of an appropriate SIBC was always more or less an ad-hoc process. In addition, most researchers have limited themselves to first order (Leontovich) SIBCs. By doing so, SIBCs could only take into account field variations perpendicular to conducting surfaces, and, with some exceptions, flat surfaces. Yet, a more general approach, one that eliminates these limitations has

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been available for some 70 years in the form of the Rytov perturbation method [1]. The method is based on power series expansion and, as such, allows implementation of arbitrary order SIBCs. By using higher order terms, curved surfaces as well as variation of fields along the surfaces of conductors can be incorporated in computation, increasing accuracy and expanding the range of applicability. Significantly, the method also allows calculation of fields and related quantities within the skin layer itself. The method has additional advantages. First, it incorporates other existing SIBCs such as the Leonntovich method (which becomes the first order term in Rytovs expansion). Second, because of the series form, implementation of higher order approximations require virtually no modifications to existing software only addition of a higher order module. Rytovs method however is limited to time-harmonic applications.

The work described here is based on the Rytov expansion method but goes beyond it in a number of significant ways [2]. First, we have extended the method into the time domain whereby the limitations of time-harmonic representation have been overcome. Second, treatment of problems with nonlinear and nonhomogeneous material properties follows from the general approach and, within the constraints imposed by material behavior, follows the same general approach. Third, the inclusion of the toolbox approach allows the user to develop his/her own boundary conditions to suit formulations, numerical methods and implementation preferences. The SIBCs described here are based on simple characteristics of the problem the characteristic dimension of the problem (i.e. minimum thickness, smallest radius of curvature, smallest distance between members of the geometry, etc.) and either the skin depth (time harmonic problems) or pulse width (time dependent problems).

The development of SIBCs, beyond the classical Leontovich method is usually an arduous task, requiring considerable skill. In addition, the value of the SIBC to the problem at hand must be ascertained beforehand, that is, one has to be reasonably certain that SIBCs can in fact be used to some advantage in the first place. Then one must be able to estimate the errors involved in the use of SIBCs of various order of approximation before implementation commences. To this end, the present suite of SIBCs not only analyzes the errors but laso includes a simple procedure that allows the user to decide a-priori the order of the SIBC and, indeed if an SIBC can be used based on the characteristic properties of the problem and error tolerance.

2 The Concept of Surface Impedance Boundary Condition

The concept of surface impedance boundary conditions is simple. Assuming a plane wave impinging on a conducting surface and calculation of the electric and magnetic field intensities inside the conductor leads to the well known relation for the intrinsic impedance in the conductor

$$Z_c = \frac{1+j}{\sigma\delta}\ [\Omega] \tag{1}$$

where σ is the skin depth. Since the electromagnetic field is continuous across the conductors surface the intrinsic impedance of the wave remains the same at the interface. Therefore, the relation in (1) may be used as a surface impedance boundary condition since it contains all the necessary information about the field distribution inside the conductor. Eq. (1) however comes with a set of assumptions under which it is valid. First, we assumed a high loss tangent ($\sigma \gg \omega\varepsilon$). More importantly, the wave was assumed to propagate perpendicular to the interface and because it is a plane wave the fields cannot vary on the surface. The sources of the wave are at infinity and, of course, materials are simple (linear, homogeneous, isotropic). The conductor itself is infinitely thick (half space). Nevertheless this little exercise shows how simple an SIBC can be. In addition, some of the requirements can be relaxed relatively easily. For example, the conductor does not have to be thick the only requirement is that reflections within the conductor must die out before they reach the surface. For this to happen, the conductor must only be 2-3δ thick or that $\delta \ll D$ where D is the characteristic size of the geometry. Eq. (1) is the SIBC due to Leontovich [3], although it is more often written in terms of complex permeability and complex permittivity and often in terms of electric and magnetic surface current densities:

$$\vec{K}_m = Z_c\,\vec{n}\times\vec{K}_e, \qquad Z_c = \sqrt{\frac{\mu_c}{\varepsilon_c}} \tag{2}$$

3 The Proposed Suite of SIBCS

The SIBCs proposed here were developed starting from Rytovs method but they go beyond in a number of significant ways [2].

1. They apply to time-harmonic as well as time-dependent problems.
2. Apply to linear an nonlinear media
3. May be incorporated within any formulation and any numerical technique.
   (a) We have developed higher order SIBCs (up to 4th order)
   (b) Incorporated within the Boundary Integral Equations method (BIE), Finite Difference Time Domain method (FDTD), the Finite Integration Technique (FIT) and the Finite Element Method (FEM). Other variations are equally possible

One of the main advantages of our method is in that the error in the SIBC can be estimated a-priori based on the characteristic properties of the problem. This allows the user to select the appropriate order SIBC for a given problem and, more importantly, to decide if an SIBC can be used and the consequences of its use. A simple methodology has been developed to facilitate this evaluation prior to implementation.

4 Development and Application of SIBCS

To demonstrate the development of SIBCs, we first discuss the basics starting with no more than the definition of skin depth. Then, we introduce the perturbation approach and discuss the process of identifying the characteristic properties of the problem, incorporation of these into a small parameter and expansion in this parameter. The various formulations are discussed, with emphasis on the unifying concepts rather than on details of the formulations. For the purpose of presenting results we will concentrate on implementation in the BIE formulation in terms of vector potentials but others will be discussed as well.

Because of its practical importance, the calculation of errors and the methodology used to decide on a particular order of approximation will be discussed with examples. Finally, we will give specific examples of practical calculations in the time- and frequency domain to demonstrate the wide range of applicability of the suite of SIBCs described above.

Following perturbation theory, we first introduce the scale factors for the basic variables in the governing equations. Selection of the scale factors is based on knowledge of the characteristic variation of such input data as total current and dimensions of the conductor. Then the equations are re-written in terms of non-dimensional variables so that each is a function of the corresponding dimensional variable and its scale factor. As a result, combination(s) of the scale factors will appear in the governing equations as parameters.

In conducting media, the distribution of various electromagnetic quantities in the conducting domain can be described by the equation of diffusion:

$$\nabla \times (\nabla \times \vec{f}) + \sigma\mu \frac{\partial \vec{f}}{\partial t} = 0 \tag{3}$$

In lossy dielectrics the governing equation is the telegraphers equation but, since the process of defining the SIBCs is the same, we will only discuss here Eq. (3) [2]. Here the vector function $\vec{f}$ may denote various electromagnetic fields and potentials. The purpose of the SIBC toolbox is transformation of this equation with the use of perturbation techniques and derivation of a set of approximate relationships between components and derivatives of $\vec{f}$ at the interface. Those results can be applied not only to electromagnetics, but to any other area where functions under consideration are governed by the diffusion equation. Local to the surface we define an inward normal coordinate η and tangential orthogonal coordinates ξ_1 and ξ_2. Non-dimensional variables $\tilde{\xi}_1$, $\tilde{\xi}_2$ and $\tilde{\eta}$ are introduced that have the variation ranges of the same order of magnitude and are related to ξ and η, respectively, as follows

$$\tilde{\xi}_i = \frac{\xi_i}{D}, \qquad \tilde{\eta} = \frac{\eta}{\delta} \tag{4}$$

Based on the condition in (1) we introduce a small parameter proportional to the ratio of the penetration depth and the characteristic size of the conductors surface:

$$\tilde{p} = \frac{\delta}{D} = \sqrt{\frac{\tau}{\sigma\mu D^2}} \ll 1 \tag{5}$$

The local radii of curvature d_i are directly related to the variation of the function $\vec{f}$ in the directions tangential to the surface of the conductor. This leads to the following representation

$$\tilde{d}_i = \frac{d_i}{D} \tag{6}$$

We then first define the required components of the diffusion equation in the local coordinates, that is, define f, its tangential and normal derivatives as well as any other necessary function (such as the curl) at the interface in the time or frequency domain in non-dimensional variable. Of course, one can always transform back to dimensional variables using the scale factors. These operations lead to the following general relations, in the frequency and time domain, which are applicable for any function f at the interface (denoted by b).

$$\begin{aligned} f_\eta^b =& \tilde{p}\frac{1-j}{2}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\Big[\dot{f}_{\xi_i}^b - \tilde{p}\frac{1-j}{2}\dot{f}_{\xi_i}^b\frac{\tilde{d}_i-\tilde{d}_{3-i}}{2\tilde{d}_i\tilde{d}_{3-i}} \\ &+\tilde{p}^2\frac{j}{2}\Big(\dot{f}_{\xi_i}^b\frac{\tilde{d}_i^2+2\tilde{d}_i\tilde{d}_{3-i}-3\tilde{d}_{3-i}^2}{4\tilde{d}_i^2\tilde{d}_{3-i}^2} - \frac{1}{2}\frac{\partial^2\dot{f}_{\xi_i}^b}{\partial\tilde{\xi}_i^2} + \frac{1}{2}\frac{\partial^2\dot{f}_{\xi_i}^b}{\partial\tilde{\xi}_{3-i}^2} - \frac{\partial^2\dot{f}_{\xi_{3-i}}^b}{\partial\tilde{\xi}_i\partial\tilde{\xi}_{3-i}}\Big)\Big] \end{aligned} \tag{7}$$

$$\begin{aligned} f_\eta^b =& \tilde{p}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\Big[f_{\xi_i}^b * \tilde{T}_2^b - \tilde{p}\frac{\tilde{d}_i-\tilde{d}_{3-i}}{2\tilde{d}_i\tilde{d}_{3-i}}f_{\xi_i}^b * \tilde{T}_3^b \\ &-\tilde{p}^2\Big(f_{\xi_i}^b\frac{\tilde{d}_i^2+2\tilde{d}_i\tilde{d}_{3-i}-3\tilde{d}_{3-i}^2}{4\tilde{d}_i^2\tilde{d}_{3-i}^2} - \frac{1}{2}\frac{\partial^2 f_{\xi_i}^b}{\partial\tilde{\xi}_i^2} + \frac{1}{2}\frac{\partial^2 f_{\xi_i}^b}{\partial\tilde{\xi}_{3-i}^2} - \frac{\partial^2 f_{\xi_{3-i}}^b}{\partial\tilde{\xi}_i\partial\tilde{\xi}_{3-i}}\Big) * \tilde{T}_4^b\Big] \end{aligned} \tag{8}$$

$$\tilde{T}_2^b = (\pi\tilde{t})^{-\frac{1}{2}} \qquad \tilde{T}_3^b = \tilde{U}(\tilde{t}) \qquad \tilde{T}_4^b = 2\tilde{t}^{\frac{1}{2}}\pi^{-\frac{1}{2}} \tag{9}$$

where $U(t)$ is the Heaviside function, $\tilde{t} = t/\tau$, and $*$ denotes a convolution product.

The relations in (7) and (8) are third order (Rytov) relations as can be seen from the order of $\tilde{p}$. If the term preceded by $\tilde{p}^2$ in the square brackets is removed, we obtain the Mitzner (second order) SIBC and if we remove the term preceded by $\tilde{p}$, the Leontovich relation is obtained. Setting $\tilde{p} = 0$ (that is, $\delta = 0$) reduces the relation to the perfect electric conductor (PEC) condition.

To see how this toolbox may be used to advantage, consider the formulation in terms of the magnetic scalar potential ϕ in the absence of sources. Under these conditions ϕ can be introduced as follows:

$$\vec{H} = -\nabla\phi \tag{10}$$

In the free space outside the conductor we have

$$\nabla^2\phi = 0 \tag{11}$$

The boundary conditions at the dielectric/conductor interface follow directly from (10):

$$\vec{n}\times\nabla\phi\big|_{diel} = \vec{n}\times\vec{H}\big|_{cond} \qquad \frac{\partial\phi}{\partial\vec{n}}\Big|_{diel} = \vec{n}\cdot\vec{H}\Big|_{cond} \tag{12}$$

The use of the conditions in (12) requires that the problem inside the conductor in terms of the magnetic field be considered together with (11). To eliminate the conducting region from the numerical procedure we use the surface impedance "toolbox" discussed above and write (replacing f with H) [2]

$$\begin{aligned} H_\eta^b =& \tilde{p}\frac{1-j}{2}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\Big[\dot{H}_{\xi_i}^b - \tilde{p}\frac{1-j}{2}\dot{H}_{\xi_i}^b\frac{\tilde{d}_i-\tilde{d}_{3-i}}{2\tilde{d}_i\tilde{d}_{3-i}} \\ &+\tilde{p}^2\frac{j}{2}\Big(\dot{H}_{\xi_i}^b\frac{\tilde{d}_i^2+2\tilde{d}_1\tilde{d}_{3-i}-3\tilde{d}_{3-i}^2}{8\tilde{d}_i^2\tilde{d}_{3-i}^2} - \frac{1}{2}\frac{\partial^2\dot{H}_{\xi_i}^b}{\partial\tilde{\xi}_i^2} + \frac{1}{2}\frac{\partial^2\dot{H}_{\xi_i}^b}{\partial\tilde{\xi}_{3-i}^2} - \frac{\partial^2\dot{H}_{\xi_{3-i}}^b}{\partial\tilde{\xi}_i\partial\tilde{\xi}_{3-i}}\Big)\Big] \end{aligned} \tag{13}$$

$$\begin{aligned} H_\eta^b =& \tilde{p}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\Big[H_{\xi_i}^b*\tilde{T}_2^b - \tilde{p}\frac{\tilde{d}_i-\tilde{d}_{3-i}}{2\tilde{d}_i\tilde{d}_{3-i}}H_{\xi_i}^b*\tilde{T}_3^b - \frac{\tilde{d}_i-\tilde{d}_{3-i}}{2\tilde{d}_i\tilde{d}_{3-i}} \\ &-\tilde{p}^2\Big(H_{\xi_i}^b\frac{\tilde{d}_i^2+2\tilde{d}_i\tilde{d}_{3-i}-3\tilde{d}_{3-i}^2}{8\tilde{d}_i^2\tilde{d}_{3-i}^2} - \frac{1}{2}\frac{\partial^2H_{\xi_i}^b}{\partial\tilde{\xi}_i^2} + \frac{1}{2}\frac{\partial^2H_{\xi_i}^b}{\partial\tilde{\xi}_{3-i}^2} - \frac{\partial^2H_{\xi_{3-i}}^b}{\partial\tilde{\xi}_i\partial\tilde{\xi}_{3-i}}\Big)*\tilde{T}_4^b\Big] \end{aligned} \tag{14}$$

Clearly this is not only simple but very useful as one does not have to derive an ad-hoc SIBC for each formulation. The relations in (13) and (14) may be transformed back to dimensional variables to obtain:

$$H_\eta^b = \frac{1}{\sqrt{j\omega\sigma\mu}} \sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \Big[\dot{H}_{\xi_i}^b - \frac{1}{\sqrt{j\omega\sigma\mu}} \frac{d_i - d_{3-i}}{2d_i d_{3-i}} \dot{H}_{\xi_i}^b - \frac{1}{j\omega\sigma\mu} \Big(\dot{H}_{\xi_i}^b \frac{d_i^2 + 2d_i d_{3-i} - 3d_{3-i}^2}{8d_i^2 d_{3-i}^2} - \frac{1}{2} \frac{\partial^2 \dot{H}_{\xi_i}^b}{\partial \xi_i^2} + \frac{1}{2} \frac{\partial^2 \dot{H}_{\xi_i}^b}{\partial \xi_{3-i}^2} - \frac{\partial^2 \dot{H}_{\xi_{3-i}}^b}{\partial \xi_i \partial \xi_{3-i}} \Big) \Big] \tag{15}$$

$$H_\eta^b = \frac{1}{\sqrt{\sigma\mu}} \sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \Big[H_{\xi_i}^b * \tilde{T}_2^b - \frac{1}{\sqrt{\sigma\mu}} \frac{\tilde{d}_i - \tilde{d}_{3-i}}{2\tilde{d}_i \tilde{d}_{3-i}} H_{\xi_i}^b * \tilde{T}_3^b - \frac{1}{\sigma\mu} \Big(H_{\xi_i}^b \frac{d_i^2 + 2d_1 \tilde{d}_{3-i} - 3d_{3-i}^2}{8d_i^2 d_{3-i}^2} - \frac{1}{2} \frac{\partial^2 H_{\xi_i}^b}{\partial \xi_i^2} + \frac{1}{2} \frac{\partial^2 H_{\xi_i}^b}{\partial \xi_{3-i}^2} - \frac{\partial^2 H_{\xi_{3-i}}^b}{\partial \tilde{\xi}_i \partial \xi_{3-i}} \Big) * \tilde{T}_4^b \Big] \tag{16}$$

where the time functions T are those in (9) after return to dimensional variables. However, it is a simple matter of use of Eq. (10) to obtain the SIBC in trems of the scalar potential itself. From (10) it directly follows that

$$H_{\xi_i} = \frac{\partial \phi}{\partial \xi_i}, \quad H_\eta = \frac{\partial \phi}{\partial \eta}. \tag{17}$$

Substituting (17) into (15)-(16) yields the boundary relations between the normal and tangential derivatives of the scalar potential in the following form (in dimensional variables):

$$\left(\frac{\partial \phi}{\partial \vec{n}}\right)^b = \frac{1}{\sqrt{j\omega\sigma\mu}} \sum_{i=1}^{2} \frac{\partial^2}{\partial \xi_i^2} \Big[\dot{\phi}^b - \frac{1}{\sqrt{j\omega\sigma\mu}} \frac{d_i - d_{3-i}}{2d_i d_{3-i}} \dot{\phi}^b - \frac{1}{j\omega\sigma\mu} \Big(\dot{\phi}^b \frac{d_i^2 + 2d_i d_{3-i} - 3d_{3-i}^2}{8d_i^2 d_{3-i}^2} - \frac{1}{2} \frac{\partial^2 \dot{\phi}^b}{\partial xi_i^2} - \frac{1}{2} \frac{\partial^2 \dot{\phi}^b}{\partial \xi_{3-i}^2} \Big) \Big] \tag{18}$$

$$\left(\frac{\partial \phi}{\partial \vec{n}}\right)^b = \frac{1}{\sqrt{\sigma\mu}} \sum_{i=1}^{2} \frac{\partial^2}{\partial \xi_i^2} \Big[\phi^b * T_2^b - \frac{1}{\sqrt{\sigma\mu}} \frac{d_i - d_{3-i}}{2d_i d_{3-i}} \phi^b * T_3^b - \frac{1}{\sigma\mu} \Big(\phi^b \frac{d_i^2 + 2d_i d_{3-i} - 3d_{3-i}^2}{8d_i^2 d_{3-i}^2} - \frac{1}{2} \frac{\partial^2 \phi^b}{\partial \xi_i^2} - \frac{1}{2} \frac{\partial^2 \phi^b}{\partial \xi_{3-i}^2} \Big) * T_4^b \Big] \tag{19}$$

Since the tangential derivative of a function can be approximated using the function calculated at nodes located along the conductors surface, the expressions in (18)-(19) are SIBCs of various orders of approximation relating the scalar potential and its normal derivative at the surface of the conductor. Again, this is a third order SIBC and can be reduced to lower orders by removing appropriate terms. A similar process is used to obtain SIBCs for the E-H, A-ψ and other formulations [2]. For example, the E-H formulation results in the following (again using the toolbox approach):

$$\tilde{E}^b_{\xi_{3-k}} = (-1)^k (1+j)\tilde{p}\Big[\dot{\tilde{H}}^b_{\xi_k} - \tilde{p}\frac{1-j}{2}\dot{\tilde{H}}^b_{\xi_k}\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}} \tag{20}$$
$$+\tilde{p}^2\frac{j}{2}\Big(\dot{\tilde{H}}^b_{\xi_k}\frac{\tilde{d}_k^2 + 2\tilde{d}_k\tilde{d}_{3-k} - 3\tilde{d}_{3-k}^2}{8\tilde{d}_k^2\tilde{d}_{3-k}^2} - \frac{1}{2}\frac{\partial^2\dot{\tilde{H}}^b_{\xi_k}}{\partial\tilde{\xi}_k^2} + \frac{1}{2}\frac{\partial^2\dot{\tilde{H}}^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}^2} - \frac{\partial^2\dot{\tilde{H}}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_{3-k}}\Big)\Big]$$
$$+O(\tilde{p}^4)$$

$$\tilde{E}^b_{\xi_{3-k}} = (-1)^k \tilde{p}\Big[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}(\tilde{H}^b_{\xi_k} * \tilde{T}^b_2) - \tilde{p}\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}}\tilde{H}^b_{\xi_k} * \tilde{T}^b_2 \tag{21}$$
$$-\tilde{p}^2\Big(\tilde{H}^b_{\xi_k}\frac{\tilde{d}_k^2 + 2\tilde{d}_k\tilde{d}_{3-k} - 3\tilde{d}_{3-k}^2}{8\tilde{d}_k^2\tilde{d}_{3-k}^2} - \frac{1}{2}\frac{\partial^2\tilde{H}^b_{\xi_k}}{\partial\tilde{\xi}_k^2} + \frac{1}{2}\frac{\partial^2\tilde{H}^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}^2} - \frac{\partial^2\tilde{H}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_{3-k}}\Big) * \tilde{T}^b_2\Big]$$
$$+O(\tilde{p}^4)$$

In these $k = 1, 2$. These SIBCs and have been implemented in a variety of numerical methods and many applications following essentially similar steps [2, 4–10]

5 A Decision Methodology for Selection of SIBCS

The error in Rytovs approximation (20) or (21) is 4th order in skin depth or, as we have indicated previously, in the small parameter $\tilde{p}$. Lower order Mitzner, Leontovich and PEC conditions errors are 3rd, 2nd and 1st order in $\tilde{p}$ respectively. A critical question is which approximation to use for a specific application and, even more importantly, if an SIBC can in fact be used and still obtain a given level of accuracy. The required order is not always clear *a priori*, especially in transient applications. To answer this question a-priori, without the need to first solve the problem and then analyze the errors, we resort to the original scales of the problem: the characteristic dimension D and the characteristic time ???as discussed above. Based on these values we develop a methodology for selection of SIBCs [2, 11]. Using D and τ, we define the characteristic skin depth δ and characteristic dimension λ of the field variation along the bodys surface as:

$$\delta = \sqrt{\frac{\tau}{\mu\sigma}}, \qquad \lambda = c\tau \tag{22}$$

where c is the velocity of light. If the material properties are non-linear, the nonlinear characteristic permeability μ and/or conductivity σ must be used in (22). This may be the case when a nonlinear BH curve is given or when conductivity is temperature dependent. The conditions of applicability of the surface impedance concept can now be written in terms of D, $\delta$and λ:

$$\lambda \ll D \qquad \text{or} \qquad \tilde{p} = \frac{\delta}{D} \ll 1 \tag{23}$$

$$\lambda \gg D \qquad \text{or} \qquad \tilde{q} = \frac{D}{\lambda} = \frac{D}{c\tau} \ll 1 \tag{24}$$

We now take and as the basic parameters of the problem. The conditions (23) and (25) hold for all SIBCs. The approximation in (20) or (21) allows us to replace the condition in (23) by:

$$\tilde{p}^4 \ll 1 \tag{25}$$

Now we can evaluate the ranges of the characteristic values for which the SIBCs (20)-(21) are best applicable. The conditions (23)-(24) of applicability of the SIBCs involve the two characteristic scales (D and τ) and the two parameters of the problem ($\tilde{p}$ and$\tilde{q}$), therefore, the scales are `uniquely` expressed by these parameters.

From (9.8) it follows that the approximation errors in the PEC-limit, the Leontovich SIBCs, the Mitzner SIBCs and the Rytov SIBCs are $\tilde{p}$, $\tilde{p}^2$, $\tilde{p}^3$ and $\tilde{p}^4$, respectively. Based on this observation, we can define approximate ranges of the parameter $\tilde{p}$, for which the SIBCs of these classes can be best applied:

(a) For the PEC approximation: $\tilde{p} < 0.06$

(b) For SIBCs in the Leontovich approximation: $\tilde{p} = 0.06 \div 0.25$ ($\tilde{p}^2 \simeq 0.003 \div 0.06$)

(c) For SIBCs in the Mitzner approximation: $\tilde{p} = 0.25 \div 0.4$ ($\tilde{p}^3 \simeq 0.02 \div 0.06$)

(d) For SIBCs in the Rytov approximation: $\tilde{p} = 0.4 \div 0.5$ ($\tilde{p}^4 \simeq 0.03 \div 0.06$)

(e) The range of $\tilde{q}$ the parameter can be defined as: $\tilde{q} < 0.05$

With these definitions in the approximation error due to using the specific SIBCs will not exceed 6%.

These relations may be used as follows: From (22), (23) and (25) we can write:

$$D = \frac{\sqrt{\tau}}{\tilde{p}\sqrt{\sigma\mu}} \qquad \text{and} \qquad D = c\tau\tilde{q} \tag{26}$$

Now, by substituting the extreme values of the parameters $\tilde{p}$ and $\tilde{q}$ from (a) through (e) into (26), the desired ranges of the scales D and τ can be obtained for a given problem. The relations in (26) are written in terms of specific material properties. To obtain more universal relations we introduce the following non-dimensional variables

$$\tilde{D} = \sigma\mu c D \qquad \tilde{\tau} = \sigma\mu c^2 \tau \tag{27}$$

With the variables (27) the functions in (26) can be written in the following form:

$$\tilde{D} = \tilde{q}\tilde{\tau} \qquad \tilde{D} = \frac{\sqrt{\tilde{\tau}}}{\tilde{p}} \tag{28}$$

These are more convenient than the relations in (26) because they are non-dimensional. Now we can plot these for the extreme values of ∂ and $\tilde{q}$ given in (a)- (e). A representation of these values is shown in Fig. 1.

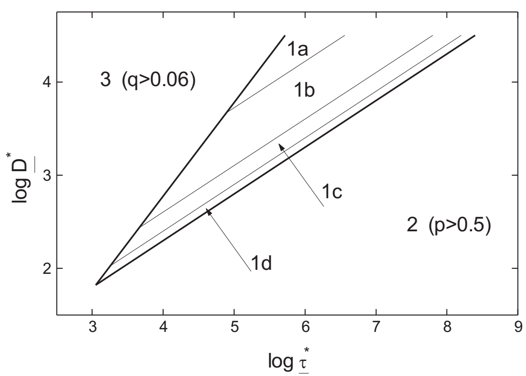

Fig. 1. The $\tilde{D} - \tilde{\tau}$ plane.

Region (1a) in Fig. 1 shows the application area of the SIBCs in the PEC-limit. Region (1b) is the application range in the Leontovich approximation. Similarly, the Mitzner approximation and the Rytov approximation ranges are shown as (1c) and (1d) respectively. If the point in the $\tilde{D} - \tilde{\tau}$ plane lies in regions 2 and 3, the surface impedance concept cannot be applied because the conditions in (a)-(e) break down.

Now we can summarize the methodology for selection of an appropriate SIBC for a given physical problem as follows:

1. Identify the characteristic values D and τ and the material properties σ and μ;
2. Calculate or estimate the non-dimensional values $\tilde{D}$ and $\tilde{\tau}$ using (27);
3. Find the appropriate point in the $\tilde{D} - \tilde{\tau}$ plane;
4. If this point lies in the regions 1a-1d, choose the corresponding SIBCs from (20) or (21) or the PEC boundary conditions by selecting those terms indicated in (a)-(e).

To demonstrate the application of the foregoing methodology, we consider here a problem in the time domain made of a pair of identical parallel copper conductors with circular cross section in which equal and oppositely directed pulses of current of magnitude 1 A flow from an external source as shown in Fig. 2. The radius of each conductor and the distance between the conductors were taken equal to 0.1 m (characteristic value $D = 0.1$m). Under these conditions the current density has only one component directed along the conductors.

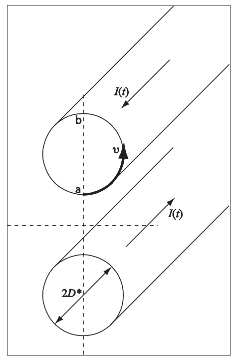

Fig. 2. Physical geometry of the problem.

The SIBCs in (21) (of various orders) were coupled with the surface integral equation using the formulation and solved using the boundary element method [11].

The distributions of the surface current density over one half of the cross section of one conductor were calculated for a current pulse of width $\tau = 0.1$s ($\tilde{p} = 0.37$ and $\tilde{q} = 3.3 \times 10^{-9}$). Fig. 3 shows that use of the Leontovich SIBC leads to unacceptable computational errors (about 18%). On the other hand, the difference between the curves obtained in the Mitzner and Rytov approximations does not exceed 3%. Therefore, in this problem it is necessary to use the Mitzner SIBC as the methodology predicts.

If the current pulses are shorter, lower order approximations are sufficient. For example, for $\tau = 10^{-3}$s, a PEC condition would suffice (4% error) whereas for a pulse width $\tau = 10^{-2}$s the Leontovich approximation is optimal (2% error).

6 Applications

The previous sections discussed the issues and some of the theory of SIBCs. But the importance of these methods is in the applications. These applications are divided into two separate parts. The first deals with the classical issue of propagation into conducting media. These are essentially low frequency applications. One of the features of the methods discussed here is that they extend the classical SIBCs into lower frequencies by allowing lower errors through use of higher order SIBCs [2, 12–15]. The second part deals with high frequency applications in

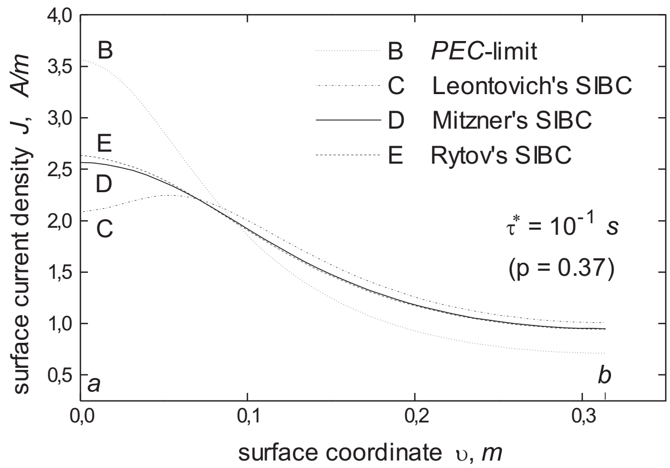

Fig. 3. Distribution of surface current density over the conductors surface ($\tau = 0.1$ s).

lossless dielectrics, again, all being dictated by the skin depth [2]. Additional applications, which may span the whole range of applicability include ninlinear and nonhomogeneous media [2] as well as treatment of edges and corners [16–18]. In this section we describe some of the applications.

To illustrate some of these application we first consider the problem in Fig. 1 with a current pulse. The is cast steel (with the *BH*-curve given in [19]]), $I = 100$A and $\tau = 0.1$s and a nonlinear SIBc in the time-domain is now needed. The calculations are performed using four alternatives and the numerical results are shown in Fig. 4: the PEC boundary conditions (curve *A*), the proposed non-linear SIBC (curve *B*) and two linear SIBCs (curves *C* and *D*). Curve *D* requires a preliminary non-linear run to determine the maximum field reluctivity. This max-field method leads to a 40% error in the calculated power losses. The accuracy of the power losses computed using a linear SIBC can be improved if the characteristic reluctivity ν^* is used (curve *C*) instead of the max-field reluctivity ν. However, the difference in the surface current density between curves *B* and *C* is larger than that between curves *B* and *D*. Details of the formulation and the SIBC can be found in [2].

The next example is a multi conductor transmission line made of three conductors and a common, outer “ground” conductor as shown in Fig. 5. The purpose here is the calculation of . The results obtained using the SIBC formulation [14] are tabulated in Tables 1 and 2. Table 1 shows the self inductance and self resistance of

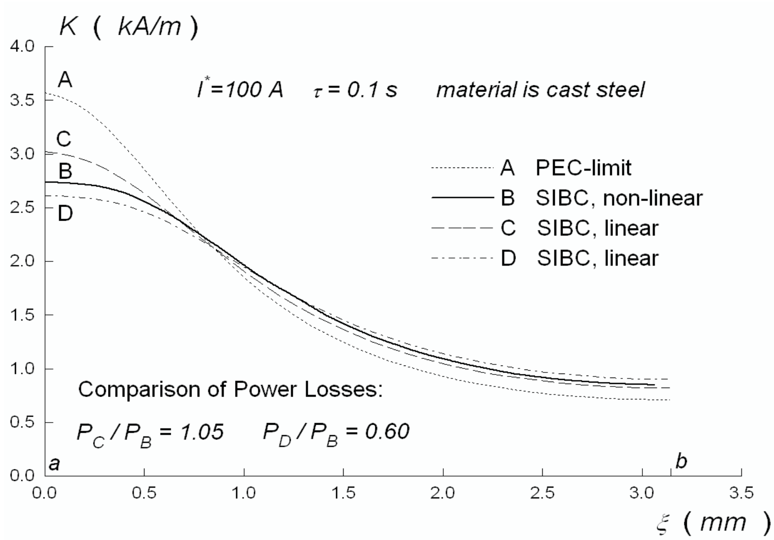

Fig. 4. Distributions of the surface current density along half the contour around the cross section of one conductor shown in Fig. 1.

conductor 1. Table 2 shows the mutual inductances and resistances between conductor 1 and 2. In all cases the frequency is varied from 50 Hz to 100 kHz and the results are compared with numerical results obtained using an FEM solution.

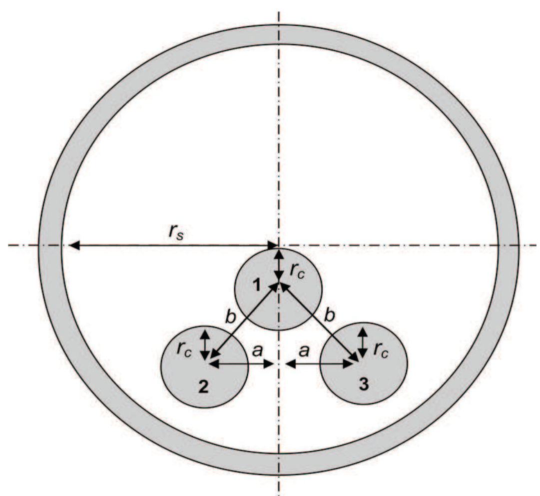

Fig. 5. Simulated three-phase power cable ($r_s = 100$mm, $r_c = 17$mm, $a = 1.5r_c$, $b = 3r_c$; copper conductors with $\sigma_c = 5.8 \times 10^7$ S/m, aluminum shield with $\sigma_a = 3.8 \times 10^7$ S/m; all media have relative permeability $\mu_r = 1$).

Table 1. Per-unit-length self-resistance and self-inductance of conductor No. 1 for the three-phase system of Fig. 5.

	FEM		*BEM – SIBC*	
Frequency [Hz]	$R_{11}[\Omega/\mathrm{m}]$	$L_{11}[\mathrm{H/m}]$	$R_{11}[\Omega/\mathrm{m}]$	$\boldsymbol{L_{11}}$ [H/m]
50	3.2534×10^{-5}	3.8605×10^{-7}	3.3674×10^{-5}	$\mathbf{3.8828\times10^{-7}}$
100	4.5510×10^{-5}	3.6337×10^{-7}	4.6398×10^{-5}	$\mathbf{3.6302\times10^{-7}}$
200	6.4437×10^{-5}	3.4360×10^{-7}	6.4894×10^{-5}	$\mathbf{3.4340\times10^{-7}}$
300	7.8817×10^{-5}	3.3449×10^{-7}	7.9249×10^{-5}	$\mathbf{3.3437\times10^{-7}}$
500	1.0174×10^{-4}	3.2524×10^{-7}	1.0214×10^{-4}	$\mathbf{3.2515\times10^{-7}}$
$\mathbf{10^3}$	1.4516×10^{-4}	3.1525×10^{-7}	1.4440×10^{-4}	$\mathbf{3.1574\times10^{-7}}$
$\mathbf{10^4}$	4.5792×10^{-4}	3.0011×10^{-7}	4.5762×10^{-4}	$\mathbf{3.0004\times10^{-7}}$
$\mathbf{10^5}$	$\mathbf{1.4812\times10^{-3}}$	$\mathbf{2.9511\times10^{-7}}$	$\mathbf{1.4492\times10^{-3}}$	$\mathbf{2.9506\times10^{-7}}$

Table 2. Per-unit-length mutual-resistance and mutual-inductance between conductors No. 1 and No. 2 for the three-phase system of Fig. 5.

	FEM		*BEM – SIBC*	
Frequency [Hz]	R_m [Ω/m]	L_m [H/m]	R_m [Ω/m]	$\boldsymbol{L_m}$ [H/m]
50	6.8090×10^{-6}	1.1826×10^{-7}	7.5031×10^{-6}	$\mathbf{1.1623\times10^{-7}}$
100	1.0478×10^{-5}	1.0954×10^{-7}	1.0978×10^{-5}	$\mathbf{1.0929\times10^{-7}}$
200	1.5447×10^{-5}	1.0384×10^{-7}	1.6024×10^{-5}	$\mathbf{1.0392\times10^{-7}}$
300	1.9360×10^{-5}	1.0134×10^{-7}	1.9937×10^{-5}	$\mathbf{1.0146\times10^{-7}}$
500	2.5608×10^{-5}	9.8796×10^{-8}	2.6177×10^{-5}	$\mathbf{9.8942\times10^{-8}}$
$\mathbf{10^3}$	3.8502×10^{-5}	9.6235×10^{-8}	3.7691×10^{-5}	$\mathbf{9.6378\times10^{-8}}$
$\mathbf{10^4}$	1.2288×10^{-5}	9.1939×10^{-8}	1.2305×10^{-4}	$\mathbf{9.2099\times10^{-8}}$
$\mathbf{10^5}$	$\mathbf{4.2558\times10^{-5}}$	$\mathbf{9.0575\times10^{-8}}$	$\mathbf{3.9325\times10^{-4}}$	$\mathbf{9.0740\times10^{-8}}$

7 Conclusions

The SIBCs developed here and the applications discussed point to a systematic approach to the development and use of SIBCs covering the entire range of low penertration problems. Those methods and formulation that have not been defined so far are covered under the toolbox approach which, in effect allows development of additional, new SIBCs to suit the users need. The assumptions needed were minimal. A method of evaluation of suitability of SIBCs for specific problems allows the user an informed decision before embrakation on development and use of the method.

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