FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 22, no. 1, April 2009, 105-116

Some Criteria for Chaos and no Chaos in the Quadratic Map of the Plane

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Abstract: This paper gives some criteria for the existence and the non-existence of chaotic attractors in the general 2-D quadratic map.

Keywords: 2-D quadratic map, choas, no chaos.

1 Introduction

THE MOST general 2-D quadratic map is given by

$$f(x,y) = \begin{bmatrix} a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy \\ b_0 + b_1x + b_2y + b_3x^2 + b_4y^2 + b_5xy \end{bmatrix} = \begin{bmatrix} z_a(x,y) \\ z_b(x,y) \end{bmatrix} \tag{1}$$

where $(a_i, b_i)_{0\leq i\leq 5} \in \mathbb{R}^{12}$ are the bifurcation parameters. Some special cases of the map (1) can be used in potential applications in several different ways and types of studies [1–5]. Some important results about the dynamical properties, bifurcations, and stability of some special cases of the 2-D map (1) are given in [6–13]. However, there are a few papers that focus on the general case of this map. For example, in [14] some solutions of low-dimensional, low-order polynomial maps were classified numerically as either fixed point, limit cycle, chaotic, or unstable using Lyapunov exponent calculations, with the result that a few percent are chaotic. For the 2-D quadratic maps, this percentage is about $11.10 \pm 0.36\%$. Furthermore, in [15] the correlation dimension was calculated for the strange attractors obtained numerically for some cases of the map (1), and it was found that the average correlation dimension scales approximately as the square root of the dimension of the

Manuscript received on December 23, 2008.

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system with a small variation. In [14–17] a systematic search for chaotic orbits of the general 2-D quadratic map (1) with randomly chosen coefficients was described using a simple computer program that gives different attractors. Some simple special cases of the general 2-D quadratic map (1) were studied in detail in [6, 18–22], with analytical results in [6, 18, 19]. In [12] the number of possible chaotic attractors for the map (1) was reduced to 30 types, and the existence of unbounded and bounded orbits was investigated analytically with analytical predictions of some system orbits. Furthermore, a classification of the possible chaotic orbits was given according to the number of nonlinearities, showing how to reduce all the dynamics of the general case (1) to a finite number of maps with well known formulas. On the other hand, in [13] a rigorous proof of the hyperchaoticity of the general map (1) is given using the so-called *second-derivative test* defined for real functions.

This paper offers a similar rigorous proof for the chaoticity and the non-chaoticity of the general map (1) using the so-called second-derivative test defined for real functions. Indeed, the notions of critical points and the second-derivative test are well defined for functions of two variables. The critical points of function $f(x,y)$ are solutions of the equations $\frac{\partial f(x,y)}{\partial x} = 0$ and $\frac{\partial f(x,y}{\partial y} = 0$, which must be solved simultaneously. Let $(x_c, y_c))$ be a critical point, and define

$$d_f(x_c,y_c) = \frac{\partial^2 f(x,y)}{\partial x^2}(x_c,y_c)\frac{\partial^2 f(x,y)}{\partial y^2}(x_c,y_c) - \left[\frac{\partial^2 f(x,y)}{\partial x \partial y}(x_c,y_c)\right]^2. \tag{2}$$

We have the following cases: If $d_f(x_c,y_c) > 0$ and $\frac{\partial^2 f(x,y)}{\partial x^2}(x_c,y_c) < 0$, then $f(x,y)$ has a relative maximum at (x_c,y_c). If $d_f(x_c,y_c) > 0$ and $\frac{\partial^2 f(x,y)}{\partial x^2}(x_c,y_c) > 0$, then $f(x,y)$ has a relative minimum at (x_c,y_c). If $d_f(x_c,y_c) < 0$, then $f(x,y)$ has a saddle point at (x_c,y_c). If $d_f(x_c,y_c) = 0$, then the second-derivative test is inconclusive.

The Jacobian matrix of the map (1) is given by

$$J(x,y) = \begin{bmatrix} a_1 x + 2a_3 x + a_5 y & a_2 y + 2a_4 y + a_5 x \\ b_1 x + 2b_3 x + b_5 y & b_2 y + 2b_4 y + b_5 x \end{bmatrix} \tag{3}$$

For the map (1) assume that

$$\Omega_1 : \begin{cases} a_3 > 0, & a_4 > 0, \quad 4a_3 a_4 > a_5^2 \\ b_3 < 0, & b_4 < 0, \quad 4b_3 b_4 > b_5^2, \end{cases} \tag{4}$$

where Ω_1 defines a subset of the elements $(a_i, b_i)_{0 \le i \le 5} \in \mathbb{R}^{12}$. If the second-derivative test for both $z_a(x,y)$ and $z_b(x,y)$ is used separately, then one has for all

$(x,y) \in \mathbb{R}^2$ that

$$\begin{aligned} z_a(x,y) &\geq \frac{a_0a_5^2 - a_1a_2a_5 - 4a_0a_3a_4 + a_1^2a_4 + a_2^2a_3}{a_5^2 - 4a_3a_4} = L_a \\ z_b(x,y) &\leq \frac{b_0b_5^2 - b_1b_2b_5 - 4b_0b_3b_4 + b_1^2b_4 + b_2^2b_3}{b_5^2 - 4b_3b_4} = L_b, \end{aligned} \tag{5}$$

i.e., for all iterations $(x,y) \in \mathbb{R}^2$ of the map (1), one has

$$x \geq L_a \text{ and } y \leq L_b. \tag{6}$$

It is shown in [23] that a system $x_{k+1} = g(x_k)$, $x_k \in \Omega \subset \mathbb{R}^n$, such that the derivative $g'(x)$ of the function $g(x)$ satisfies the following inequality

$$\|g'(x)\| = \|J)\| = \sqrt{\lambda_{\max}(J^TJ)} \leq N < +\infty, \tag{7}$$

with a smallest eigenvalue of J^TJ that satisfies

$$\lambda_{\min}(J^TJ)) \geq \theta > 0, \tag{8}$$

where $N^2 \geq \theta$, then, for any $x_0 \in \Omega$, all the Lyapunov exponents at x_0 are located inside $\left[\frac{\ln\theta}{2}, \ln N\right]$. That is,

$$\frac{\ln\theta}{2} \leq l_i(x_0)) \leq \ln N, i = 1, 2, ..., n, \tag{9}$$

where $l_i(x_0))$ are the Lyapunov exponents for the map g.

In [13] we use inequalities (7) and (8) with $\theta = 1$ and $N > 1$ to give sufficient conditions for the existence of hyperchaotic attractors in the general 2-D quadratic map (1) in terms of the parameters $(a_i, b_i))_{0\leq i\leq 5} \in \mathbb{R}^{12}$. In this paper we use only the inequality (7) and search for some real N such that $0 < N \leq 1$ for which the map (1) has no chaotic attractors. This result permits us to use the notion of compliment defined for ensembles to determine rigorously all regions of the parameters $(a_i, b_i))_{0\leq i\leq 5} \in \mathbb{R}^{12}$ for the occurrence of chaos in the quadratic map of the plane (1).

For the map (1) one has

$$J^TJ = \begin{bmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{bmatrix} \tag{10}$$

where $J_{12} = J_{21}$ because J^TJ is symmetric and

$$\begin{aligned} J_{11} =& [(a_1+2a_3)x+a_5y)]^2+[(b_1+2b_3)x+b_5y]^2 \\ J_{12} =& [(a_1+2a_3)x+a_5y][a_5x+(a_2+2a_4)y] \\ &+[(b_1+2b_3)x+b_5y)(b_5x+(b_2+2b_4)y] \\ J_{22} =& [a_5x+(a_2+2a_4)y]^2+[b_5x+(b_2+2b_4)y)]^2. \end{aligned} \tag{11}$$

Because J^TJ is at least a positive semi-definite matrix, then all its eigenvalue are real and positive, i.e., $\lambda_{\max}(J^TJ)) \geq \lambda_{\min}(J^TJ)) \geq 0$. Hence the eigenvalues of J^TJ are given by

$$\begin{aligned} \lambda_{\max}(J^TJ)) &= \frac{J_{11}+J_{22}+\sqrt{J_{11}^2-2J_{11}J_{22}+4J_{12}^2+J_{22}^2}}{2} \\ \lambda_{\min}(J^TJ)) &= \frac{J_{11}+J_{22}-\sqrt{J_{11}^2-2J_{11}J_{22}+4J_{12}^2+J_{22}^2}}{2}. \end{aligned} \tag{12}$$

We have

$$\begin{aligned} J_{11} &= C_1x^2+C_2y^2+C_3xy \\ J_{12} &= \frac{1}{2}C_3x^2+C_4y^2+C_5xy \\ J_{22} &= C_2x^2+C_6y^2+2C_4xy \end{aligned} \tag{13}$$

where

$$\begin{aligned} C_1 &= (2a_3+a_1)^2+(2b_3+b_1)^2 \geq 0 \\ C_2 &= a_5^2+b_5^2 \geq 0 \\ C_3 &= 2[(a_1+2a_3)a_5+(b_1+2b_3)b_5] \\ C_4 &= (a_2+2a_4)a_5+(b_2+2b_4)b_5 \\ C_5 &= (a_1+2a_3)(a_2+2a_4)+(b_1+2b_3)(b_2+2b_4)+a_5^2+b_5^2 \\ C_6 &= (2a_4+a_2)^2+(2b_4+b_2)^2 \geq 0. \end{aligned} \tag{14}$$

The 2-D quadratic map (1) is non-chaotic if there exist a real N satisfying inequality (7) such that

$$\begin{aligned} &0 < N \leq 1 \\ &\xi_1x^2+\xi_2y^2+\xi_3xy-2N \leq 0 \\ &\xi_4x^4+\xi_5y^4+\xi_6x^3y+\xi_7xy^3+\xi_8x^2y^2 \\ &\quad -N^2\xi_1x^2-N^2\xi_2y^2-N^2\xi_3xy+1+N^4 \geq 0, \end{aligned} \tag{15}$$

where

$$\begin{array}{ll} \xi_1 = C_1 + C_2 \geq 0 & \xi_5 = C_4^2 - C_2C_6 \\ \xi_2 = C_2 + C_6 \geq 0 & \xi_6 = C_3C_5 - C_2C_3 - 2C_1C_4 \\ \xi_3 = C_3 + 2C_4 & \xi_7 = 2C_4C_5 - C_3C_6 - 2C_2C_4 \\ \xi_4 = \frac{1}{4}C_3^2 - C_1C_2 & \xi_8 = C_5^2 - C_1C_6 - C_3C_4 - C_2^2. \end{array} \tag{16}$$

Assume first that

$$\Omega_2 : \xi_3 < 0. \tag{17}$$

The aim of the following investigation is to determine an interval for the quantity $0 < N \leq 1$ such that (7) holds for all $x \geq L_a$ and $y \leq L_b$. For this purpose, begin with the second condition of (15) and consider the function $m(x,y) = \xi_1 x^2 + \xi_2 y^2 + \xi_3 xy - 2N$, assuming that

$$\Omega_3 : a_1 < 0. \tag{18}$$

Then from (4) and (5) one has

$$L_a \leq x \leq -\frac{a_3x^2}{a_1} - \frac{a_4y^2}{a_1} - \frac{a_5xy}{a_1} - \frac{a_2y}{a_1} + \frac{L_a}{a_1} - \frac{a_0}{a_1}. \tag{19}$$

Thus we can choose

$$x_1 \leq L_a \leq x \leq -\frac{a_3x^2}{a_1} - \frac{a_4y^2}{a_1} - \frac{a_5xy}{a_1} - \frac{a_2y}{a_1} + \frac{L_a}{a_1} - \frac{a_0}{a_1} \leq x_2 \tag{20}$$

where x_1 and x_2 are the roots of the equation $m(x,y) = 0$ with respect to x, i.e., its discriminant is $8N\xi_1 + (\xi_3^2 - 4\xi_1\xi_2)y^2 > 0$ for all $y \in \mathbb{R}$. Then one has

$$\begin{aligned} x_1 &= \frac{-\xi_3 y - \sqrt{8N\xi_1 + (\xi_3^2 - 4\xi_1\xi_2)y^2}}{2\xi_1} \\ x_2 &= \frac{-\xi_3 y + \sqrt{8N\xi_1 + (\xi_3^2 - 4\xi_1\xi_2)y^2}}{2\xi_1}. \end{aligned} \tag{21}$$

The inequality $x_1 \leq L_a$ holds for all $y \leq L_b$ if

$$\Omega_4 : L_b \leq \frac{-2\xi_1 L_a}{\xi_3}, \tag{22}$$

and the inequality

$$-\frac{a_3x^2}{a_1} - \frac{a_4y^2}{a_1} - \frac{a_5xy}{a_1} - \frac{a_2y}{a_1} + \frac{L_a}{a_1} - \frac{a_0}{a_1} \leq x_2$$

holds for all $y \leq L_b$ if

$$w_1(x,y) + w_2(x,y) + \xi_{21} \leq 0 \tag{23}$$

where

$$\begin{aligned} w_1(x,y) &= \xi_9 x^4 + \xi_{10} y^4 + \xi_{11} x^2 y^2 + \xi_{12} x^3 y + \xi_{13} x y^3 + \xi_{14} y^3 \\ w_2(x,y) &= \xi_{15} x^2 y + \xi_{16} x y^2 + \xi_{17} x^2 + \xi_{18} y^2 + \xi_{19} x y + \xi_{20} y \end{aligned} \tag{24}$$

and

$$\begin{aligned}
\xi_9 &= \frac{4a_3^2\xi_1^2}{a_1^2} \\
\xi_{10} &= \frac{4a_4^2\xi_1^2}{a_1^2} \\
\xi_{11} &= \frac{4(2a_3a_4 + a_5^2)\xi_1^2}{a_1^2} \\
\xi_{12} &= \frac{8a_3a_5\xi_1^2}{a_1^2} \\
\xi_{13} &= \frac{8a_4a_5\xi_1^2}{a_1^2} \\
\xi_{14} &= \frac{-4(a_1\xi_3 - 2a_2\xi_1)a_4\xi_1}{a_1^2} \\
\xi_{15} &= \frac{-4(a_1\xi_3 - 2a_2\xi_1)a_3\xi_1}{a_1^2} \\
\xi_{16} &= \frac{-4(a_1\xi_3 - 2a_2\xi_1)a_5\xi_1}{a_1^2} \\
\xi_{17} &= \frac{-8(L_a - a_0)a_3\xi_1^2}{a_1^2} \\
\xi_{18} &= \frac{-4(a_1a_2\xi_3 - 2a_0a_4\xi_1 + 2a_4\xi_1L_a - a_1^2\xi_2 - a_2^2\xi_1)\xi_1}{a_1^2} \\
\xi_{19} &= \frac{-8(L_a - a_0)a_5\xi_1^2}{a_1^2} \\
\xi_{20} &= \frac{4(L_a - a_0)(a_1\xi_3 - 2a_2\xi_1)\xi_1}{a_1^2} \\
\xi_{21} &= \frac{4(a_0^2\xi_1 - 2Na_1^2 - 2a_0\xi_1L_a + \xi_1L_a^2)\xi_1}{a_1^2}.
\end{aligned} \tag{25}$$

Now consider the function $w(x,y) = w_1(x,y) + w_2(x,y) + \xi_{21}$. The critical points of w are the solutions of the system

$$\begin{aligned} &4\xi_9 x^3 + 3\xi_{12} y x^2 + (2\xi_{17} + 2\xi_{15} y + 2y^2 \xi_{11})x + \xi_{13} y^3 + \xi_{16} y^2 + \xi_{19} y = 0 \\ &4\xi_{10} y^3 + (3\xi_{14} + 3\xi_{13} x) y^2 + (2\xi_{18} + 2\xi_{16} x + 2\xi_{11} x^2) y \\ &\qquad + \xi_{12} x^3 + \xi_{15} x^2 + \xi_{19} x + \xi_{20} = 0. \end{aligned} \tag{26}$$

Assume that

$$\Omega_5 : \xi_9 \neq 0,\ \xi_{10} \neq 0. \tag{27}$$

Then both equations in (26) are cubic, and the first equation of (26) has at least one real solution $s_c^{(1)}$ for all values of y, and at most three roots $(s_c^{(i)})_{1\leq i\leq 3}$ for all values of y. The second equation of (26) has at least one real solution $q_c^{(1)}$ for all values of x, and at most three roots $(q_c^{(i)})_{1\leq i\leq 3}$ for all values of x. Thus there are still solutions $(s_c^{(i)}, q_c^{(i)})$ of equation (26) that are critical points of the function h. On the other hand, one has

$$\begin{aligned} &\frac{d^2 w}{dx^2}(x,y) = 12\xi_9 x^2 + 2\xi_{11} y^2 + 6\xi_{12} xy + 2\xi_{15} y + 2\xi_{17} \\ &d_w(x,y) = d_1(x,y) + d_2(x,y) \end{aligned} \tag{28}$$

where

$$\begin{aligned} d_1(x,y) &= \xi_{22} x^4 + \xi_{23} y^4 + \xi_{24} x^2 y^2 + \xi_{25} x^3 y + \xi_{26} x y^3 + \xi_{27} y^3 + \xi_{28} x^2 y \\ d_2(x,y) &= \xi_{29} x y^2 + \xi_{30} y^3 + \xi_{31} x^2 + \xi_{32} y^2 + \xi_{33} xy + \xi_{34} x + \xi_{35} y + \xi_{36} \end{aligned} \tag{29}$$

and

$$\begin{aligned} \xi_{22} &= 24\xi_9\xi_{11} & \xi_{26} &= 12\xi_{11}\xi_{13} + 72\xi_{10}\xi_{12} \\ \xi_{23} &= 24\xi_{10}\xi_{11} & \xi_{27} &= 24\xi_{10}\xi_{15} + 12\xi_{11}\xi_{14} \\ \xi_{24} &= 144\xi_9\xi_{10} + 36\xi_{12}\xi_{13} + 4\xi_{11}^2 & \xi_{28} &= 12\xi_{16}\xi_{12} + 4\xi_{11}\xi_{15} + 72\xi_9\xi_{14} \\ \xi_{25} &= 72\xi_9\xi_{13} + 12\xi_{11}\xi_{12} & \xi_{29} &= 4\xi_{16}\xi_{11} + 36\xi_{12}\xi_{14} + 12\xi_{13}\xi_{15} \end{aligned} \tag{30}$$

and

$$\begin{aligned} \xi_{30} &= 24\xi_{10}\xi_{15} + 12\xi_{11}\xi_{14} & \xi_{33} &= 12\xi_{17}\xi_{13} + 12\xi_{18}\xi_{12} + 4\xi_{16}\xi_{15} \\ & & &\quad - 48\xi_{11}\xi_{13} \\ \xi_{31} &= 24\xi_9\xi_{18} + 4\xi_{17}\xi_{11} - 16\xi_{11}^2 & \xi_{34} &= 4\xi_{16}\xi_{17} - 16\xi_{16}\xi_{11} \\ \xi_{32} &= 24\xi_{17}\xi_{10} + 4\xi_{18}\xi_{11} + 12\xi_{14}\xi_{15} & \xi_{35} &= 12\xi_{17}\xi_{14} - 24\xi_{16}\xi_{13} + 4\xi_{18}\xi_{15} \\ &\quad - 36\xi_{13}^2 & \xi_{36} &= 4\xi_{17}\xi_{18} - 4\xi_{16}^2 + 24\xi_{16}\xi_9. \end{aligned} \tag{31}$$

If one root $(s_c^{(1)}, q_c^{(1)})$ exists for equation (26), then assume that $\frac{d^2w}{dx^2}(s_c^{(1)}, q_c^{(1)}) < 0$ and $d_w(s_c^{(1)}, q_c^{(1)}) > 0$, i.e.,

$$\Omega_6 : \begin{cases} 12\xi_9(s_c^{(1)})^2 + 2\xi_{11}(q_c^{(1)})^2 + 6\xi_{12}s_c^{(1)}q_c^{(1)} + 2\xi_{15}q_c^{(1)} + 2\xi_{17} < 0 \\ d_1(s_c^{(1)}, q_c^{(1)}) + d_2(s_c^{(1)}, q_c^{(1)}) > 0. \end{cases} \tag{32}$$

Hence the function w has a relative maximum at $(s_c^{(1)}, q_c^{(1)})$, i.e., $w(x,y \leq w(s_c^{(1)}, q_c^{(1)})$ for all $(x,y) \in \mathbb{R}^2$, and in this case we choose $w(s_c^{(1)}, q_c^{(1)}) < 0$, i.e.,

$$w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)}) + \xi_{21} < 0 \tag{33}$$

or

$$\Omega_7 : \frac{[w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)}]a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0L_a)\xi_1^2}{8\xi_1 a_1^2} = N_1 < N \tag{34}$$

because only the coefficient ξ_{21} depends on N.

If equation (26) has more than one root, then one calculates $\frac{d^2w}{dx^2}(s_c^{(i)}, q_c^{(i)})$, $d_w(s_c^{(i)}, q_c^{(i)})$, and $w(s_c^{(i)}, q_c^{(i)})$ and determines the type of each point by imposing some conditions as above, and according to the values of $w(s_c^{(i)}, q_c^{(i)})$, one can determine the global maximum of the function h, and finally make the quantity $w(s_c^{(i)}, q_c^{(i)})$ strictly negative.

For the third condition of (15), consider the function $v(x,y) = v_1(x,y) + v_2(x,y)$, where

$$\begin{aligned} v_1(x,y) &= \xi_4x^4 + \xi_5y^4 + \xi_6x^3y + \xi_7xy^3 + \xi_8x^2y^2 + 1 \\ v_2(x,y) &= N^2[N^2 - (\xi_1x^2 + \xi_2y^2 + \xi_3xy)] \end{aligned} \tag{35}$$

The critical points of the function v are the solutions of the system

$$\begin{aligned} 4\xi_4x^3 + (3\xi_6y)x^2 + (2\xi_8y^2 - 2N^2\xi_1)x + \xi_7y^3 - N^2\xi_3y = 0 \\ 4\xi_5y^3 + (3\xi_7x)y^2 + (2\xi_8x^2 - 2N^2\xi_2)y + \xi_6x^3 - N^2\xi_3x = 0. \end{aligned} \tag{36}$$

With the same analysis as above, there are still solutions $(k_c^{(i)}, l_c^{(i)})$ of equation (36) that are critical points for the function v. On the other hand, one has

$$\begin{aligned} &\frac{d^2v}{dx^2}(x,y) = 12\xi_4x^2 + 2\xi_8y^2 + 6\xi_6xy - 2N^2\xi_1 \\ &d_v(x,y) = p_1(x,y) + p_2(x,y), \end{aligned} \tag{37}$$

where

$$\begin{aligned} p_1(x,y) &= \xi_{51}x^4 + \xi_{52}y^4 + \xi_{53}x^3y + \xi_{54}xy^3 + \xi_{55}x^2y^2 \\ p_2(x,y) &= N^2h_1(x,y) + h_2(x,y) + 4\xi_1\xi_2N^4, \end{aligned} \tag{38}$$

and

$$\begin{aligned} h_1(x,y) &= 12(\xi_2\xi_6 - \xi_1\xi_7)xy - 4(6\xi_2\xi_4 + \xi_1\xi_8)x^2 - 4(6\xi_1\xi_5 + \xi_2\xi_8)y^2 \\ h_2(x,y) &= -(16\xi_8^2 x^2 + 36\xi_7^2 y^2 + 48\xi_7\xi_8 xy), \end{aligned} \tag{39}$$

and

$$\begin{array}{ll} \xi_{37} = 24\xi_4\xi_8 & \xi_{41} = 144\xi_4\xi_5 + 36\xi_6\xi_7 + 4\xi_8^2 \\ \xi_{38} = 24\xi_5\xi_8 & \xi_{42} = -16\xi_8^2 - 4N^2(6\xi_2\xi_4 + \xi_1\xi_8) \\ \xi_{39} = 72\xi_4\xi_7 + 12\xi_6\xi_8 & \xi_{43} = -36\xi_7^2 - 4N^2(6\xi_1\xi_5 + \xi_2\xi_8) \\ \xi_{40} = 12\xi_7\xi_8 + 72\xi_5\xi_6 & \xi_{44} = 12N^2(\xi_2\xi_6 - \xi_1\xi_7) - 48\xi_7\xi_8 \\ & \xi_{45} = 4N^4\xi_1\xi_2. \end{array} \tag{40}$$

If one root $(k_c^{(1)}, l_c^{(1)})$ exists for equation (36), then assume that $\frac{d^2v}{dx^2}(k_c^{(1)}, l_c^{(1)}) > 0$ and $d_v(k_c^{(1)}, l_c^{(1)}) > 0$, i.e.,

$$\Omega_8 : \begin{cases} N < \sqrt{\dfrac{12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)}}{2\xi_1}} = N_2 \\ 4\xi_1\xi_2 N^4 + h_1(k_c^{(1)}, l_c^{(1)})N^2 + [p_1(k_c^{(1)}, l_c^{(1)}) + h_2(k_c^{(1)}, l_c^{(1)})] > 0. \end{cases} \tag{41}$$

The first condition of (41) is possible if

$$\Omega_9 : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} > 0, \tag{42}$$

and the second condition of (41) is possible for all $N \in \mathbb{R}$ if

$$\Omega_{10} : h_1^2(k_c^{(1)}, l_c^{(1)}) - 16p_1(k_c^{(1)}, l_c^{(1)})\xi_1\xi_2 - 16h_2(k_c^{(1)}, l_c^{(1)})\xi_1\xi_2 < 0 \tag{43}$$

because $\xi_1\xi_2 > 0$, and from the first condition of (15), and conditions (34) and (41) one has that $N_i, i = 1,2$ must satisfy the inequalities

$$\max(0, N_1) < N < \min(1, N_2). \tag{44}$$

We have the following cases:

(a) If $N_1 \leq 0$ and $N_2 \geq 1$, i.e.,

$$\begin{cases} \Omega_{11} : [w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)})]a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0L_a)\xi_1^2 \leq 0 \\ \Omega_{12} : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} - 2\xi_1 \geq 0, \end{cases} \tag{45}$$

then one has $0 < N < 1$.

(b) If $N_1 \leq 0$ and $N_2 \leq 1$, i.e.,

$$\begin{cases} \Omega_{11} : [w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)})]a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0 L_a)\xi_1^2 \leq 0 \\ \bar{\Omega}_{12} : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} - 2\xi_1 \leq 0 \end{cases} \tag{46}$$

where $\bar{\Omega}_{12}$ is the compliment of the subset Ω_{12}, then there exists an N such that $0 < N < N_2 \leq 1$.

(c) If $N_1 \geq 0$ and $N_2 \geq 1$, i.e.,

$$\begin{cases} \Omega_{13} : [w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)})]a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0 L_a)\xi_1^2 \geq 0 \\ \Omega_{12} : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} - 2\xi_1 \geq 0, \end{cases} \tag{47}$$

then there exists an N such that $0 \leq N_1 < N \leq 1$, with the condition $N_1 < 1$, i.e.,

$$\Omega_{14} : (w_1(s_c^{(1)}, q_c^{(1)}) + w_2(s_c^{(1)}, q_c^{(1)}) - 8\xi_1)a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0 L_a)\xi_1^2 < 0. \tag{48}$$

(d) If $N_1 \geq 0$ and $N_2 \leq 1$, i.e.,

$$\begin{cases} \Omega_{15} : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} - 2\xi_1 \leq 0 \\ \bar{\Omega}_{12} : 12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)} - 2\xi_1 \leq 0, \end{cases} \tag{49}$$

then one has $0 \leq N_1 < N < N_2 \leq 1$, with the condition $N_1 < N_2$, i.e.,

$$\Omega_{16} : \frac{(w_1 + w_2)a_1^2 + (4a_0^2 + 4L_a^2 - 8a_0 L_a)\xi_1^2}{8\xi_1 a_1^2} < \sqrt{\frac{12\xi_4(k_c^{(1)})^2 + 2\xi_8(l_c^{(1)})^2 + 6\xi_6 k_c^{(1)} l_c^{(1)}}{2\xi_1}} \tag{50}$$

where $w_1 + w_2 = (w_1 + w_2)(s_c^{(1)}, q_c^{(1)})$.

Therefore, for all the above cases there exists an N such that $0 < N \leq 1$ in which inequality (15) holds for all $x \geq L_a$ and $y \leq L_b$.

Finally, the general map (1) has no chaotic attractors if all the above inequalities hold. Hence we have proved the following theorem:

Theorem 1 *If* $\cap_{i=1}^{i=12}\Omega_i \neq \emptyset$, *or* $\cap_{i=1}^{i=11}\Omega_i \cap \bar{\Omega}_{12} \neq \emptyset$, *or* $\cap_{i=1, i\neq 11}^{i=14}\Omega_i \neq \emptyset$, *or* $\cap_{i=1}^{i=10}\Omega_i \cap \bar{\Omega}_{12} \cap \Omega_{15} \cap \Omega_{16} \neq \emptyset$, *then the general quadratic map of the plane given by equation (1) has no chaotic attractors* (x, y) *with the condition* $x \geq L_a$ *and* $y \leq L_b$, *where* L_a *and* L_b *are given by (5).*

An immediate and fundamental result of the Theorem 1 is given by

Theorem 2 *If* $(a_i,b_i)_{0\leq i\leq 5} \in \cup_{i=1}^{i=12}\overline{\Omega}_i$, *or* $(a_i,b_i)_{0\leq i\leq 5} \in \cup_{i=1}^{i=11}\overline{\Omega}_i \cup \Omega_{12}$, *or* $(a_i,b_i)_{0\leq i\leq 5} \in \cup_{i=1,i\neq 11}^{i=14}\Omega_i$, *or* $(a_i,b_i)_{0\leq i\leq 5} \in \cup_{i=1}^{i=10}\Omega_i \cup \Omega_{12} \cup \Omega_{15} \cup \Omega_{16}$, *then the general quadratic map of the plane given by equation (1) has possible (or other types of solutions, especially, unbounded orbits) chaotic attractors* (x,y) *with the condition* $x \geq L_a$ *and* $y \leq L_b$, *where* L_a *and* L_b *are given by (5).*

We conclude with the following remarks:

(a) The above inequalities do not guarantee the boundedness of the attractors.

(b) Not all chaotic or non-chaotic attractors are obtained from the above conditions.

(c) Finding a specific example is not simple because at each step the solution of third-degree equations and very complicated inequalities with 12 unknown variables are required.

(d) It may be possible to convert the proof to a numerical algorithm.

(e) Some of the above chaotic or non-chaotic attractors can be infinitely or very large.

At the end of this paper, let us anounce the following open problems:

1. Find sufficient conditions (in the same direction of this paper) that guarantee the boundedness of the attractors.
2. Find a specific example where the conditions of Theorem 1 or 2 holds.
3. Convert the proof to a numerical algorithm.

2 Conclusion

We have given a rigorous proof of the existence and non-existence of chaos in the general quadratic map of the plane. The proof shows how to locate specific types of orbits in some cases.

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