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Remarks on the Development and Recent Results in the Theory of Gibbs Derivatives

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Abstract: This tutorial paper discusses the development of the theory of Gibbs differential operators, and highlights some important older and more and recent results in this area.

Keywords: Gibbs derivates, Walsh functions, finite dyadic group.

1 Introduction

FOURIER ANALYSIS and differential calculus are two very important tools in signal processing and related areas. From a group theoretic point of view, these two essential concepts can be related by the observation that classical Newton-Leibniz derivative can be viewed as a closed linear operator on the real line R that maps each group character of R $e^{2\pi ikx}$ into the scalar multiple $2\pi ike^{2\pi ikx}$ of that character, and by recalling that exponential functions, eigenfunctions of differential operators, are at the same time kernels in the Fourier representations of signals.

Attempts towards extending classical Fourier analysis to function systems different than group characters of R emerged in part also due to convergence problems in approximation of real valued functions. In this context, different function systems, such as the Walsh functions [77] and the Haar functions [35], have been intentionally invented to resolve related problems.

Fine [15] and Vilenkin [74] observed independently that Walsh functions can be identified with group characters of the dyadic or Cantor group. In this way, the Walsh functions have been fully incorporated in the Walsh-Fourier analysis as a particular example of harmonic analysis on groups.

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In late 60's and early 70' there emerged a high interest in the discrete counterpart of Walsh functions due to their compatibility with binary signals and bistable circuits, and for the simplicity of the calculation of the spectral Walsh coefficients without multiplication of complex numbers that is necessary in the discrete Fourier analysis.

Ideas to extend the theory to the discrete Walsh functions by linking differential operators and group characters naturally emerged in the research community. In particular, such an initiative was suggested to J. Edmund Gibbs by Dr Alastair Gebbie, at that time the Head of the Basic Physic Division of National Physical Laboratory in Teddington, Middlessex, England. The first complete formulation of the definition of the Gibbs derivatives has been made by James Edmund Gibbs on January 13, 1967. It has been completely presented in [19].

The especially simple case of the Gibbs derivative on a finite dyadic group, called initially logic derivative or finite dyadic derivative, is of importance not only for its own sake but also how this conceptually new approach to differentiation first emerged [25–28]. The subsequent development of this theory and highlighting some important results are the subject of this paper. For the limited space, we have to restrict the selection of references to the publications where certain results were reported for the first time and in exceptional cases to some other relevant publications. An extensive bibliography of Gibbs derivatives can be found in [69]. In [69], we provide some insight into the bibliography on Gibbs derivatives.

2 Notation and Definitions

For each positive integer n, let G_n denote the finite dyadic group whose elements are n-tuples $x = (x_0, x_1, \ldots, x_{n-1})$ with $x \in \{0,1\}$. The group operation $\oplus$ is coordinatewise addition modulo 2, that is, for each $x, \xi, \in G_n$, $x \oplus \xi = (x_0 \oplus \xi_0, \ldots, x_{n-1} \oplus \xi_{n-1})$. Denote by B_n the set of non-negative integers less than 2^n: $B_n = \{0,1,\ldots,2^n - 1\}$.

With each element x of G_n we associate a unique element of B_n by means of a function $V_n : G_n \to B_n$ defined by

$$V_n(x) = \sum_{i=0}^{n-1} 2^{n-i-1} x_i. \tag{1}$$

The element of B_n thus defined is usually called, in switching theory, the "decimal index" of x. The function V_n is bijective, so to each integer of B_n corresponds under V_n^{-1} a unique element of G_n whose coordinates are the coefficients in the dyadic expansion of that integer. The existence of the bijection V_n enables us safely to ignore the distinction between G_n and B_n. Thus we shall denote by x indifferently the n-tuple $(x_0, \ldots, x_{n-1})$ or its decimal index $V_n((x_0, \ldots, x_{n_1}))$. We denote by

$\oplus$ indifferently the operation already defined in G_n or the operation in B_n defined by

$$x \oplus \xi = V_n(V_n^{-1}(x) \oplus V_n^{-1}(\xi)), \quad (x, \xi \in B_n).$$

In the right member of this equality, $\oplus$ is the operation in G_n; in the left member, $\oplus$ is the operation in B_n. Thus $V_n : G_n \to B_n$ is a group isomorphism.

The space of all complex-valued functions f on G_n (or on B_n) will be denoted by L_n, or, since it is not usually necessary to express the dependence on n, simply L.

Notice that the finite dyadic group of order 2^n is equal to the direct n-th power of the cyclic group $G_1 = (\{0,1\}, \oplus)$. This enables us to write (the value-vectors of) the discrete Walsh functions $wal(w,\cdot) : B_n \to \{1,-1\}$, $(w \in B_n)$ (the characters of the group G_n) as the rows (or columns) of a matrix $\mathbf{W}_n$ defined by

$$\mathbf{W}_n = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^n, \tag{2}$$

where $[\cdot]^n$ denotes the continued Kronecker product of n copies of the matrix $[\cdot]$.

Using the discrete Walsh functions as an orthogonal basis in the space L_n, we may define the discrete Walsh transform as a particular case of the Fourier transform on groups. Thus, the Walsh transform coefficients $S_f(w)$ $(w \in B_n)$ of $f \in L$ are defined by

$$S_f(w) = \sum_{x=0}^{2^n-1} f(x) wal(w,x). \tag{3}$$

The inverse Walsh transform is given by

$$f(x) = 2^{-n} \sum_{w=0}^{2^n-1} S_f(w) wal(w,x). \tag{4}$$

If we express the Walsh spectral coefficients $S_f(w)$ and the original function f as vectors of order 2^n, $\mathbf{S}_f = [S_f(0), \ldots, S_f(2^n-1)]^T$ and $\mathbf{F} = [f(0), \ldots, f(2^n-1)]^T$, then the equalities (3) and (4) may be written

$$\mathbf{S}_f = \mathbf{W}_n \mathbf{F}, \tag{5}$$

$$\mathbf{F} = 2^{-n} \mathbf{W}_n \mathbf{S}_f. \tag{6}$$

The discrete dyadic convolution product $f * g$ of two functions $f, g \in L$ is defined by

$$(f * g) = \sum_{u=0}^{2^n-1} f(x \oplus u) g(u), \quad (x \in B_n).$$

The dyadic convolution theorem states that, if the Walsh transforms of f and g are S_f and S_g, respectively, then the transform of $f \cdot g$ is $S_f * S_g$ and the transform of $f * g$ is $S_f \cdot S_g$.

3 The Gibbs Derivative on Finite Dyadic Groups

The initial definition of the Gibbs derivative has been formulated for functions on finite dyadic group G_n as follows [19].

Definition 1. *To each function $f \in L$ we assign a function $f^{[1]} \in L$ defined by*

$$f^{[1]}(x) = -\frac{1}{2}\sum_{r=0}^{n-1}(f(x \oplus 2^r) - f(x))2^r, \quad (x \in B_n). \tag{7}$$

We call $f^{[1]}$ the (first-order) Gibbs derivative of f. The corresponding operator $D_n : L \to L$, defined by $D_n f = f^{[1]}$ $(f \in L)$, will be called the Gibbs differentiator $B_n \to C$, where C is the filed of complex numbers.

Nomenclature of this type enables us to distinguish each of the various species of Gibbs differentiator, by indicating the domain and range of the functions on which it operates. To simplify the notation we shall generally abbreviate D_n as D.

If $f : B_n \to C$ is identified with the corresponding function [1] defined on G_n, and if we use the same symbol f for this function $f : G_n \to \mathbf{K}$, then Definition 1 may be written

$$f^{[1]}(x) = -\frac{1}{2}\sum_{i=0}^{n-1}2^{n-i-1}(D_i f)(x), \tag{8}$$

where $x = (x_0, \ldots, x_{n-1})$, and D_i $(i = 0, \ldots, n-1)$ denotes the (i-th coordinate) partial Gibbs differentiator, defined [2] by

$$\begin{aligned}(D_i f)((x_0, \ldots, x_{n-1})) \quad &= \quad f((x_0, \ldots, x_i \oplus 1, \ldots x_{n-1})) \\ &\qquad - f((x_0, \ldots, x_{n-1})).\end{aligned}$$

The following theorem summarises some of the chief properties of the Gibbs differentiator $B_n \to \mathbf{K}$ (or $G_n \to C$).

Theorem 1. *The operator $D : L \to L$ has the following properties:*

[1] This function should strictly be denoted by the symbol fV_n, where juxtaposition denotes composition.

[2] Let the complement $x \oplus 1$ of $x \in \{0, l\}$ be denoted by $\overline{x}$. Then the i-th coordinate partial Gibbs derivative may be written

$$\begin{aligned}(D_i f)((x_0, \ldots, x_{n-1})) \quad &= \quad f((x_0, \ldots, \overline{x}_i, \ldots, x_{n-})) \\ &\qquad - f((x_0, \ldots, x_i, \ldots, x_{n-1})).\end{aligned}$$

.

1. *Linearity:* $D(a_1 f_1 + a_2 f_2) = a_1 f_1 + a_2 f_2$, $(a_1, a_2 \in C, f_1, f_2 \in L)$.
2. $Df = O \in L$ *iff* $f \in L$ *is a constant function.*
3. *If the Walsh transform of* $f \in L$ *is* S_f*, then the Walsh transform of* Df *is given by*

$$S_{Df} = wS_f. \tag{9}$$

4. $D(f_1 * f_2) = (Df_1) * f_2 = f_1 * (Df_2), (f_1, f_2 \in L)$.
5. *For each* $\alpha \in B_n$*, the operator* D *commutes with the translation operator* T_α *defined by* $(T_\alpha f)(x) = f(x \oplus \alpha)$, $(f \in L, x \in B_n)$*:*

$$DT_\alpha = T_\alpha D.$$

6. *The operator* D *does not obey the product rule satisfied by the Newton- Leibniz differentiator* D_{NL}*, namely,*

$$\begin{aligned} D_{NL}(f_1 f_2) &= f_1(D_{NL} f_2) + (D_{NL} f_1) f_2, \\ &\qquad (f_1, f_2 : R \to R, say). \end{aligned}$$

In other words, for some $f_1, f_2 \in L$*,*

$$D(f_1 f_2) \neq f_1(Df_2) + (Df_1) f_2.$$

Proof. The properties 1 to 6 are easily derived from Definition 1 and properties of the Walsh transform.

By using the property 3 of Theorem 1 it is easy to prove that the set of 2^n discrete Walsh functions defined on B_n is the set of eigenfunctions of the Gibbs differentiator $B_n \to \mathbf{K}$; that is, if we write f_w for $wal(w, \cdot)$ $(w \in B_n)$, then

$$f^{[1]} = w f_w. \tag{10}$$

In other words, the discrete Walsh functions are the 2^n solutions of the eigenvalue problem

$$D_n f = w f, \tag{11}$$

of which the eigenvalues are $w = 0, 1, \ldots, 2^n - 1$, and the corresponding eigenfunctions are $wal(0, \cdot)$, $wal(1, \cdot)$ $\ldots$ $wal(2^n - 1, \cdot)$, respectively [21].

In this context we distinguish Gibbs derivatives of the first and second kinds, corresponding respectively to the Walsh-Paley [59] and Walsh-Kaczmarz [41] orderings of the Walsh functions [20].

The Gibbs derivative can be extended to arbitrary complex order p by way of the definition of the discrete δ-function

$$\delta(x) = 2^{-n} \sum_{w=0}^{2^n-1} wal(w,x) \quad (x \in B_n).$$

The function thus defined has the properties

$$\delta(x) = \begin{cases} 1, & (x = 0), \\ 0, & (x \neq 0), \end{cases}$$

and

$$f = \delta * f, \quad (f \in L).$$

The derivative of order $p \in \mathbf{K}$ of the δ-function is defined by an extension of the equality (10):

$$\delta^{[p]}(x) = 2^{-n} \sum_{w=0}^{2^n-1} w^p wal(w,x). \tag{12}$$

The derivative of order p of an arbitrary function $f \in L$ may be conveniently obtained as the dyadic convolution product of $\delta^{[p]}$ and f. For, by an easily proved extension of the property 4,

$$D^p f = f^{[p]} = \delta * f^{[p]} = \delta^{[p]} * f. \tag{13}$$

The *Gibbs derivative of fractional order* is obtained in the case that p is real.

4 Generalizations of the Gibbs Derivative Through Spectral Interpretation

We note first that the property 3 of Theorem 1 serves as an alternative definition of the Gibbs derivative. More precisely, the Gibbs derivative of a function $f \in L$ is the function $f^{[1]} \in L$ such that

$$S_{f^{[1]}}(w) = wS_f(w), \quad (w \in B_n).$$

The differentiator operating on the space of complex-valued functions on each locally compact Abelian group has the characters of that group as eigenfunctions.

The first attempt at such a definition, restricted to finite Abelian groups, was made by Gibbs and Ireland [29]. These authors regarded such a group as the direct product of indecomposable cyclic groups, and tried to define the differentiator on such a cyclic group consistently with the recapture, as a limiting case, of the classical derivative (see [27]). The differentiator on a finite direct product of indecomposable cyclic groups is defined as a linear combination of the partial differentiators with respect to the direct factors, the numerical coefficients being chosen to yield a convenient set of eigenvalues (essentially, for a group of order g, the least g non-negative integers).

Onneweer [46] independently gave the analogous definition on a countable direct product of groups of prime order, effectively amending and extending the definition of the earlier authors.

Ren Fu-xian, Su Weiyi, and Zheng Wei-xing [16] in 1978 and Zheng Wei-jing and Su Weiyi in 1981 [79] gave a definition (reproduced as **Definition B** by Zelin He in 1983 [36]) on the p-adic groups. This result by Zelin He, has been referred in the further work by the same author [37].

Moraga [44], has discussed the particular case of Gibbs differentiation on finite direct powers of a group of prime order: in this case the eigenfunctions are the discrete Levy-Vilenkin-Chrestenson functions [10, 42, 74].

The extension of Gibbs differentiation to non-Abelian groups. Such a definition is obtained [65] where the dyadic derivative of a function is expressed in terms of its Walsh-Fourier coefficients. The role of the group characters in the definition of the Gibbs derivatives on Abelian groups is taken over, in the case of the derivatives on non-Abelian groups, by the unitary irreducible representations of these groups [65]. The definition applies also to functions with codomain a finite field, provided that this field admits a Fourier transform [67]. The Gibbs derivatives on finite non-Abelian groups may be regarded as linear harmonic translation-invariant systems [64]. An extension of the notion of Gibbs differentiation to matrix-valued functions on finite non-Abelian groups has been done in [72].

4.1 Butzer-Wagner Dyadic Derivative

A major contribution to the theory of dyadic differentiation has been made by Butzer and Wagner in a sequence of publications starting with [6] and [9]. These authors were foremost in recognising the value of the Gibbs derivative in analysis, just as Pichler, from 1970 onwards, led the way in applications to system theory [61].

The dyadic group G is (isomorphic to) the set of all sequences $x = (x_1, x_2, \ldots)$ with the group operation $\oplus$ being termwise addition modulo 2. Let $X(G)$ denote one of the spaces $C(G)$ (the space of dyadic continuous functions on G) and $L^p(G)$ $(1 \leq p < \infty)$ (that of p-th power Lebesgue-integrable functions on G). The series used to define the derivative in either the pointwise or the norm sense [9], [75] is

$$(Df)(x) = \frac{1}{2}\sum_{j=0}^{\infty} 2^j (f(x) - f(x \oplus e_{j+1})),$$

where $f \in X(G)$, $x \in G)$, $e_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,k}, \ldots)$. For a function $f \in X(G)$, if there exist $g \in X(G)$, such that

$$\lim_{m\to\infty} \||\frac{1}{2}\sum_{j=0}^{m} (f(x) - f(x \oplus e_{j+1})) - g(x)\||_{X(G)} = 0,$$

then $g(x)$ is the first strong dyadic derivative of f.

For a function f defined on $[0,1)$, if

$$\frac{1}{2}\sum_{j=0}^{\infty} 2^j (f(x) - f(x \oplus e_{j+1})) = c < \infty,$$

for $x \in [0,1)$, then c is called the first poitwise dyadic derivative of f at x.

Butzer and Wagner [6] use the Fine map to transfer the definition of the strong derivative to periodic functions on the real group, of period 1.

In a further paper, the same authors [7] define Gibbs derivatives of the second kind (corresponding to the Walsh-Kaczmarz ordering of the Walsh functions) for functions defined on G, in both the norm sense and the pointwise sense. In the following year [8], they extend to functions on the non-negative real line the definitions of both strong and pointwise derivatives (of the first kind, corresponding to the Walsh-Paley ordering). A further paper [60] provides a readable summary of the results up to that time. The booklet of Wagner [75] gives a helpful introduction to many analytical aspects of dyadic differentiation, treating in parallel the differentiators of the first and second kinds. A generalization of Butzer-Wagner differentiation was considered in [57, 58] and few other publications [69].

The class of dyadically differentiable functions was greatly extended by Butzer, Engels, and Wipperfürth [4], by their definition of an *extended dyadic (ED) derivative*. They defined the ED-derivative as the result of applying the Euler summation process to the series obtained by applying a certain sequence of multipliers to the coefficients of the Walsh-Fourier series of the conventional dyadic derivative. This remarkable definition is justified by its success in extending the class of differentiable functions, for example, to piecewise polynomial functions and to the products of such functions with Dirichlet's function. Butzer and Engels [3] subsequently defined the ED-derivative directly in the original function space, not in the transform space.

4.2 Dyadic derivative on R_+

Differentiation of functions on R_+ has been initiated in [53], and [56, 76].

The dyadic differentiators on $[0,1)$ and on R_+ may be defined by using the Walsh-Fourier coefficients in a manner that is an extension of the equality (9) above. Zelin He in 1983 [36] has generalised this approach to define the dyadic derivatives of fractional orders $\alpha \in R$. This definition, in case $\alpha = r \in N$, reduces to that of Butzer and Wagner for the dyadic derivative of integer order; and, in case $\alpha = -r$, $(r \in N)$, to that of Butzer and Wagner for the dyadic antiderivative of integer order. The extended dyadic derivative of Butzer, Engels, and Wipperfürth [4] is defined at the outset for fractional orders $\alpha \in R$. The definition therefore covers, by the case $\alpha < 0$, antidifferentiation.

Zelin He [38] further related the Gibbs derivative and integral to Walsh-Fourier integral operators, and defined the concept of the p-adic differential-integral-type operator with order-type (λ_1, λ_2). This is a far-reaching generalisation, in a certain direction, of the concepts of Gibbs derivative and integral, and leads to further applications of the Gibbs derivative to p-adic approximation theory. See, for instance, a related discussion in [69].

The definition of dyadic differentiation on the real half-line [8] has been extended by Engels and Splettstösser [14] to square-integrable stochastic processes, in particular, dyadic-stationary processes. Iff such a derivative exists for a process X, X is said to be dyadically differentiable i.m. (in the mean). A necessary and sufficient condition for a Walsh-harmonisable [3] dyadic-stationary process to be dyadically differentiable has been given by Endow in 1987 [13].

F. Weisz studied in [82, 83] a general summability method of different orthogonal series by using an integral function θ. The results derived have been verified also for miltidimensional dyadic derivatives [84].

4.3 Modified dyadic derivatives and dyadic distributions

A modified definition of the strong dyadic derivative has been discussed in [30]. This research has been continued in [32], where also the fractional derivative of order $\alpha > 0$ have been considered. There have been shown criteria for the existence of these differential operators and their inverse integral operators, and determined a countable set of eigenfunctions of these operators. Some related applications of these results were reported in [31].

The paper [33], reviews results related to the pointwise and strong dyadic

[3] A dyadic-stationary process is said to be Walsh harmonisable iff it assumes its spectral representation in terms of the generalised Walsh functions. A characterization of such processes has been given by Endow [12] in 1984.

derivative and integral for functions on R_+. Based on these results introduced are the modified dyadic strong and pointwise derivatives and integrals of fractional orders on R_+ and some of their properties studied. Study of properties of modified fractional dyadic derivative and integral has been continued in [34].

4.4 Differentiation on *p*-adic groups and generalizations

The definition of dyadic differentiation has been further discussed by Onneweer in 1977, 1978, and 1979 [46, 47, 49], [52] and [53] and [54]. Every acceptable definition has the property that the eigenfunctions of the dyadic differentiator are the Walsh functions, but the corresponding eigenvalues are, and therefore the definition is, to some extent arbitrary, depending on, among other things, the chosen enumeration (Paley or Kaczmarz) of the Walsh functions. Onneweer gives a definition such that the set of Walsh functions is partitioned into classes each having a common eigenvalue: this partition is independent of the choice of enumeration, and consequently so is the definition. The eigenvalues of the respective classes of Walsh functions are analogues of frequency [22–24]. This suggests that this type of definition may be better adapted to applications in the exact sciences. This definition has subsequently been extended to functions defined on *p*-adic and *p*-series fields [48, 50], *p*-adic groups [49], and a local field by Onneweer [51], [52].

In [80] and [81], presented is the definition of a derivative in terms of pseudo-differential operators.

There have been defined also L^r-weak *p*-adic derivative, the adjacent *p*-adic derivative, the partial *p*-adic derivative [16, 39, 78]. Extensions of Gibbs differentiation to *a*-adic groups has been provided in [40], and [43].

Further contributions to the theory of Gibbs differential operators on locally compact Vilenkin groups are done in [85, 86].

4.5 Gibbs derivatives with respect to Haar functions and generalizations

By analogy with the conventional dyadic calculus (related to Walsh functions), derivatives and antiderivatives have been introduced related to the Haar system of orthogonal functions. These definitions enable a calculus to be built up that plays the same role in Haar-Fourier analysis as the conventional dyadic calculus plays in Walsh-Fourier analysis [5, 55].

A Haar derivative for complex-valued functions on a finite dyadic group has been defined by Stanković and Stojić [73], using an analogue of the matrix representation of the discrete dyadic derivative, see, for instance, [71]. These authors (1987) use the same approach to define a family of differentiators having the discrete generalised Haar functions [1] as eigenfunctions. A generalisation of the Gibbs derivative to an arbitrary orthogonal basis in the space of functions from an arbitrary finite Abelian group to the complex, or a finite (Galois), field is given in [66, 68]. Extension of Gibbs differentiation to functions with finite fields as codomains had been discussed earlier by Cohn-Sfetcu and Gibbs [11]. The Gibbs derivatives for multiple-valued logic functions have been discussed in [70].

5 Recent Results

Recent development of Gibbs differentiation have been done in two-parameter differentiation, as for instance, the fundamental theorem of two-parameter pointwise derivative on Vilenkin groups [17] has been presented in [18, 45]. The monograph [82] is the first monograph which considers the theory of more-parameter dyadic and classical Hardy spaces, including also d-dimensional dyadic derivatives. See, also [83, 84]. Different generalizations of the Gibbs dyadic derivative have been used in study of local fields [87–89].

From the point of view of practical applications, research have been concentrated in applications of various generalizations of Gibbs differential operators in theory of fractals and fractal functions. Notice, for instance, that in [62], it is introduced a kind of Weierstrass-like function in p-series local field and discussed their its p-adic derivative with targeted applications in determination of the ”rate of change” of fractal functions in local fields. In [63], the 3-adic Cantor functions on 3-series field is constructed and its 3-adic derivative evaluated, and shown that it has at most $\ln 2/\ln 3$ order. A recently compiled bibliography on Gibbs derivatives comprises 313 bibliographic items [69].

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