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p-Adic Calculus and its Applications to Fractal Analysis and Medical Science

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Abstract: Recently, we reconsider continuously definitions and properties of Gibbs derivatives on p adic field $\mathbb{Q}_p$, by using so called pseudu-differential operators T^{α}, we find the kernel κ_{α} of T^{α}, as well as the spectra of it. Moreover, we show some applications of T^{α} to fractal analysis and medical science.

Keywords: Fractal analysis, Gibbs derivatives, p-adic calculus, p-adic field, medical sciences.

1 Introduction

SINCE MATHEMATICIAN J.E.Gibbs defined the Gibbs derivatives in 60' to 70's last century, lots of mathematicians in all of the world pay their great attention to this new concept and have done many many interesting research work in the new topics, see [1,2], including Chinese mathematician ([3]). Then, we do our research continuously on the Gibbs calculus, and pay attention to differential equations in Gibbs derivatives sense, we called them p-adic calculus.

We concentrate our mind in the following 3 problems in this note:

1. history and new topics of p-adic calculus in Section 2
2. new development of p-adic calculus over p-adic field $\mathbb{Q}_p$ in Section 3
3. applications of p-adic calculus to fractal analysis and medical sciences in Section 4

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2 History and New Topics of p-Adic Calculus

Recall that since 60's - 70's last century J.E.Gibbs introduced the concept of Gibbs derivatives ([4]), mathematicians P.L.Butzer et al ([5]), J.Pal et al ([6, 7]), C.W.Onneweer ([8, 9]), F.X.Ren et al ([10]) published many interesting papers over dyadic compact groups and p-adic compact groups, as well as about properties of Walsh function system. Then, at 80's, R.S.Stankovic ([11]) had important contribution for non-Abelian groups. Moreover, for p-adic locally compact groups, for local fields, some fundamental papers appeared ([9, 12, 13]) in 80's - 90's. Anyway, mathematicians in China who pay the attention to the topics about Gibbs derivatives since they find that this kind of derivatives and integrals (called " p-adic calculus" by Chinese mathematicians) is very useful in the study of fractal analysis and medical science.

Recently, R.S.Stankovic gives a nice survey ([1, 2]) and the survey about the contribution of Chinese given by Su ([3]) show that the study of p-adic calculus is very active and the new calculus has very strong vitality.

For a local field K, denote $\bigoplus$ its addition operation, and $\bigotimes$ its multiplication operation.

To develop p-adic calculus, we bring forward the following open problems:

1. Development for the study of p-adic calculus on p-adic field $\mathbb{Q}_p$, mainly, p-adic partial differential equations, for example, spectrum theory, kernels, fundamental solutions, and so on. then, develop the p-adic PDE theory to a general local field, including to that of finite algebraic extensions of p-series field and p-adic field to establish the p-adic differential equation theory on general local fields.

2. Development for the study of p-adic calculus on the product group $(K^*, \bigotimes)$ of a local field K, including Fourier analysis (such as Mellin-Fourier transform, function spaces, distribution theory, and so on). Moreover, for that of a mapping $f : K \to K$ and its p-adic calculus, as well as some corresponding problems.

3. Development for the applications of p-adic calculus to fractal analysis, life science and medical science. We are interested in that of to determine malign of liver's tumors, whether it is cancer or not by virtue of computing the Hausdorff dimensions of a boundary of a tumor. Moreover, how to determine which genes act to liver's cancer, so that doctors can determine a suitable scheme to patients.

3 New Development of p-Adic Calculus Over $\mathbb{Q}_p$

To develop fractal PDE on a local field, we start from p-adic PDE theory on the p-adic field $\mathbb{Q}_p$ (see [14]).

Let $p \geq 2$ be a prime, $\mathbb{Q}$ the rational number field. For x in $\mathbb{Q}$, its non-archimedean norm $|x|_p$ is defined as: $|x|_p = 0$ if $x = 0$; $|x|_p = p^{-r}$ if $x = p^r \frac{m}{n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $|m|$ and n are not divisible by p. The completion of the rational number field $\mathbb{Q}$ with respect to the non-archimedean norm $|x|_p$ is the p-adic field $\mathbb{Q}_p$.

$\mathbb{Q}_p$ can be also expressed as

$$\mathbb{Q}_p = \{x = (x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0, x_1, \cdots) : x_j \in \{0, 1, \cdots, p-1\}, s \in \mathbb{P}\}$$

with addition coordinately $\oplus$ mod p, $\mathbb{P} = \{0\} \bigcup \mathbb{N}$, and multiplication $\bigotimes$ mod p. If we endow a topology $\tau = \{B_k\}_{k \in \mathbb{P}}$ to $\mathbb{Q}_p$, where

$$B_k = \{x = (0, \cdots, 0, x_k, x_{k+1}, \cdots) \in \mathbb{Q}_p : x_j \in \{0, 1, \cdots, p-1\}, j \geq k, x_k \neq 0\}$$

is a k-neighborhood of $0 \in \mathbb{Q}_p$, then $\mathbb{Q}_p$ becomes a non-discrete, totally disconnected, locally compact topological field; and the Walsh system

$$\Gamma = \mathbb{Q}_p{}^{\wedge} = \{w_p(x,y) = \exp \frac{2\pi i}{p} x \otimes y : x, y \in \mathbb{Q}_p\}, \quad x \otimes y = \sum x_{1-j} y_j$$

is the character group of $\mathbb{Q}_p$.

If $f : \mathbb{Q}_p \to \mathbb{C}$ is a complex valued Haar measurable function on $\mathbb{Q}_p$.

We try to give principles to define ”rate of change” of f, or call it ”derivative”, denote by $f^{<\alpha>}$, $\alpha \geq 0$, then, it should satisfy the following :

1. operation property — ”derivative” as an operation, it has inverse operation ”integral”;
2. spectrum property — ”derivative” as a signal, its Fourier transform $[f^{<\alpha>}]^{\wedge}$ has a relationship between the Fourier transform $f^{\wedge}$ of f;
3. smoothness property — ”derivative” as a tool to describe smoothness, it satisfies the equivalent theorem in approximation theory, i.e., the Jackson theorem and Bernstein theorem;
4. dual property — ”derivative” as an eigen-function, it belongs to the character group of $\mathbb{Q}_p$, and has corresponding eigen-equation and eigen-value.

We define p-adic pseudo-differential operator on $\mathbb{Q}_p$

Definition 1. A complex-valued function φ defined on $\mathbb{Q}_p$ is called locally-constant if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x), \quad \forall x' \in B_{l(x)}.$$

Denote by $\mathscr{E} = \mathscr{E}(\mathbb{Q}_p)$ and $\mathscr{S} = \mathscr{S}(\mathbb{Q}_p)$ the linear spaces of locally-constant functions on $\mathbb{Q}_p$ and locally-constant functions with compact supports, respectively. $\mathscr{S}$ is also called the test function class. Denote by $\mathscr{S}' = \mathscr{S}'(\mathbb{Q}_p)$ the distribution space on $\mathscr{S}$. $\mathscr{S}'$ is a complete topological space.

Definition 2. For $\xi \in \mathbb{Q}_p$, denote $\langle\xi\rangle = \max\{1, |\xi|_p\}$, then $\langle\xi\rangle \in \mathscr{E}$. For $\alpha \in \mathbb{R}$, $\varphi \in \mathscr{S}$, T^α is a pseudo-differential operator with the symbol $\langle\xi\rangle^\alpha$ owing to the formula that

$$T^\alpha\varphi = (\langle\xi\rangle^\alpha\varphi^\wedge)^\vee.$$

It is easy to check that $T^\alpha\varphi$ exists in $\mathscr{S}$. The definition domain of T^α can be extended to the distribution space $\mathscr{S}'$ by the relation

$$(T^\alpha f, \varphi) = (f, T^\alpha\varphi), \quad f \in \mathscr{S}', \quad \varphi \in \mathscr{S}.$$

Hence for $f \in \mathscr{S}'$, we also have

$$T^\alpha f = (\langle\xi\rangle^\alpha f^\wedge)^\vee.$$

And also $T^\alpha f$ exists in $\mathscr{S}'$.

The operator T^α is said to be a p-adic derivative operator on $\mathscr{S}'$ of order α for $\alpha > 0$, and a p-adic integral operator on $\mathscr{S}'$ of order $-\alpha$ for $\alpha < 0$. For $\alpha = 0$, $T^0 f = f$ for all $f \in \mathscr{S}'$, T^0 is the identity operator.

The new results for operator T^α over the field $\mathbb{Q}_p$ is obtained recently ([14]).

Theorem 1. Let $\mathbb{Q}_p$ be the p-adic field, then we have

(i) convolution kernel κ_α of T^α — for $\alpha \in \mathbb{R}$, and $\Delta_k(x) = 1_{B_k}$, $k \in \mathbb{Z}$, $x \in \mathbb{Q}_p$, 1_E the characteristic function of E, then

(a) $\kappa_\alpha = (\frac{1-p^\alpha}{1-p^{-\alpha-1}}\pi_{-\alpha} + \frac{p^\alpha-1}{p^{\alpha+1}-1})\Delta_0$,
for $\alpha \neq 0, -1$, where π_α is defined by

$$(\pi_\alpha, \varphi) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}(\varphi(x) - \varphi(0))dx, \quad \varphi \in \mathscr{S}, \quad \alpha \in \mathbb{R}, \alpha \neq 0;$$

(b) $\kappa_0 = \delta$, where $\delta \in \mathscr{S}'$ is the Dirac distribution;

(c) $\kappa_{-1} = (1 - \frac{1}{p})(1 - \log_p|x|_p)\Delta_0$.
And the kernel κ_α has the properties.

(d) semi-group property: $\kappa_\alpha * \kappa_\beta = \kappa_{\alpha+\beta}$;

(e) Fourier transform: $\kappa_\alpha^\wedge = \langle\xi\rangle^\alpha$;

(f) continuous properties:

$$\lim_{\alpha\to 0}\kappa_\alpha = \delta, \text{ in } \mathscr{S}',$$
$$\lim_{\alpha\to -1}\kappa_\alpha = \kappa_{-1}, \text{ in } \mathscr{S}'.$$

(ii) convolution operator —

(g) $T^\alpha f = \kappa_\alpha * f, \quad \forall f \in \mathscr{S}'$;

(h) $T^\alpha T^\beta f = T^{\alpha+\beta} f = T^\beta T^\alpha f$.

Theorem 2. The eigenvalues of the convolution operator T^α are

(a) $\{\lambda_N\}_{N\in\mathbb{Z}^+} = \{1, p^\alpha, p^{2\alpha}, \cdots\}$, for $\alpha > 0$;

(b) $\{\lambda_N\}_{N\in\mathbb{Z}^+} = \{1\}$, for $\alpha = 0$;

(c) $\{\lambda_N\}_{N\in\mathbb{Z}^+} = \{\cdots, p^{2\alpha}, p^\alpha, 1\}$, for $\alpha < 0$.

And the eigenfunctions of T^α has the following property:

(d) the set of test functions $\{\psi_{NjI}\}$ is an orthonormal basis of eigen-functions of T^α in L^2, where for $j = 1, 2, \cdots, p-1$

$$\psi_{NjI}(x) = p^{-\frac{N}{2}}\chi_p(p^{N-1}jx)\Delta_0(p^N x - z_I), \quad N \in \mathbb{Z}, \quad I = z_I + B_0 \in \mathbb{Q}_p/B_0.$$

Moreover,

$$T^\alpha \psi_{1-N,j,I}(x) = \begin{cases} p^{N\alpha}\psi_{1-N,jI}(x), & \text{for } N > 0, \\ \psi_{1-N,jI}(x), & \text{for } N \le 0. \end{cases}$$

4 Applications of *p*-Adic Calculus to Fractal Analysis and Medical Sciences

1. To fractal analysis —

(a) Establish function spaces on local fields ([3, 15, 16]). Triebel B-type spaces $B_s^{p,q}(K)$, F-type spaces: $F_S^{p,q}(K)$, including Bosov type spaces, Sobolev type spaces, Lebesgue type spaces are established over local fields. Moreover, the Holder type spaces $C^\alpha(K)$, Lipschtiz type classes $Lip(\alpha, K)$ over local fields play very important role in the fractal analysis, and as we know that *p*-adic calculus plays a core stone role.

(b) Define several dimensions for fractals lived in local fields ([17, 18]), and construct some fractal sets, fractal functions, as well as evaluate dimensions of them ([19–21])

(i) define measures and dimensions: Hausdorff measure and Hausdorff dimension, Packing measure and Packing dimension, box-counting dimension, distribution dimension, and study properties of these measures and dimensions.

(ii) construct 3-adic Cantor set, 3-adic Cantor function, and Weierstrass-like function; moreover, evaluate the p-adic derivatives and p-adic integrals.

(c) discuss p-adic partial differential equations with fractal boundaries, search solutions of this kind of PED([22]), and try to establish fractal dynamics over local fields (see Fig.1).

$$\begin{cases} \dfrac{\partial^{<2>}u}{\partial t^{<2>}} = \dfrac{\partial^{<2>}u}{\partial x^{<2>}} + \dfrac{\partial^{<2>}u}{\partial y^{<2>}}, & t>0, (x,y)\in D, \\ u|_{t=0} = \varphi(x,y), & (x,y)\in D, \\ \dfrac{\partial^{<1>}u}{\partial t^{<1>}}\Big|_{t=0} = \psi(x,y), & (x,y)\in D, \\ u\big|_{\gamma} = 0, & t>0, \end{cases}$$

where $\dfrac{\partial^{<2>}u}{\partial t^{<2>}}, \ldots,$ are p-adic partial derivatives, $\varphi(x,y)$, $\psi(x,y)$ are defined in D, and can be express in convergent series

$$\varphi(x,y) = \sum_{k=1}^{\infty} \varphi_k(x,y), \quad \psi(x,y) = \sum_{k=1}^{\infty} \psi_k(x,y),$$

and $\gamma = \partial D$ is a p-adic Von Koch type curve.

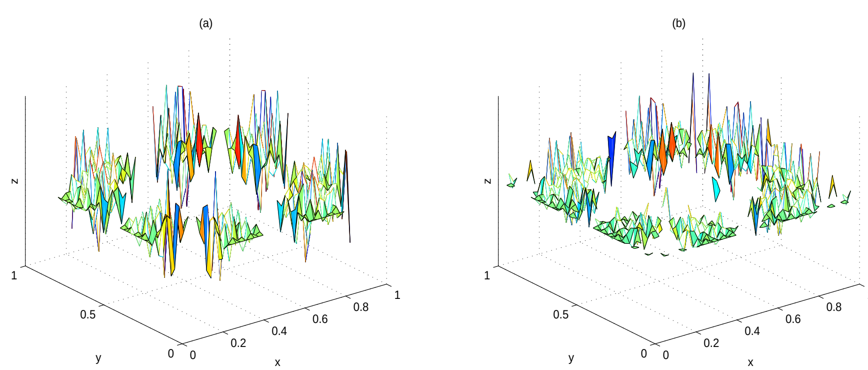

Fig. 1. Graph of a formal solution (Weierstrass type function)

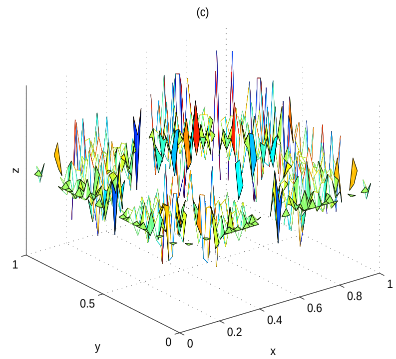

Fig. 1. Continue: Graph of a formal solution (Weierstrass type function)

2. To medical sciences —

(a) To determine malignant cases of liver's cancer we evaluate fractal dimensions (including Hausdorff dimensions, box-counting dimensions) of slices of liver's cancer (see MRI of some patients in fig.2) to help surgeons for analyzing situations of illness of a patient, and determining malignant of liver's cancer.

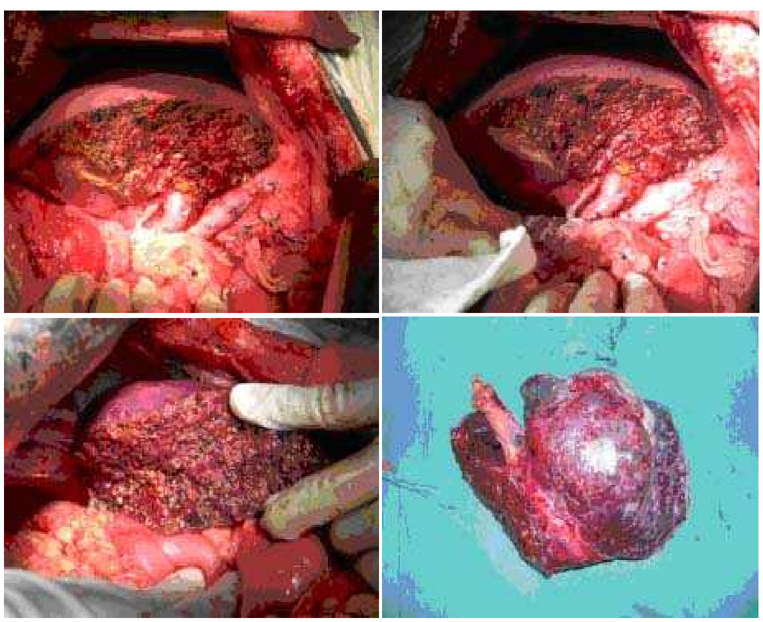

Fig. 2. MRI of liver's cancer of some patients.

(b) To determine volumes of livers for auxiliary partial orthotopic liver transplantation we evaluate volume of supplying original part, so that the transplantation has a best effect, that is, the supplying body and accepting body both keep the best states (see fig.3).

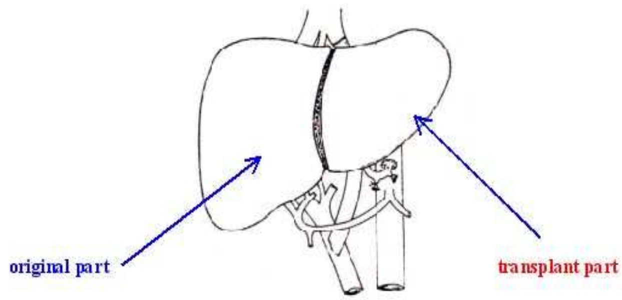

Fig. 3. Auxiliary partial orthotopic liver transplantation.

(c) To determine genes which control liver's cancer we evaluate percents of effects of every gene to liver's cancer of a patient, the data come from CMOS chips.

The references for that of local fields, Vilenkin groups and fractals, we refer to [3, 13, 15, 16, 22, 28–31].

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