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Quasi-Measures and Walsh Series

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Abstract: Properties of quasi-measures on the dyadic group G and on the product group G^d are considered and applications of this properties to the theory of the uniqueness of Walsh series are discussed.

Keywords: Walsh series, dyadic group, product group.

1 Preliminaries

LET G be *the dyadic group* [1–3]. The dyadic group is a set of sequences $t = \{t_i\}_{i=0}^{\infty}$ where $t_i = 0$ or 1. The mapping $\phi(t) = \sum_{i=0}^{\infty} t_i 2^{-i-1}$ establishes the one-one correspondence between G and the so-called *modified segment* J^*. The modified segment $J^* = [0,1]^*$ can be interpreted as the closed segment $[0,1]$ in which the dyadic rational points are counted twice: the 'left' point $p/2^k - 0$ corresponds to the infinite dyadic expansion and the 'right' point $p/2^k + 0$ corresponds to the finite expansion. The topology in G is defined by the system of neighborhoods $V_k = \{t = \{t_i\} : t_i = a_i, i \leq k-1\}$. The corresponding neighborhoods in J^* are the segments $[p/2^k + 0, (p+1)/2^k - 0]$. We shall identify G and J^*.

Let $\{\omega_n(t)\}_{n=0}^{\infty}$ be the *Walsh-Paley system* on G [2–4]. Fix natural $d \geq 1$. If $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+$, and $\mathbf{t} = (t^1, \ldots, t^d) \in G^d$, then the *d-dimensional Walsh function* $\omega_{\mathbf{n}}(\mathbf{t})$ is defined by

$$\omega_{\mathbf{n}}(\mathbf{t}) = \prod_{i=1}^{d} \omega_{n_i}(t^i).$$

Let

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} c_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{t}) \tag{1}$$

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be a *d-dimensional Walsh series* on G^d with real coefficients $c_{\mathbf{n}}$. For $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{Z}_+$ the **N**-*th rectangular partial sum* $S_{\mathbf{N}}$ of the series (1) at a point $\mathbf{t}$ is

$$S_{\mathbf{N}}(\mathbf{t}) = \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_d=0}^{N_d-1} c_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{t}).$$

The series (1) *rectangularly converges* to sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \text{ as } \min_i \{N_i\} \to \infty.$$

Let $\rho \in (0,1]$; then the series (1) *ρ-regularly converges* to sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } \min(N_i/N_j) \geq \rho.$$

Consider intervals

$$\Delta = \left[\frac{p_1}{2^{k_1}} + 0, \frac{p_1+1}{2^{k_1}} - 0\right] \times \ldots \times \left[\frac{p_d}{2^{k_d}} + 0, \frac{p_d+1}{2^{k_d}} - 0\right] \subset (J^*)^d \qquad (2)$$

where $k_s = 0, 1, \ldots,$ $p_s = 0, \ldots, 2^{k_s} - 1$. We call those intervals *dyadic intervals of rank* $\mathbf{k} = (k_1, \ldots, k_d)$. If Δ is a dyadic interval of rank $\mathbf{k}$, then $|\Delta|$ denotes its Haar measure, i.e. $2^{-(k_1+\ldots+k_d)}$. By $\operatorname{reg}\Delta$ we understand the *parameter of a regularity* of the dyadic interval Δ [5], i.e.

$$\operatorname{reg}\Delta = \min_{i,j=1,\ldots,d} 2^{k_i - k_j}.$$

Consider a point $t^i \in J^*$. We say that the sequence $\{\Delta_{k_i}\}$ of one-dimensional dyadic intervals is the *basic sequence convergent to t^i* [6] if $t^i \in \Delta_{k_i}$ for all k_i and rank of Δ_{k_i} equals k_i. Then the d-multiple sequence $\{\Delta_{\mathbf{k}}\}$ of d-dimensional dyadic intervals is the *basic sequence convergent to* $\mathbf{t} = (t^1, \ldots, t^d) \in (J^*)^d$ if

$$\Delta_{\mathbf{k}} = \Delta_{k_1} \times \ldots \times \Delta_{k_d} \qquad (3)$$

where $\{\Delta_{k_i}\}$ is the one-dimensional basic sequence convergent to t^i.

2 Quasi-Measures on G^d.

Let $\mathscr{B}$ denotes the family of all dyadic intervals (2). We consider some properties of *$\mathscr{B}$-interval functions* $\tau : \mathscr{B} \to \mathbb{R}$. By a quasi-measure on G^d we mean a finitely additive $\mathscr{B}$-interval function [3]. If $\mathbf{k} = (k_1, \ldots, k_d)$, then we denote by $2^{\mathbf{k}}$ the vector

$(2^{k_1},\dots,2^{k_d})$. For series (1) we define $\mathscr{B}$-interval function ψ associated with this series via

$$\psi(\Delta) = S_{2^{\mathbf{k}}}(\mathbf{t})|\Delta| \tag{4}$$

where Δ denotes the dyadic interval of rank $\mathbf{k}$ such that $\mathbf{t} \in \Delta$. It is known that ψ is a quasi-measure. The correspondence established by formula (4) between series (1) and quasi-measures is one-to-one (it is even a linear isomorphism if the set of the series (1) and the set of quasi-measures are naturally endowed with the structure of a vector space). It is known that any series (1) is the Fourier-Stieltjes series for the quasi-measure associated with this series. The next fact is also well-known.

Theorem 1 *Let $f \in L_1(G^d)$, (S) be a Fourier series of the function f, and ψ be the quasi-measure associated with this series. Then*

$$\psi(\Delta) = \int_\Delta f(\mathbf{t})\,d\mathbf{t}$$

for every diadic interval Δ.

Recall some definition [5]. Let τ be a quasi-measure, and $\rho \in (0,1]$. *Upper dyadic ρ-regular derivative* of the quasi-measure τ at a point $\mathbf{t} \in G^d$ is defined by

$$\overline{D}_d^\rho \tau(\mathbf{t}) \stackrel{def}{=} \overline{\lim} \frac{\tau(\Delta)}{|\Delta|} \quad \text{as} \quad |\Delta| \to 0, \quad \operatorname{reg}\Delta \geq \rho, \quad \mathbf{t} \in \Delta.$$

3 A Continuity of Quasi-Measures.

Consider different types of a continuity of quasi-measures. A quasi-measure τ is called *continuous in the sense of Saks* [7] if

$$\lim \tau(\Delta) \to 0 \ \text{ as } \ |\Delta| \to 0. \tag{5}$$

A $\mathscr{B}$-interval function τ is *strongly continuous at a point* $\mathbf{t} \in G^d$ [5] if

$$\lim \tau(\Delta) \to 0 \ \text{ as } \ |\Delta| \to 0, \ \mathbf{t} \in \Delta. \tag{6}$$

Let $\rho \in (0,1]$; then we say that a function τ is *ρ-continuous at a point* $\mathbf{t} \in G^d$ if

$$\lim \tau(\Delta) \to 0 \ \text{ as } \ |\Delta| \to 0, \ \operatorname{reg}\Delta \geq \rho, \ \mathbf{t} \in \Delta. \tag{7}$$

It is clear that in the one-dimensional case $(6) \Leftrightarrow (7)$. It is obviously that for all $\rho \in (0,1]$ and $t \in G^d$ $(5) \Rightarrow (6) \Rightarrow (7)$.

4 Quasi-Measures and the Coefficients of Walsh Series

In the case $d = 1$ we consider the following conditions for coefficients and partial sums of the series (1):

$$\lim_{n\to\infty} 2^{-n} S_{2^n}(t) = 0 \tag{8}$$

(Crittenden-Shapiro condition [8]);

$$\lim_{n\to\infty} c_n = 0. \tag{9}$$

The next results follows from (4).

Proposition 1 *Let (S) be a series of the form (1), ψ be the quasi-measure associated with this series, and $t \in G$. If the partial sums $S_{2^n}(t)$ of the series (S) satisfies the condition (8), then the quasi-measure ψ is strongly continuous at the point t. If the coefficients of the series (S) satisfies the condition (9), then the quasi-measure ψ is continuous in the sense of Saks. Assume that the series (S) converges to a finite sum at some point $\mathbf{t}_0 \in G$; then the quasi-measures ψ is continuous in the sense of Saks.*

The next statement was proved in [9].

Proposition 2 *Assume that for $d = 2$ the series (1)* ***rectangularly*** *converges to a finite sum at every point of a 'cross' $(\{a\} \times [0,1]) \bigcup ([0,1] \times \{b\})$. Then the quasi-measure ψ associated with this series is continuous in the sense of Saks.*

In the case of ρ-regular convergence the statements of the last theorems can fail to hold even for everywhere convergence of the appropriate series. This fact follows from the next theorem [10].

Theorem 2 *For every $\rho \in (0,1]$ there exists a double Walsh series which is ρ-regularly convergent to a finite sum everywhere on G^d, but the quasi-measure ψ associated with this series is not $\rho/4$-continuous at some point $\mathbf{t} \in G^d$. As corollary this quasi-measure is not continuous in the sense of Saks.*

The continuity in the sense of Saks was used for the solving the problem of recovery the coefficients of rectangularly convergent multiple Walsh series [11,12].

5 Σ_d-Continuity and Uniqueness Problems for Multiple Walsh Series.

The next type of continuity was offered in [13, 14]. Put

$$\Sigma_d = \{\sigma = (\sigma_1, \dots, \sigma_d) : \sigma_i = 0 \text{ or } 1 \text{ for all } i = 1, \dots, d\}; \quad |\sigma| = \sum_{i=1}^{d} |\sigma_i|.$$

Let $\{\Delta_{\mathbf{k}}\}$ be the basic sequence of the form (3) convergent to a point $\mathbf{t} \in G^d$. Put

$$\Delta^0_{k_i} = \Delta_{k_i+1}, \;\; \Delta^1_{k_i} = \Delta_{k_i} \setminus \Delta_{k_i+1}; \;\; \text{if } \sigma \in \Sigma_d, \text{ then } \; \Delta^{\sigma}_{\mathbf{k}} = \Delta^{\sigma_1}_{k_1} \times \ldots \times \Delta^{\sigma_d}_{k_d}.$$

We say that a function τ is Σ_d*-continuous at a point* $\mathbf{t}$ if

$$\lim_{k_1 = \ldots = k_d \to \infty} \sum_{\sigma \in \Sigma_d} (-1)^{|\sigma|} \tau(\Delta^{\sigma}_{\mathbf{k}}) = 0. \tag{10}$$

It can be proved that if $\rho \le 1/2$ then $(7) \Rightarrow (10)$ at every point $t \in G^d$.

Theorem 3 *If $d = 1$ then* $(6) \Leftrightarrow (7) \Leftrightarrow (10)$.

Thus a study of Σ_d-continuity is important only in the multidimensional case. The next theorem [13] establishes the connection between this continuity and the coefficients of series (1).

Theorem 4 *Let (S) be a series of the form (1), ψ be the quasi-measure associated with this series, and $\rho \in (0, 1/2]$. If the coefficients of the series (S) satisfies the condition*

$$\lim c_{n_1,\dots,n_d} = 0 \quad \textit{as} \quad \min\{n_1, \dots, n_d\} \to \infty, \quad \min_{i,j=1,\dots,d} \{n_i/n_j\} \ge \rho,$$

then the quasi-measure ψ is Σ_d-continuous at every point $t \in G^d$. Assume that the series (S) ρ-regularly converges to a finite sum at some point $\mathbf{t}_0 \in G$; then the quasi-measures ψ is Σ_d-continuous at every point $t \in G^d$.

In the multidimensional case Σ_d-continuity was used for a study of questions of uniqueness for ρ-regular convergent multiple Walsh series. The next 'monotonicity theorem' for quasi-measures was proved in [10].

Theorem 5 *Suppose that the quasi-measure τ satisfies*

$$\overline{D}^1_d \tau(\mathbf{t}) \ge 0 \tag{11}$$

at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Let the function τ be Σ^*_d-continuous at every point $\mathbf{t} \in G^d$. Then $\tau(\Delta) \ge 0$ for every dyadic interval Δ.

The theorem 5 may be used for the proof the following fact concerning sets of uniqueness. Recall that a set L is called the *set of uniqueness* (or in short: a U-set) for a system $\{\varphi_{\mathbf{n}}\}$ if from the convergence of a series $\sum_{\mathbf{n}} c_{\mathbf{n}} \varphi_{\mathbf{n}}$ to zero outside the set L it follows that $c_{\mathbf{n}} = 0$ for all $\mathbf{n}$. The following statement for d-multiple Walsh series was obtained.

Theorem 6 *(See [10, 13]). Let a number $\rho \in (0, 1/2]$ be chosen. Then any finite or countable set $L \subset G^d$ is a U-set for the multiple Walsh system with ρ-regular convergence.*

The concept of Σ_d-continuity also was used for the solving the problem of recovery the coefficients of multiple Walsh series [10]. This concept is also helpful in the theory of the uniqueness of Haar series [14].

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