Quasi-Measures and Walsh Series

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Abstract: Properties of quasi-measures on the dyadic group $G$ and on the product group $G^d$ are considered and applications of this properties to the theory of the uniqueness of Walsh series are discussed.

Keywords: Walsh series, dyadic group, product group.

1 Preliminaries

Let $G$ be the dyadic group [1–3]. The dyadic group is a set of sequences $t = \{t_i\}_{i=0}^\infty$ where $t_i = 0$ or $1$. The mapping $\phi(t) = \sum_{i=0}^\infty t_i 2^{-i-1}$ establishes the one-one correspondence between $G$ and the so-called modified segment $J^*$. The modified segment $J^* = [0,1]^*$ can be interpreted as the closed segment $[0,1]$ in which the dyadic rational points are counted twice: the 'left' point $p/2^k - 0$ corresponds to the infinite dyadic expansion and the 'right' point $p/2^k + 0$ corresponds to the finite expansion. The topology in $G$ is defined by the system of neighborhoods $V_k = \{t = \{t_i\} : t_i = a_i, i \leq k - 1\}$. The corresponding neighborhoods in $J^*$ are the segments $[p/2^k + 0, (p + 1)/2^k - 0]$. We shall identify $G$ and $J^*$.

Let $\{\omega_n(t)\}_{n=0}^\infty$ be the Walsh-Paley system on $G$ [2–4]. Fix natural $d \geq 1$. If $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+$, and $t = (t^1, \ldots, t^d) \in G^d$, then the $d$-dimensional Walsh function $\omega_n(t)$ is defined by

$$\omega_n(t) = \prod_{i=1}^d \omega_{n_i}(t^i).$$

Let

$$\sum_{n=0}^\infty c_n \omega_n(t) \quad (1)$$

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be a \(d\)-dimensional Walsh series on \(G^d\) with real coefficients \(c_n\). For \(N = (N_1, \ldots, N_d) \in \mathbb{Z}_+^d\), the \(N\)-th rectangular partial sum \(S_N\) of the series (1) at a point \(t\) is

\[
S_N(t) = \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_d=0}^{N_d-1} c_n \omega_n(t).
\]

The series (1) \textit{rectangularly converges} to sum \(S(t)\) at a point \(t\) if

\[
S_N(t) \to S(t) \quad \text{as} \quad \min_i \{N_i\} \to \infty.
\]

Let \(\rho \in (0, 1]\); then the series (1) \(\rho\)-\textit{regularly converges} to sum \(S(t)\) at a point \(t\) if

\[
S_N(t) \to S(t) \quad \text{as} \quad \min_i \{N_i\} \to \infty \quad \text{and} \quad \min(N_i/N_j) \geq \rho.
\]

Consider intervals

\[
\Delta = \left[ \frac{p_1}{2^{k_1}} + 0, \frac{p_1 + 1}{2^{k_1}} - 0 \right] \times \cdots \times \left[ \frac{p_d}{2^{k_d}} + 0, \frac{p_d + 1}{2^{k_d}} - 0 \right] \subset (J^*)^d
\]

(2)

where \(k_s = 0, 1, \ldots, p_s = 0, \ldots, 2^{k_s} - 1\). We call those intervals \textit{dyadic intervals of rank} \(k = (k_1, \ldots, k_d)\). If \(\Delta\) is a dyadic interval of rank \(k\), then \(|\Delta|\) denotes its Haar measure, i.e. \(2^{-(k_1+\ldots+k_d)}\). By \(\text{reg} \Delta\) we understand the \textit{parameter of a regularity} of the dyadic interval \(\Delta\) [5], i.e.

\[
\text{reg} \Delta = \min_{i,j=1,\ldots,d} 2^{k_i-k_j}.
\]

Consider a point \(t^i \in J^*\). We say that the sequence \(\{\Delta_k\}\) of one-dimensional dyadic intervals is the basic sequence convergent to \(t^i\) [6] if \(t^i \in \Delta_k\) for all \(k_i\) and rank of \(\Delta_k\) equals \(k_i\). Then the \(d\)-multiple sequence \(\{\Delta_k\}\) of \(d\)-dimensional dyadic intervals is the basic sequence convergent to \(t = (t^1, \ldots, t^d) \in (J^*)^d\) if

\[
\Delta_k = \Delta_{k_1} \times \ldots \times \Delta_{k_d}
\]

(3)

where \(\{\Delta_k\}\) is the one-dimensional basic sequence convergent to \(t^i\).

### 2 Quasi-Measures on \(G^d\).

Let \(\mathcal{B}\) denotes the family of all dyadic intervals (2). We consider some properties of \(\mathcal{B}\)-interval functions \(\tau : \mathcal{B} \to \mathbb{R}\). By a quasi-measure on \(G^d\) we mean a finitely additive \(\mathcal{B}\)-interval function [3]. If \(k = (k_1, \ldots, k_d)\), then we denote by \(2^k\) the vector
For series (1) we define $B$-interval function $\psi$ associated with this series via

$$\psi(\Delta) = S_{2^k}(t)|\Delta|$$

where $\Delta$ denotes the dyadic interval of rank $k$ such that $t \in \Delta$. It is known that $\psi$ is a quasi-measure. The correspondence established by formula (4) between series (1) and quasi-measures is one-to-one (it is even a linear isomorphism if the set of the series (1) and the set of quasi-measures are naturally endowed with the structure of a vector space). It is known that any series (1) is the Fourier-Stieltjes series for the quasi-measure associated with this series. The next fact is also well-known.

**Theorem 1** Let $f \in L_1(G^d)$, $(S)$ be a Fourier series of the function $f$, and $\psi$ be the quasi-measure associated with this series. Then

$$\psi(\Delta) = \int_\Delta f(t) \, dt$$

for every diadic interval $\Delta$.

Recall some definition [5]. Let $\tau$ be a quasi-measure, and $\rho \in (0, 1]$. Upper dyadic $\rho$-regular derivative of the quasi-measure $\tau$ at a point $t \in G^d$ is defined by

$$D^\rho_d \tau(t) \overset{\text{def}}{=} \lim_{|\Delta| \to 0} \frac{\tau(\Delta)}{|\Delta|} \quad \text{as} \quad |\Delta| \to 0, \quad \text{reg} \Delta \geq \rho, \quad t \in \Delta.$$ 

### 3 A Continuity of Quasi-Measures.

Consider different types of a continuity of quasi-measures. A quasi-measure $\tau$ is called **continuous in the sense of Saks** [7] if

$$\lim \tau(\Delta) \to 0 \quad \text{as} \quad |\Delta| \to 0.$$ 

(5)

A $B$-interval function $\tau$ is **strongly continuous at a point** $t \in G^d$ [5] if

$$\lim \tau(\Delta) \to 0 \quad \text{as} \quad |\Delta| \to 0, \quad t \in \Delta.$$ 

(6)

Let $\rho \in (0, 1]$; then we say that a function $\tau$ is $\rho$-**continuous at a point** $t \in G^d$ if

$$\lim \tau(\Delta) \to 0 \quad \text{as} \quad |\Delta| \to 0, \quad \text{reg} \Delta \geq \rho, \quad t \in \Delta.$$ 

(7)

It is clear that in the one-dimensional case (6) $\iff$ (7). It is obviously that for all $\rho \in (0, 1]$ and $t \in G^d$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7).
4 Quasi-Measures and the Coefficients of Walsh Series

In the case $d = 1$ we consider the following conditions for coefficients and partial sums of the series (1):

$$\lim_{n \to \infty} 2^{-n} S_{2^n}(t) = 0$$

(8)  

(Crittenden-Shapiro condition [8]);

$$\lim_{n \to \infty} c_n = 0.$$  

(9)

The next results follows from (4).

**Proposition 1** Let $(S)$ be a series of the form (1), $\psi$ be the quasi-measure associated with this series, and $t \in G$. If the partial sums $S_{2^n}(t)$ of the series $(S)$ satisfies the condition (8), then the quasi-measure $\psi$ is strongly continuous at the point $t$. If the coefficients of the series $(S)$ satisfies the condition (9), then the quasi-measure $\psi$ is continuous in the sense of Saks. Assume that the series $(S)$ converges to a finite sum at some point $t_0 \in G$; then the quasi-measures $\psi$ is continuous in the sense of Saks.

The next statement was proved in [9].

**Proposition 2** Assume that for $d = 2$ the series (1) rectangularly converges to a finite sum at every point of a 'cross' $(\{a\} \times [0, 1]) \cup ([0, 1] \times \{b\})$. Then the quasi-measure $\psi$ associated with this series is continuous in the sense of Saks.

In the case of $\rho$-regular convergence the statements of the last theorems can fail to hold even for everywhere convergence of the appropriate series. This fact follows from the next theorem [10].

**Theorem 2** For every $\rho \in (0, 1]$ there exists a double Walsh series which is $\rho$-regularly convergent to a finite sum everywhere on $G^d$, but the quasi-measure $\psi$ associated with this series is not $\rho/4$-continuous at some point $t \in G^d$. As corollary this quasi-measure is not continuous in the sense of Saks.

The continuity in the sense of Saks was used for the solving the problem of recovery the coefficients of rectangularly convergent multiple Walsh series [11,12].
5 $\Sigma_d$-Continuity and Uniqueness Problems for Multiple Walsh Series.

The next type of continuity was offered in [13, 14]. Put

$$\Sigma_d = \{ \sigma = (\sigma_1, \ldots, \sigma_d) : \sigma_i = 0 \text{ or } 1 \text{ for all } i = 1, \ldots, d \}; \ |\sigma| = \sum_{i=1}^{d} |\sigma_i|.$$  

Let $\{\Delta_k\}$ be the basic sequence of the form (3) convergent to a point $t \in G^d$. Put

$$\Delta_{k_i}^0 = \Delta_{k_i+1}, \ \Delta_{k_i}^1 = \Delta_{k_i} \setminus \Delta_{k_i+1}; \ \text{if } \sigma \in \Sigma_d, \text{ then } \Delta_{k_i}^\sigma = \Delta_{k_i}^{\sigma_1} \times \cdots \times \Delta_{k_i}^{\sigma_d}.$$  

We say that a function $\tau$ is $\Sigma_d$-continuous at a point $t$ if

$$\lim_{k_1=\ldots=k_d \to \infty} \sum_{\sigma \in \Sigma_d} (-1)^{|\sigma|} \tau(\Delta_{k_i}^\sigma) = 0. \quad (10)$$

It can be proved that if $\rho \leq 1/2$ then (7) $\Rightarrow$ (10) at every point $t \in G^d$.

**Theorem 3** If $d = 1$ then (6) $\iff$ (7) $\iff$ (10).

Thus a study of $\Sigma_d$-continuity is important only in the multidimensional case. The next theorem [13] establishes the connection between this continuity and the coefficients of series (1).

**Theorem 4** Let $(S)$ be a series of the form (1), $\psi$ be the quasi-measure associated with this series, and $\rho \in (0,1/2]$. If the coefficients of the series $(S)$ satisfies the condition

$$\lim_{n_1,\ldots,n_d \to \infty} \min\{n_1,\ldots,n_d\} = \infty, \ \min_{i,j=1,\ldots,d} \{n_i/n_j\} \geq \rho,$$

then the quasi-measure $\psi$ is $\Sigma_d$-continuous at every point $t \in G^d$. Assume that the series $(S)$ $\rho$-regularly converges to a finite sum at some point $t_0 \in G$; then the quasi-measures $\psi$ is $\Sigma_d$-continuous at every point $t \in G^d$.

In the multidimensional case $\Sigma_d$-continuity was used for a study of questions of uniqueness for $\rho$-regular convergent multiple Walsh series. The next ‘monotonicity theorem’ for quasi-measures was proved in [10].

**Theorem 5** Suppose that the quasi-measure $\tau$ satisfies

$$\mathcal{D}_\tau \tau(t) \geq 0 \quad (11)$$

at every point $t \in G^d$ except possibly a countable set $L$. Let the function $\tau$ be $\Sigma_d$-continuous at every point $t \in G^d$. Then $\tau(\Delta) \geq 0$ for every dyadic interval $\Delta$. 
The theorem 5 may be used for the proof the following fact concerning sets of uniqueness. Recall that a set $L$ is called the set of uniqueness (or in short: a $U$-set) for a system $\{\varphi_n\}$ if from the convergence of a series $\sum_n c_n \varphi_n$ to zero outside the set $L$ it follows that $c_n = 0$ for all $n$. The following statement for $d$-multiple Walsh series was obtained.

**Theorem 6** (See [10, 13]). Let a number $\rho \in (0, 1/2]$ be chosen. Then any finite or countable set $L \subset G^d$ is a $U$-set for the multiple Walsh system with $\rho$-regular convergence.

The concept of $\Sigma_d$-continuity also was used for the solving the problem of recovery the coefficients of multiple Walsh series [10]. This concept is also helpful in the theory of the uniqueness of Haar series [14].

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**References**


