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Upper Bounds of Performances Guided Robust Hybrid Controller

Vojislav Filipović

Abstract: In this paper the multiple models and switching controllers is used for control of system with unmodeled dynamics. The analog part of the system is described by finite set of continuous models with unmodeled dynamics. As a set of controllers is used a finite set of LQ controllers with the prescribed degree of stability. Using linear matrix inequalites it is derived LQ controllers and upper bounds for index of performances. The set of upper bounds is used for creation of switching sequence. For so given switching system the robust asymptotic stability is proved.

Keywords: Multiple models, unmodeled dynamics, switching controllers, stability

1 Introduction

THE HYBRID dynamical system is a dynamical system that involved the interaction of discrete and continuous dynamics. Continuous variables take the values from the set of real numbers and discrete variables take the values from a finite set of symbols. Analog part of the hybrid system, in this paper, is described with differential equation [1]. The discrete part of hybrid system belongs to the area of discrete event system such as automata, max-plus algebra or Petry nets [2].

From the clasical control theory point of vew hybrid system can be considered as a switching control between analog feedback loops [3] and [4].

In the area of hybrid control system now we have different approaches. In [5] mixed logical model for hybrid system is proposed. The model is described by linear differential equation subject to linear mixed-integer inequalites. The system with inequality constraints can be described with complementarity class of hybrid

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The author is with Regional Center for Talents, PO Box 126, 15300 Loznica, Serbia (e-mail vfilip@eunet.yu).

system [6]. The very large class of hybrid system can be described as a picewise linear systems [7]. Equivalence between above classes of hybrid system is presented in [8].

All above system have the deterministic character. In some situation systems have the stochastic character. The stochastic frame for hybrid systems is introduced recently [9]. In that case the theory of point stochastic processes is used.

During the last decade the new approach for adaptive control is introduced [10–12]. When we have large parameters errors the clasic adaptive control results in a slow convergence with large transient errors. To overcome such problems in the new theory of adaptive control the concept of multiple models is proposed. So higher level of adaptive contol is introduced. Similar approach is taken, also, in [13] where mapping of hybrid state to hybrid control is based on system performance.

Theory of hybrid control system can be use in the field of system with quantization [14], Internet congestion control [15] and field of wireless communication networks [16].

In this paper we use the concept of multi model system for hybrid contol of hybrid dinamic system with the analog uncertainty. The unmodeled dynamic is described in the continuous domain. The discrete part of hybrid system is determined using upper bounds of index of performances. Finally, in the form of theorem robust stability of the closed-loop hybrid system is established.

2 System Description by Multiple Models

In this part of paper we consider multiple model description of process. It will be assumed that the process model is a member of admissible process models

$$\boldsymbol{F} = \bigcup_{p \in \boldsymbol{P}} \boldsymbol{F}_p \tag{1}$$

where $\boldsymbol{P}$ is matrix index set which represents the range of parametric uncertainty so that for each fixed $p \in \boldsymbol{P}$ the subfamily $\boldsymbol{F}_p$ accaunts for unmodeled dynamics. Usually, $\boldsymbol{P}$ is compact subset of finite-dimensional normed vector space [17].

In this paper we will suppose that process can be described with collection of linear time invariant system with unmodeled dynamics

$$\dot{x}(t) = A_p(w)\,x(t) + B_p u(t), \quad p = 1, 2, \ldots, s \tag{2}$$

where $x \in R^n$ and $u \in R^p$ are state and control signal of the system respectively. In relation (2) matrix $A_p(w)$ describes the unmodeled dynamics for the system in the form of matrix perturbation

$$A_p(w) = A_p + \Delta_p(w), \quad w \in W \tag{3}$$

where A_p is known matrix and $A_p(w)$ is a matrix perturbation with finite norm

$$\|\Delta_p(w)\|_q < \delta, \quad \delta > 0, \quad q = 1, 2, \ldots, \infty \tag{4}$$

Relation (2) describes the continuous part of the system. The event driven part can be described in the next form

$$p^+(t) = \varphi(p(t), \sigma(t)) \tag{5}$$

where $p(t)$ is a discrete event variable, $\sigma(t)$ is a discrete input and $\varphi(\cdot,\cdot)$ is a function which describes behaviour of $p(t)$. It is important to note that

$$p^+(t) = p(t_{n+1}), \quad p(t) = p(t_n), \quad t_n < t_{n+1} \tag{6}$$

Specific form of switching sequence will be described in the next part of the paper.

Remark 1: In [18] continuous-time model with unmodeled dynamics is considered

$$\dot{x}(t) = (A_p + \Delta A_p(w(t)))x(t) + (B_p + \Delta B_p(w(t)))u(t)$$

where uncertainty vector $w(t)$ is Lebesque mesaurable and within an allowable bounding set $\Omega \in R^p$ for all $t \in [0, \infty]$. But design of robust hybrid controller is completely different then in this paper.

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Remark 2: Uncertainty in model (2) can be much complex [9]. The very general description can be made using notion of polytop [19].

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3 The Switching Controller

In the general case, no single controller is capable of solving the regulation problem for the entire set of process models (1). Because we will use the family of controllers [11]

$$\{C_q : q \in \boldsymbol{D}\} \tag{7}$$

where D is index set. It is supposed that the family is sufficiently rich so that every admissible process model can be stabilized by controller C_q for some index $q \in$ D. In this paper will be considered the case

$$\boldsymbol{F} = \boldsymbol{D} \tag{8}$$

We will finde the member of family C_q using linear matrix inequalities (LMI) tool. The controllers will be robust LQ controllers with prescribed degree of stability.

Let us first introduce optimal LQ controller with prescribed degree of stability for the fixed p and $\Delta_p(w) = 0$ For cuch case system has the form

$$\dot{x}(t) = A_p x(t) + B_p u(t) \tag{9}$$

The index of performance is [20]

$$J = \int_{t_0}^{\infty} e^{2\alpha t} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) \mathrm{d}t, \quad \alpha > 0 \tag{10}$$

Optimal controller is

$$u(t) = -R^{-1} B_p^T P_p x(t) \tag{11}$$

whereby matrix P_p is a solution of algebraic Riccati equation

$$P_p(A_p + \alpha I) + \left(A_p^T + \alpha I\right) P_p - P_p B_p R^{-1} P_p + Q = 0 \tag{12}$$

It is well known fact that for LQ controller (10) - (12) the Lyapunov function has the form [21]

$$V(x) = x^T(t) P_p x(t) \tag{13}$$

We will use that fact for system with unmodeled dynamics. It will be used for closed-loop system

$$\dot{x}(t) = A_p(w) x(t) + B_p u(t), \quad w \in W \tag{14}$$

$$u(t) = -R^{-1} B_p^T P_p x(t) \tag{15}$$

The next goal is to finde matrix P_p without the help of Riccati equation (12). Instead, we will use the LMI tool. Result will be formulated in the form of theorem. Before that we will formulate known result from matrix theory which are necessery for proof of theorem.

Lemma 1 [21]. (Quadratic functional). Let us suppose that the $x(t)$ is solution for system $\dot{x}(t) = Ax(t)$, $x(0) = x_0$. Matrix A is stable and matrix $U > 0$. Then value of functional J_p

$$J_u = \int_0^\infty x^T(t) U x(t) \, \mathrm{d}t$$

equal to $x_0^T Y x_0$ whereby Y is solution of Lyapunov equation

$$A^T Y + YA = -U$$

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Lemma 2 [22]. (Lyapunov inequality). Let the a Hurvitz matrix, coulpe A, B is controlable and $Z_- > 0$ is a solution of Lyapunov equation

$$AZ + ZA^T = -T, \quad T = B^T B$$

The matrix Lyapunov inequality

$$AZ + ZA^T \leq -BB^T$$

has a solution and for every solution Z of Lyapunov inequality is

$$Z \geq Z_-$$

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Now we will, for fixed p, formulate theorem.

Theorem 1: Let us suppose that for the closed-loop system (14) - (15) is satisfied

1. For fixed p-th subsystem, couple

$$[A_p + \alpha I, B_p]$$

   is controlable
2. Matrices Q and R are positive definite
3. $X_p(\gamma_p)$ is solution of the next LMI

$$\begin{bmatrix} C_p & D_p \\ E_p & F_p \end{bmatrix} \leq 0$$

$$C_p = (A_p(w) + \alpha I) X_p + X_p [A_p(w) + \alpha I]^T + \gamma_p^{-2} B_p R^{-1} B_p^T$$

$$D_p = \gamma_p^{1/2} X_p Q^{1/2}, \quad E_p = \gamma_p^{1/2} Q^{1/2} X_p, \quad F_p = -I$$

$$X_p > 0, \quad w \in W$$

4. For $\forall\gamma_p > 0$

$$\gamma_p^* = \arg\min_{\gamma_p} \varphi(\gamma_p), \quad \varphi(\gamma_p) = \gamma_p^{-1} x_0^T X_p(\gamma_p) x_0$$

Then for feedback law

$$u(t) = -R^{-1} B_p^T \left(X_p\left(\gamma_p^*\right)\right)^{-1} x(t)$$

uper bound for index of performance is

$$J_p \leq \varphi\left(\gamma_p^*\right), \quad \forall w \in W$$

Proof: Let us introduce next transformation

$$\dot{\hat{x}}(t) = e^{\alpha t} x(t), \quad \hat{u}(t) = e^{\alpha t} u(t) \tag{16}$$

Now relations (14) and (16) have the form

$$\dot{\hat{x}}(t) = (A_p(w) + \alpha I)\hat{x}(t) + B_p \hat{u}(t) \tag{17}$$
$$\hat{u}(t) = -R^{-1} B_p^T P_p \hat{x}(t) \tag{18}$$
$$\dot{\hat{x}}(t) = \tilde{A}_p(w), \quad \tilde{A}_p(w) = A_p(w) + \alpha I - B_p R^{-1} B_p^T P_p \tag{19}$$

Using opservation (13) we have

$$\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = x^T(t)\left[\tilde{A}_p^T(w) P_p + P_p \tilde{A}_p(w) x(t)\right] x(t) \tag{20}$$

Condition $\dot{V} < 0$ will be satisfied if

$$\tilde{A}_p^T(w) P_p + P_p \tilde{A}_p(w) < 0 \tag{21}$$

From (19) and (21) we have

$$(A_p(w) + \alpha I)^T P_p + P_p(A_p(w) + \alpha I) - 2P_p B_p R^{-1} B_p^T P_p < 0 \tag{22}$$

Let us multipling last inequality from left and right side with the matrix $X_p = P_p^{-1}$. Follows

$$(A_p(w) + \alpha I)^T X_p + X_p(A_p(w) + \alpha I) - 2B_p R^{-1} B_p^T < 0, \quad X > 0 \tag{23}$$

Last relation is linear matrix inequality and unknown variable is matrix X. Let us introduce parameter $\gamma_p > 0$ and from (23) we have

$$\begin{aligned}&(A_p(w)+\alpha I)^T X_p + X_p(A_p(w)+\alpha I) - 2B_pR^{-1}B_p^T\\&+\gamma_p\left(B_pR^{-1}B_p^T + X_pQX_p\right), \quad X_p > 0\end{aligned} \tag{24}$$

Now we will use Lemma 1. Let us put in Lemma 1

$$A_p = \tilde{A}_p(w) = A_p(w) + \alpha I - B_pR^{-1}B_p^TX_p^{-1} \tag{25}$$

$$U = Q + X_p^{-1}B_pR^{-1}B_p^TX_p^{-1} \tag{26}$$

From Lemma 1 follows

$$\tilde{A}_p^T(w)Y_p + Y_p\tilde{A}_p(w) = -\left(Q + X_p^{-1}B_pR^{-1}B_p^TX_p^{-1}\right) \tag{27}$$

$$J_p = x_0^TX_px_0 \tag{28}$$

Let us multiplay inequality by (24) from left and right side with the matrix X_p^{-1}. We have

$$\begin{aligned}&(A_p(w)+\alpha I)^T X_p^{-1} + X_p^{-1}(A_p(w)+\alpha I) - 2X_p^{-1}B_pR^{-1}B_p^TX_p^{-1}\\&\le -\gamma_p\left(X_p^{-1}B_pR^{-1}B_p^TX_p^{-1}+Q\right)\end{aligned} \tag{29}$$

Last relation can be rewritten in the next form

$$\tilde{A}_p^T(w)\left(\frac{1}{\gamma_p}X_p^{-1}\right) + \left(\frac{1}{\gamma_p}X_p^{-1}\right)\tilde{A}_p(w) \le -Q + X_p^{-1}B_pR^{-1}B_p^TX_p^{-1} \tag{30}$$

If we substract relation (27) from relation (30) we have

$$\tilde{A}_p^T(w)\left(\frac{1}{\gamma_p}X_p^{-1} - Y_p\right) + \left(\frac{1}{\gamma_p}X_p^{-1} - Y_p\right)\tilde{A}_p(w) \le 0 \tag{31}$$

Using Lemma 2 one can conclude

$$\frac{X_p^{-1}}{\gamma_p} - Y_p \ge 0 \quad \text{and} \quad \frac{X_p^{-1}}{\gamma_p} \ge Y_p \tag{32}$$

From (28) and (32) one can get

$$J_p \le \frac{1}{\gamma_p}x_0^TX_p^{-1}x_0 \tag{33}$$

Using Schur's lemma [23] inequality (24) can be rewritten in the next form

$$\begin{bmatrix} C_p & D_p \\ E_p & F_p \end{bmatrix} \leq 0 \tag{34}$$

where $C_p = (A_p(w) + \alpha I) X_p + X_p (A_p(w) + \alpha I)^T + (\gamma_p - 2) B_p R^{-1} B_p^T$, $D_p = \gamma_p^{1/2} X_p Q^{1/2}$, $E_p = \gamma_p^{1/2} Q^{1/2} X_p$, $F_p = I$, $X_p > 0$

From inequality (34) one can finde solution $X_p(\gamma_p)$. After that we can construct function

$$\varphi(\gamma_p) = \gamma_p^{-1} x_0^T X(\gamma) x_0 \tag{35}$$

Then we finde

$$\gamma_p^* = \arg\min_{\gamma_p} \varphi(\gamma_p) \tag{36}$$

From relation (36) we can construct feedback law

$$u(t) = -R^{-1} B_p^T \left(X_p \left(\gamma_p^* \right) \right)^{-1} x(t) \tag{37}$$

and upper bound for index of performance is

$$J_p \leq \varphi\left(\gamma_p^*\right) \tag{38}$$

Theorem is proved.

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From Theorem 1 follows that problem to finde the LQ controller when, in the analog part of the hybrid system , exists unmodeled dynamic, is to solve the system of linear matrix inequalities. For such problem exists software tool. When the number of LMI is large one can use one of the known iterative procedures.

4 Robust Stability of Hybrid Control System

Using relation (14) system can be described in the form

$$\begin{aligned} &\dot{x}(t) = A_p x(t) + B_p u(t), + \Delta(x(t), p(t), w) \\ &\Delta(x(t), p(t), w) = \Delta_p(w) x(t), \\ &p = 1, 2, \ldots, s \end{aligned} \tag{39}$$

Now we will formulate hybrid control law.

The analog feedback

$$u(t) = -R^{-1} B_p^T (X_p(\gamma_p^*))^{-1} x(t), \quad p = 1, 2, \ldots, s \tag{40}$$

whereby $\left(X_p\left(\gamma_p^*\right)\right)^{-1}$ is solution of LMI which is defined in Theorem 1.

The discrete feedback

$$\begin{aligned} p_1 &= \arg\min\left\{\varphi\left(\gamma_{p_1}^*\right)\right\}, \\ \varphi\left(\gamma_p^*\right) &= \arg\min_{\gamma_p}\varphi\left(\gamma_p\right), \\ \varphi\left(\gamma_p\right) &= \gamma_p^{-1}x^T\left(t\right)X_p\left(\gamma_p\right)x^T\left(t\right), \\ p &= 1,2,\ldots,s \end{aligned} \tag{41}$$

Now we will formulate theorem in which is proved robust stability of hybrid systems.

Theorem 2: Let us suppose that for dynamic hybrid system (39) - (41) is valid

1° $\left\|A_p+\alpha I-B_pR^{-1}B_p^T\left(X_p\left(\gamma_p^*\right)\right)^{-1}\hat{x}\left(t\right)\right\| \leq k_1\varphi\left(\gamma_p^*\right)+c$

$\left\|A_p+\alpha I-B_pR^{-1}B_p^T\left(X_p\left(\gamma_p^*\right)\right)^{-1}\right\| \geq 1$

$k_1>0$, $c>0$, $p=1,2,\ldots,s$

2° $\left\|\hat{x}\left(t\right)\right\| \leq k_2\varphi\left(\gamma_p^*\right)$

$k_2>0$, $\left\|\hat{x}\left(t\right)\right\| \geq 1$, $p=1,2,\ldots,s$

Then

$\left\|x\right\|_\infty \leq \left(k_1+k_2\right)\varphi\left(\gamma_p^*\right)+1+c+r_\Delta$

$r_\Delta = \sup_{w\in W}\left\|\Delta\left(x\left(t\right),p\left(t\right),w\right)\right\|$

$\Delta\left(\hat{x}\left(t\right),p\left(t\right),w\right) = \Delta_p\left(w\right)\hat{x}\left(t\right)$

Proof: From relation (17) for any $\tau\in\left[t,t+1\right]$ we have

$$\begin{aligned} \hat{x}\left(t\right) = \hat{x}\left(\tau\right) &- \int_t^\tau\left[A_p+\alpha I-B_pR^{-1}B_p^T\left(X_p\left(\gamma_p^*\right)\right)^{-1}\right]\hat{x}\left(\theta\right)\mathrm{d}\theta \\ &+ \int_t^\tau\Delta_1\left(\hat{x}\left(\theta\right),p\left(\theta\right),w\left(\theta\right)\right)\mathrm{d}\theta \end{aligned} \tag{42}$$

Further we have

$$\begin{aligned} \left\|\hat{x}\left(t\right)\right\| \leq \left\|\hat{x}\left(\tau\right)\right\| &+ \int_t^\tau\left\|\left(A_p-\alpha I-B_pR^{-1}B_p^T\left(X_p\left(\gamma_p^*\right)\right)^{-1}\right)\hat{x}\left(\theta\right)\right\|\mathrm{d}\theta \\ &+ \int_t^\tau\left\|\Delta\left(\hat{x}\left(\theta\right),p\left(\theta\right),w\left(\theta\right)\right)\right\|\mathrm{d}\theta \end{aligned} \tag{43}$$

Let us introduce the next sets

$$\begin{aligned}\Omega_1 &= \left\{\tau \in [t,t+1] : \left\| \left(A_p - \alpha I - B_p R^{-1} B_p^T \left(X_p\left(\gamma_p^*\right)\right)^{-1}\right) \hat{x}(t) \right\| \le 1 \right\} \\ \Omega_2 &= \tau \in [t,t+1] - \Omega_1\end{aligned} \tag{44}$$

Now from (43) and condition 1° of theorem follows

$$\|x(t)\| \le \|x(\tau)\| + 1 + c + k_1 \varphi\left(\gamma_p^*\right) + r_{\Delta_1} \tag{45}$$

Using last inequality and condition 2° of Theorem we have

$$\begin{aligned}\|\hat{x}(t)\| &\le \int_t^{t+1} \left(\|\hat{x}(\theta)\| + 1 + c + k_1 \varphi\left(\gamma_p^*\right) + r_\Delta\right) \mathrm{d}\theta \\ &\le (k_2 + k_1)\, \varphi\left(\gamma_p^*\right) + 2 + c + r_\Delta\end{aligned} \tag{46}$$

Since the right-hand side is independent of t we have

$$\|\hat{x}(t)\|_\infty \le (k_1 + k_2)\, \varphi\left(\gamma_p^*\right) + 2 + c + r_\Delta \tag{47}$$

From relation (16) follows

$$\|x(t)\|_\infty \le \|\hat{x}(t)\|_\infty \tag{48}$$

Theorem is proved

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Remark 3: In this remark we will comment assumptions A) and B) of Theorem2. It is well known fact that optimally designed controllers via Riccati equations always guarantee stability. Such fact suggests that, if system performance indices are approprietelly selected, optimality of performance or boundedness of performance, will provide stability and robusness. Such idea is used in [13]. In this paper we prove that index of performanse for every closed-loop subsystem has a finite upper bound. Using that fact we generalize concept of performance dominant condition from [13]. Generalization has the form as in first two conditions in Theorem 2.

5 Conclusions

In this paper the problem of design of robust hybrid controller is considered. The approach is based on LMI tool. For fixed subsystem it is proved that index of performance has a explicit upper bound. The set of upper bounds form the basis for generation of swtching sequence. Proposed switching controller, in the presence of analog unmodeled dynamics, guarantee robust stability of feedback system.

References

[1] G. C. Goodwin, S. F. Graebe, and M. F. Salgado, *Control system Design*. New Jersey: Prentice-Hall, 2001.

[2] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*. London: Kluwer Academic Publisher, 1999.

[3] A. V. Savkin and R. J. Evans, *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*. Basel: Birkhauser, 2002.

[4] D. Liberzon, *Switching in Systems and Control*. Basel: Birkhauser, 2003.

[5] A. Bemporad and M. Morari, "Control of systems integrating logics, dynamics and constrains," *Automatica*, vol. 35, no. 3, pp. 407– 428, Mar. 1999.

[6] W. P. M. H. Heemels and B. Brogliato, "The complementarity class of hybrid dynamical systems," *European Journal of Control*, vol. 9, no. 2-3, pp. 322–360, Mar. 2003.

[7] M. Johansson, *Picewise Linear Control Systems: A Computational Approach*. Berlin: Springer-Verlag, 2003.

[8] W. P. M. H, Heemels, D. Schutter, and A. Bemporad, "Equivalence of hybrid dynamical systems," *Automatica*, vol. 37, no. 5, pp. 1085–1091, May 2001.

[9] J. Hespanaha, "Stochastic hybrid systems: Applications to communication networks," in *Hybrid systems: Computation and Control*, R. Alur and G. J. Paras, Eds. Berlin: Springer Verlag, 2004.

[10] S. R. Kulkarni and P. J. Ramadge, "Model and controller selection polices based on output prediction errors," *IEEE Trans. On Automatic Control*, vol. 41, no. 11, pp. 1594–1604, 1996.

[11] A. S. Morse, "Supervisory control of families of linear set-point controllers-part 1: Exact maching," *IEEE Trans. on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, Oct. 1996.

[12] K. S. Narendra and C. Xiang, "Adaptive control of discrete-time systems using multiple models," *IEEE Trans. on Automatic Control*, vol. 45, no. 9, pp. 1669–1686, Sept. 1996.

[13] L. Y. Wang, P. Khargonekar, and A. Beydoun, "Robust control of hybrid systems: Performance guided strategies," in *Hybrid Systems V*, P. Antsaklis, W. Kohn, M. Lemmon, A. Nerode, and S. Sastry, Eds. Berlin: Springer Verlag, 1999.

[14] D. Liberzon, "A hybrid control framework for system with quantization," in *Proc. of the 40th Conference of Decision and Control*, Orlando, Florida, USA, 2001, pp. 1217–1222.

[15] R. Srikant, *The Mathematics of Internet Congestion Control*. Basel: Birkhauser, 2004.

[16] E. A. Lee and P. Varaiya, *Structure and Interpretation of Signal and Systems*. London: Addison-Wesley, 2003.

[17] J. Hespanaha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. Brinsmead, and F. D. Bruyne, "Multiple model adaptive control. part 2: Switching," *Int. J. Robust Nonlinear Control*, vol. 11, no. 4, pp. 479–496, Apr. 2001.

[18] V. Filipovic, "Robust hybrid lq controller," in *European Control Conference*, Cambridge, UK, 2003.

[19] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge: Cambrige University Press, 2004.

[20] B. D. O. Anderson and J.B.Moore, "Linear system optimization with prescribed degree of stability," *Proc. IEE*, vol. 116, no. 12, pp. 2083–2087, Dec. 1969.

[21] B. T. Polyak and P. S. Shcherbakov, *Robust Stability and Control*. Moscow: Nauka, 2002, (in Russian).

[22] S. L. Boyd, L. E. Ghaovi, E. Feron, and V. Balakrishnan, "Linear matrix inequalites in system and control theory," *SIAM*, 1994.

[23] F. R. Gantmacher, *The Theory of Marices*. New York: Chelsea Publishing Co., 1959, vol. 1 and 2.