

## An Accurate Determination of Current Distribution Within the Earth

Miroslav Prša and Karolina Kasaš - Lažetić

**Abstract:** In electrical engineering the Earth appears frequently as a media of conducting currents. In order to protect humans of electrocuting, the utilities chassis are usually grounded, the lightning rods are grounded as well and sometime the Earth is applied as a return conductor. For that reason, it is necessary to investigate the Earth's behavior when it is used as a conducting media. The best way to do this investigation is to determine the current distribution within the Earth.

**Keywords:** Current distribution within the Earth, combination of numerical and analytical calculations, graphical presentation of current lines.

### 1 Introduction

In order to determine the current distribution in the Earth, let us assume the system shown in Figure 1. The system is composed of an overhead conductor, parallel to Earth's surface and, whereby, the Earth is having a part of the return conductor.

In the literature there are many efforts to solve the problem, but usually the authors made the approximations derived from the circuits theory, which were not admissible. Using the elements with concentrated parameters, i.e. grounding resistors, capacitors and inductances to describe the behavior in the Earth, many physical events cannot be taken into account. All those calculations are more or less accurate for basic frequency, but not so exact for higher harmonics. To avoid mathematical difficulties, they introduced some simplifications, which conceal the physical picture of the problem. The best review of applied methods is given in [1].

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The authors are with Department for Power, Electronics and Communications Engineering, Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovica 6, 21000 Novi Sad, Serbia

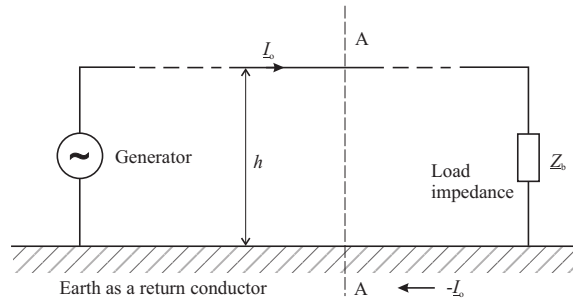


Fig. 1. The principle of the Earth as a return conductor.

For that reason, we decided to take the physical base as an essential starting-point and the applied mathematical methods will be only the consequence of such a treatment. In this paper the Earth as a return conductor is assumed to have infinite cross section to conduct the current  $-I_0$  from the load impedance  $Z_b$  towards generator. But, due to skin effect and the influence of overhead conductor, current density vector will be not uniformly distributed through this infinite cross section. Its maximal value will appear under the overhead conductor and will decrease exponentially by increasing distance in all directions. This maximal value will depend upon the following parameters: the height of the overhead conductor  $h$ , current frequency  $f$ , permeability  $\mu$  and the Earth's conductivity  $\sigma$ . These derivations will offer great possibilities in a three-phase systems treatment as well.

## 2 Theoretical Approach

The cross-section A - A from Figure 1. is presented in Figure 2.

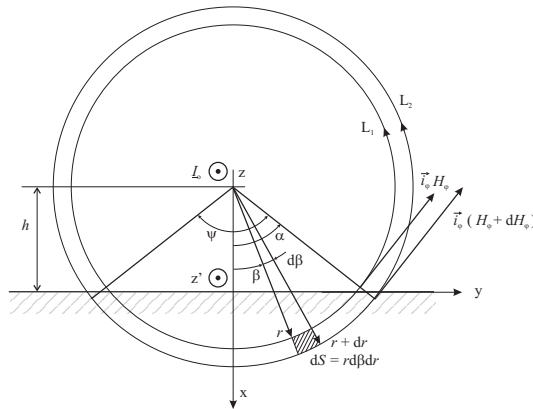


Fig. 2. The geometry of the treated problem.

The geometry suggests that the cylindrical coordinate system should be applied, where the z-axis passes through the axis of overhead conductor. In that case the current density vector has just z component, which depends only on radius r. As a matter a fact, having the infinite conductor length, the whole problem can be treated as the two-dimensional one in  $r - \varphi$  plane, so the polar coordinate system can substitute the cylindrical coordinate system.

Starting with first two Maxwell's equations in complex domain, for a quasi-stationary electromagnetic field [2, 3],

$$\nabla \times \vec{E} = -j\omega\vec{B}, \quad (1)$$

$$\nabla \times \vec{H} = \vec{J}, \quad (2)$$

expressed with complex magnetic density vector  $\vec{B}$  and complex current density vector  $\vec{J}$ ,

$$\nabla \times \vec{J} = -j\omega\sigma\vec{B}, \quad (3)$$

$$\nabla \times \vec{B} = \mu\vec{J} \quad (4)$$

where  $\sigma$  is the Earth conductivity, it can easily be shown that the current density vector in the Earth is a solution of the partial differential equation,

$$\frac{\partial^2 \underline{J}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \underline{J}_z}{\partial r} - \underline{k}^2 \underline{J}_z = 0, \quad (5)$$

where

$$\underline{k}^2 = \frac{j\omega\mu\sigma}{\pi} \arccos \frac{h}{r}. \quad (6)$$

When the coefficient  $\underline{k}^2$  is constant, the above equation is the Bessel's equation, with the solution

$$\underline{J}_z(r) = \underline{A}I_0(\underline{k}r) + \underline{C}K_0(\underline{k}r), \quad (7)$$

where  $I_0(\underline{k}r)$  is the modified Bessel function of the first kind of zero order,  $K_0(\underline{k}r)$  is the modified Bessel function of the second kind of zero order and  $\underline{A}$  and  $\underline{C}$  are the arbitrary complex constants. The proper function to represent  $\underline{J}_z(r)$  is one that vanishes at  $r \rightarrow \infty$ , since it is well known that the electric field intensity within the Earth by depth approaches zero. Therefore, the Bessel function of the first kind is discarded because of its infinite value when radius r is infinitely large. Hence, the complex constant  $\underline{A}$  is zero and the final solution for current density vector is:

$$\underline{J}_z(r) = \underline{C}K_0(\underline{k}r). \quad (8)$$

Having the coefficient  $\underline{k}^2$  not constant, we decided to apply a numerical solution in one region in Earth and analytical calculation in the remain space. Actually, as

we put as our goal to achieve the accuracy of whole calculation less than 0,5% we decided to calculate the current field numerically up to radius  $r$  less than six times penetration depth, where the penetration depth can be defined as,

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}. \quad (9)$$

For bigger distances,  $r > 6\delta$ , the constant coefficient  $\underline{k}$  can be assumed and an analytical approach gives the satisfactory accurate solution. For the numerical calculation the function  $K_0(\underline{k}r)$  is divided into real and imaginary parts,

$$\underline{J}_z(r) = \underline{C}[ker(ar) + jkei(ar)], \quad (10)$$

where

$$a = \sqrt{\frac{\omega\mu\sigma}{\pi} \arccos \frac{h}{r}}. \quad (11)$$

Verification of obtained results is very simple. Being the flux of current density vector,

$$I = \int_S \vec{J} \cdot d\vec{S}, \quad (12)$$

current within the Earth must be the same as in above conductor. Hence, the integration of current density vector over the entire Earth's cross-section, must give the current  $I_0$ . In the whole calculation, the Earth is treated as a homogeneous media with constant resistivity. This assumption is acceptable in the case when the Earth is a return conductor. Investigating the behavior of current distribution in vicinity of a single line-to-ground fault point, the local differences in Earth's resistivity should be taken into account.

### 3 Obtained Results and Discussion

The whole calculation was carried out for four different values of Earth's resistivity,

$$\rho = \frac{1}{\sigma}, \quad (13)$$

supposing that the value of current is  $I_0 = 1$  A. The results for the same value of conductor's height ( $h = 10$  m above the ground) and four different values of Earth's resistivity are presented graphically in Figure 3.

As it could be expected, in a case of the lowest Earth's resistivity,  $\rho = 50 \Omega\text{m}$ , due to the higher values of induced currents within the Earth, the skin effect is emphasized. For that reason, the intensity of current density vector  $|\vec{J}|$ , has the biggest

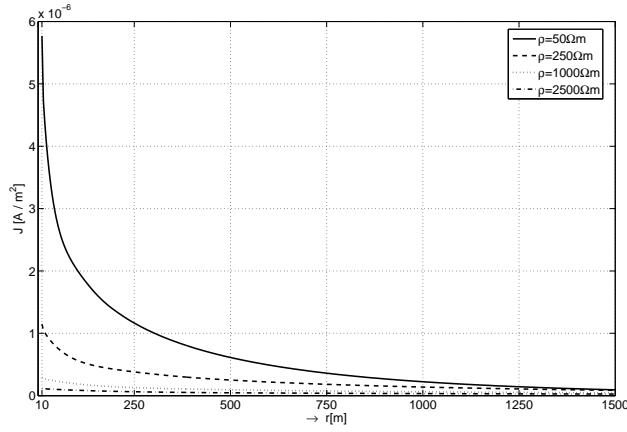


Fig. 3. Intensity of current density vector for  $h = 10$  m and four different Earth's resistivity.

value on the Earth's surface, decreasing rapidly with distance from the above conductor,  $r$ . In other cases of the Earth's resistivity,  $\rho = 250 \Omega\text{m}$ ,  $\rho = 1000 \Omega\text{m}$  and  $\rho = 2500 \Omega\text{m}$ , the skin effect is less distinct and the penetration depth is much higher. The same picture can be presented in a semi logarithmic coordinate system in order to show the expected crossing of different Earth's resistivity lines, which cannot be seen on the Figure 3. The intensity of current density vector for  $h = 10$  m in a semi logarithmic coordinate system is given in Figure 4.

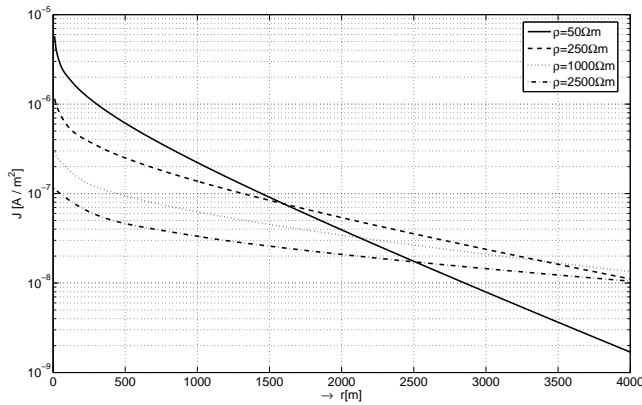


Fig. 4. Intensity of current density vector for  $h = 10$  m in a semi logarithmic coordinate system.

In order to verify the results, following the procedure described in section 2, equation (12), a numerical, Simpson's integration [4] was applied on the intensities

of current density vector with increasing distance from the above conductor, up to radius  $r$  and the results are given in Table 1 and graphically in Figure 5.

Table 1. Currents values  $I$ [A] in the Earth, up to 32 km.

$r$ [km]	50 $\Omega\text{m}$	250 $\Omega\text{m}$	1000 $\Omega\text{m}$	2500 $\Omega\text{m}$
0.5	0.2582	0.0887	0.0310	0.0147
1	0.5365	0.2275	0.0890	0.4467
2	0.8306	0.4879	0.2287	0.1247
3	0.9269	0.6737	0.3669	0.2144
4	0.9554	0.7927	0.4889	0.3034
6	0.9660	0.9082	0.6743	0.4658
8	0.9674	0.9477	0.7931	0.5969
10	0.9683	0.9607	0.8655	0.6978
12	0.9692	0.9650	0.9084	0.7733
14	0.9703	0.9665	0.9334	0.8287
16	0.9714	0.9672	0.9479	0.8689
21	0.9744	0.9682	0.9623	0.9260
27	0.9796	0.9695	0.9663	0.9529
30	0.9827	0.9702	0.9670	0.9587
32	0.9849	0.9707	0.9673	0.9612

Table 1 and Figure 5 present the dependence of current on the radius  $r$ , i.e. the current encountered in the Earth's cross-section up to radius  $r$ . As it was expected and as it can be seen in both, all curves approach the given current value of 1 A. Due to more or less emphasized skin effect, asymptotic achieving the final current value is different for different Earth's resistivities.

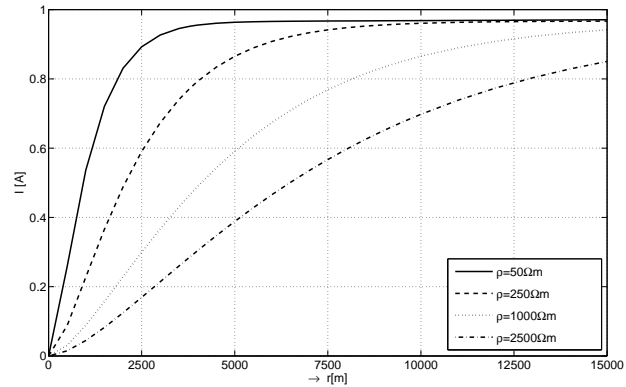


Fig. 5. Calculated current within the Earth.

The other possible manner to present the current distribution within the Earth is to draw the lines of constant intensity of current density vector, each line presenting the same flux. It means that each cross section of the Earth, between two successive lines of constant intensity of current density vector, conducts the same current. This kind of presentation is shown in Table 2 and in Figure 6, for conductor height of 10 m and for the lowest Earth's resistivity,  $\rho = 50 \Omega\text{m}$ . In order to show the lines of highest intensity of current density vector, domain presented in Figure 6. is up to 80 m in the Earth's depth intending to show how non-equidistant the lines of constant intensity of current density vector are.

Table 2. Radius  $r$  for constant  $\Delta J = 0.6 \mu \text{ A/m}^2$ .

$J[\mu \text{ A/m}^2]$	$r \text{ [m]}$
5.77	10.0
5.17	10.7
4.57	15.3
3.97	22.5
3.37	33.2
2.77	50.0
2.17	86.2
1.57	160.4

In this case the difference between two successive lines of constant intensity of current density vector is chosen to be  $\Delta J = 0.6 \mu \text{ A/m}^2$ .

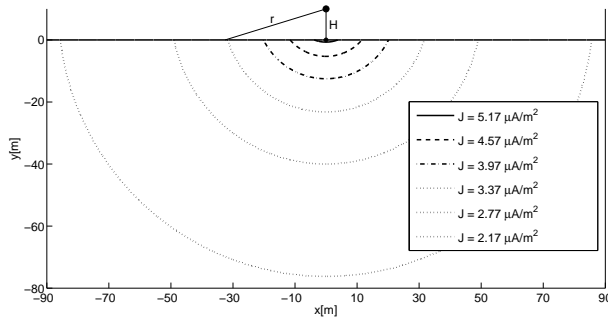


Fig. 6. Constant current density vector lines, up to 80 m within the Earth.

The only error that appears in presented calculations of current density vector is the error due to numerical procedure, especially in determination of Bessel's

functions values. Using very sophisticated recursive forms, we find the whole calculation accurate enough.

#### 4 Conclusions

The obtained results show that suggested procedure, involving mostly analytical and partly numerical calculations, gives very accurate current distribution within the Earth. Comparing with mostly applied procedures with concentrated parameters for current distribution presentation [5], this method is much more accurate and enables the determination of all physical values that appear within the Earth and on its surface. These results also show that described procedure could be applied for determination of the Earth's current distribution in a three phase systems, especially in the most common, single line-to-ground (SLG) fault case.

Moreover, suggested method will be very convenient for an accurate Earth's impedance calculation, including an exact treatment of skin effect within the Earth. We do expect much better results for all frequencies, so the higher harmonics could be taken into account as well.

#### References

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