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Statistical Dynamics of Dual-Power Law Optical Soliton

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Abstract: The dynamics of optical solitons with dual-power law nonlinearity, in presence of a stochastic perturbation term, is studied in this paper. The soliton perturbation theory is exploited to obtain the Langevin equation by virtue of which one can conclude that solitons travel down the optical fiber with a fixed mean free velocity.

Keywords: Optical fiber, dual-power law nonlinearity, stochastic preturbation, nonlinear Schrödinger equation, Langevin equation.

1 Introduction

The dynamics of pulses propagating in optical fibers has been a major area of research given its potential applicability in all optical communication systems. It has been well established [1–10] that this dynamics is described, to first approximation, by the integrable Nonlinear Schrödinger Equation (NLSE). Here the global characteristics of the pulse envelope can be fully determined by the method of Inverse Scattering Transform (IST) and in many instances, the interest is restricted to the single pulse described by the one soliton form of the NLSE. Typically though, distortions of these pulses arise due to perturbations which are either higher order corrections in the model as derived from the original Maxwell's equations [9], physical mechanisms not considered at first approximation like Raman effects or external perturbations such as the lumped effect due to the addition of bandwidth limited amplifiers in a communication line. Mathematically, these corrections are seen as perturbations of the NLSE and most of them have been studied thoroughly by regular asymptotic [4], soliton perturbation [5–7] or Lie tranform [9] methods.

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Besides the deterministic type perturbations one also needs to take into account, from practical considerations, the stochastic type perturbations. These effects can be classified into three basic types [8]:

1. Stochasticity associated with the chaotic nature of the initial pulse due to partial coherence of the laser generated radiation.
2. Stochasticity due to random nonuniformities in the optical fibers like the fluctuations in the values of dielectric constant the random variations of the fiber diameter and more.
3. The chaotic field caused by a dynamic stochasticity might arise from a periodic modulation of the system parameters or when a periodic array of pulses propagate in a fiber optic resonator.

Thus, stochasticity is inevitable in optical soliton communications [1–3, 8, 10].

2 Dual-Power Law Nonlinearity

For this law, the dimensionless form of the NLSE is

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^{2p} + \nu|q|^{4p}\right)q = 0 \tag{1}$$

In equation (1), q represents the dimensionless form of the wave profile while x and t are the independent variables that respectively represent the length of the optical fiber and the time. There are no known methods including the celebrated IST method that will integrate (1). However, (1) supports solitary waves of the form [4]

$$q(x,t) = \frac{A}{\left[1 + b\cosh\left\{B\left(x - \bar{x}(t)\right)\right\}\right]^{\frac{1}{2p}}} e^{i(-\kappa x + \omega t + \sigma_0)} \tag{2}$$

where

$$\kappa = -\nu \tag{3}$$

$$\omega = \frac{A^{2p}}{2p+2} - \frac{\kappa^2}{2} \tag{4}$$

$$B = A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2p}} \tag{5}$$

$$b = \sqrt{1 + \frac{\nu B^2}{2p^2}\frac{(1+p)^2}{1+2p}} \tag{6}$$

(7)

Here, ω is the wave number, κ is the soliton frequency whiile v is the velocity and σ_0 is the center of phase of the soliton. Also, A is the amplitude and B is the inverse width of the soliton. Finally, $\bar{x}$ is the center of the soliton so that one can write

$$v = \frac{d\bar{x}}{dt} \tag{8}$$

For dual-power law case, the solitons exist for

$$-\frac{2p^2}{B^2}\frac{1+2p}{(1+p)^2} < v < 0 \tag{9}$$

The NLSE with dual-power law nonlinearity has three known integrals of motion and the first two of which are the energy (E) and the linear momentum (M) that are repectively given by [4]

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |q|^2 dx \\ &= \frac{2A^2}{B2^{\frac{1}{p}}b^{\frac{1}{p}}} F\left(\frac{1}{p}, \frac{1}{p}; \frac{1}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} M &= \frac{i}{2}\int_{-\infty}^{\infty} \left(qq_x^* - q^* q_x\right) dx \\ &= -\frac{2\kappa A^2}{B2^{\frac{1}{p}}b^{\frac{1}{p}}} F\left(\frac{1}{p}, \frac{1}{p}; \frac{1}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \end{aligned} \tag{11}$$

2.1 Perturbation terms

Considering the effects of perturbation, [4–10] on the propagation of solitons through optical fibers, (24) is modified to

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^{2p} + \nu\,|q|^{4p}\right) q = i\epsilon R \tag{12}$$

where

$$R = \delta|q|^{2m}q + \beta q_{xx} + \sigma(x,t) \tag{13}$$

Here R is a spatio-differential operator while the perturbation parameter ϵ, with $0 < \epsilon \ll 1$, represents the relative width of the spectrum in fiber optics that arises due to quasi-monochromaticity [9]. In presence of perturbation terms, as in (12), the integrals of motion are modified. In most instances, a consequence of this is an adiabatic deformation of the soliton parameters like its amplitude, width, frequency and velocity accompanied by small amounts of radiation or small amplitude dispersive waves. For the perturbation terms, in (12), $\delta < 0$ is the nonlinear damping coefficient [4], β is the bandpass filtering term [9, 10].

The amplifiers, although needed to restore the soliton energy, introduces noise originating from amplified spontaneous emmision (ASE). To study the impact of noise on soliton evolution, the evolution of the mean free velocity of the soliton due to ASE will be studied in this paper. In case of lumped amplification, solitons are perturbed by ASE in a discrete fashion at the location of the amplifiers. It can be assumed that noise is distributed all along the fiber length since the amplifier spacing satisfies $z_a \ll 1$ [10]. In (12), $\sigma(x,t)$ represents the Markovian stochastic process with Gaussian statistics and is assumed that $\sigma(x,t)$ [8] is a function of t only so that $\sigma(x,t) = \sigma(t)$. Now, the complex stochastic term $\sigma(t)$ can be decomposed into real and imaginary parts as

$$\sigma(t) = \sigma_1(t) + i\sigma_2(t) \tag{14}$$

is further assumed to be independently delta correlated in both $\sigma_1(t)$ and $\sigma_2(t)$ with

$$\langle\sigma_1(t)\rangle = \langle\sigma_2(t)\rangle = \langle\sigma_1(t)\sigma_2(t')\rangle = 0 \tag{15}$$

$$\langle\sigma_1(t)\sigma_1(t')\rangle = 2D_1\delta(t-t') \tag{16}$$

$$\langle\sigma_2(t)\sigma_2(t')\rangle = 2D_2\delta(t-t') \tag{17}$$

where D_1 and D_2 are related to the ASE spectral density. In this paper, it is assumed that $D_1 = D_2 = D$. Thus,

$$\langle\sigma(t)\rangle = 0 \tag{18}$$

and

$$\langle\sigma(t)\sigma(t')\rangle = 2D\delta(t-t') \tag{19}$$

In soliton units, one gets,

$$D = \frac{F_n F_G}{N_{ph} z_a} \tag{20}$$

where F_n is the amplifier noise figure, while

$$F_G = \frac{(G-1)^2}{G \ln G} \tag{21}$$

is related to the amplifier gain G and finally N_{ph} is the average number of photons in the pulse propagating as a fundamental soliton.

2.2 Mathematical analysis

Using these integrals of motion, one can obtain the adiabatic parameter dynamics of the solitons as

$$\frac{dA}{dt} = \frac{\varepsilon}{pLA^{p-1}}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\int_{-\infty}^{\infty}(q^*R+qR^*)dx \tag{22}$$

$$\frac{d\kappa}{dt} = \frac{\varepsilon}{E}\left[i\int_{-\infty}^{\infty}(q_x^*R-q_xR^*)dx-\kappa\int_{-\infty}^{\infty}(q^*R+qR^*)dx\right] \tag{23}$$

where E is the energy as given by (9) while

$$\begin{aligned}L = {} & \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}} \\ & \left\{\frac{2\nu^2}{bp^3}\frac{(p+1)^3}{(b-1)(2p+1)^2}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right)\right. \\ & -\frac{2}{B^2}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right) \\ & \left.-\frac{2\nu}{bp^2}\frac{(p+1)^2}{(b-1)^2(p+2)(2p+1)}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right)\right\}\end{aligned} \tag{24}$$

Now substituting the perturbation terms R from (12) and carrying out the integrations in (21) and (22) yields

$$\begin{aligned}\frac{dA}{dt} = {} & \frac{4\varepsilon\delta A^{2m+2}}{Bb^{\frac{m+1}{p}}2^{\frac{m+1}{p}}} \\ & F\left(\frac{m+1}{p},\frac{m+1}{p},\frac{m+1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{m+1}{p},\frac{1}{2}\right) \\ & -\frac{4\varepsilon\beta A^2}{Bb^{\frac{1}{p}}2^{\frac{1}{p}}}\left[B^2F\left(2+\frac{1}{p},\frac{1}{p},\frac{3}{2}+\frac{1}{p};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right. \\ & \left.+\kappa^2F\left(\frac{1}{p},\frac{1}{p},\frac{1}{2}+\frac{1}{p};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right] \\ & +\frac{2\varepsilon}{pLA^{p-2}}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\left[\sigma_1\int_{-\infty}^{\infty}\frac{\cos\phi}{(1+a\cosh\tau)^{\frac{1}{2p}}}dx\right. \\ & \left.+\sigma_2\int_{-\infty}^{\infty}\frac{\sin\phi}{(1+a\cosh\tau)^{\frac{1}{2p}}}dx\right]\end{aligned} \tag{25}$$

$$\frac{d\kappa}{dt} = -\frac{\varepsilon\beta\kappa B^2}{4p^2A^2}\frac{F\left(2+\frac{1}{p},1+\frac{1}{p},2+\frac{1}{p};\frac{b-1}{2b}\right)}{F\left(\frac{1}{p},\frac{1}{p},\frac{1}{2}+\frac{1}{p};\frac{b-1}{2b}\right)}\frac{B(1+\frac{1}{p},1)}{B(\frac{1}{p},\frac{1}{2})}$$
$$-\frac{\varepsilon}{E}\int_{-\infty}^{\infty}\left[\frac{aB}{2p}\frac{(\sigma_2\cos\phi-\sigma_1\sin\phi\sinh\tau}{(1+a\cosh\tau)^{\frac{2p+1}{2p}}}\right.$$
$$\left.-\frac{2\kappa(\sigma_1\cos\phi+\sigma_2\sin\phi)}{(1+a\cosh\tau)^{\frac{1}{2p}}}\right]dx \tag{26}$$

where in (24) and (25), $F(\alpha,\lambda;\gamma;z)$ is the Gauss' hypergeometric function while $B(l,m)$ is the beta function. Equations (24) and (25), as it appears, is difficult to analyse. If the terms with σ_1 and σ_2 are suppressed, the resulting dynamical system has a stable fixed point, namely a sink. Now, linearizing the dynamical system about this fixed point gives, after simplification

$$\frac{dA}{dt} = -\varepsilon\left(A^{2m+1}-\frac{\xi}{A}\right) \tag{27}$$
$$\frac{d\kappa}{dt} = -\varepsilon\left[\kappa-\zeta\left(1+A-\kappa\right)\right] \tag{28}$$

where $\bar{A}$ is the fixed point of the amplitude while

$$\xi = \int_{-\infty}^{\infty}\left[\frac{\sigma_1\cos\phi}{(1+a\cosh\tau)^{\frac{1}{2p}}}+\frac{\sigma_2\sin\phi}{(1+a\cosh\tau)^{\frac{1}{2p}}}\right]dx \tag{29}$$

and

$$\zeta = \int_{-\infty}^{\infty}\left[\frac{aB}{2p}\frac{(\sigma_2\cos\phi-\sigma_1\sin\phi)\sinh\tau}{(1+a\cosh\tau)^{\frac{2p+1}{2p}}}\right.$$
$$\left.-\frac{2\kappa\left(\sigma_1\cos\phi+\sigma_2\sin\phi\right)}{(1+a\cosh\tau)^{\frac{1}{2p}}}\right]dx \tag{30}$$

Equations (26) and (27) are called the *Langevin* equations which will now be analyzed to compute the soliton mean drift velocity of the soliton. If the soliton parameters are chosen such that ζA is small, then (27) yields

$$\frac{d\kappa}{dt} = -\varepsilon\left[\kappa-\zeta\left(1-\kappa\right)\right] \tag{31}$$

One can solve (30) for κ and eventually the mean drift velocity of the soliton can be obtained. The stochastic phase factor of the soliton is defined by

$$\psi(t,y) = \int_y^t \zeta(s)ds \tag{32}$$

where $t > y$. Assuming that ζ is a Gaussian stochastic variable we arrive at

$$\langle e^{\psi(t,y)} \rangle = e^{D(t-y)} \tag{33}$$

$$\langle e^{[\psi(t,y)+\psi(t',y')]} \rangle = e^{D\theta} \tag{34}$$

where

$$\theta = 2(t + t' - y - y') - |t - t'| - |y - y'| \tag{35}$$

and

$$\langle \zeta(y) e^{-\psi(t,y)} \rangle = \frac{\partial}{\partial y} \langle e^{-\psi(t,y)} \rangle = De^{D(t-y)} \tag{36}$$

$$\langle \zeta(y)\zeta(y') e^{[-\psi(t,y)-\psi(t',y')]} \rangle = 2D\delta(y - y') e^{D\theta} + \frac{\partial^2}{\partial y \partial y'} e^{D\theta} \tag{37}$$

Now solving (30) with the initial condition as $\kappa(0) = 0$ and using equations (31)-(36) the soliton mean drift velocity is given by

$$\langle \kappa(t) \rangle = -\frac{D}{1-D} \left\{ 1 - e^{-\varepsilon(1-D)t} \right\} \tag{38}$$

¿From (37), it follows that

$$\lim_{t\to\infty} \langle \kappa(t) \rangle = -\frac{D}{1-D} \tag{39}$$

so that by virtue of (3), one can get in the limit

$$\lim_{t\to\infty} \langle v(t) \rangle = \frac{D}{1-D} \tag{40}$$

Thus, for large t, the mean drift velocity of the soliton, approaches a constant provided $D < 1$.

3 Conclusions

In this paper, the dynamics of optical solitons with dual-power law nonlinearity in presence of perturbation terms, both deterministic as well as stochastic, are studied. The Langevin equations were derived and the corresponding parameter dynamics was studied. The mean drift velocity of the soliton was obtained. In this study, it was assumed that the stochastic perturbation term σ is a function of t only, for simplicity. However, in reality σ is a function of both x and t and thus making it a far more difficult system to analyze although such kind of situations are being presently studied.

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