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Reversible Hadamard Transforms

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Abstract: A coding method which reconstruct an original digital image without distortion is called "reversible coding". In case of the classical block transform coding (Cosine, Hadamard, Haar and etc.) we have to make the number of levels of the transform coefficient very large in order to reconstruct the input signal with no distortion. In this paper we propose reversible Hadamard transform matrices. We give a recursion methods for generation of various type of real and complex reversible Hadamard transform matrices of higher order and corresponding fast transform algorithms.

Keywords: Hadamard transform, reversibile Hadanard transform matrices, image reconstruction, fast transform algorithms.

1 Introduction

In the past decade fast orthogonal transforms have been widely used in many areas, such as data compression, pattern recognition and image reconstruction, interpolation, linear filtering, spectral analysis, watermarking, cryptography and communication systems. The computation of unitary transforms is a complicated and time consuming task. However it would not be possible to use the orthogonal transforms in signal and image processing applications without effective algorithms calculating them. An important question in many applications is how to achieve the highest computation efficiency of the discrete orthogonal transforms (DOT) [1]. Among DOTs a special role plays a class of Hadamard transforms based on the Hadamard matrices ordered by Walsh and Paley, which can be obtained from the Sylvester's matrices by permutation of their rows [1]. These matrices are known

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as a non-sinusoidal orthogonal transform matrices and have found applications in digital signal processing and communication systems [1–9] as they do not require any multiplication operation in their computation.

The problem of computing a transform has been extensively studied. Methods to perform a discrete orthogonal transform with an essentially smaller number of operations than direct matrix multiplication, i.e. so-called fast transforms may be found in many publications.

In general, a fast transform $T_N f$ may be achieved by factoring the transform matrix T_N by the multiplication of k sparse matrices. Typically, $N = 2^n$, $k = \log_2 N = n$, and

$$T_{2^n} = F_n F_{n-1} \cdots F_1,$$

where F_i are very sparse matrices so that the complexity of multiplying by F_i is $O(N)$, $i = 1, 2, \ldots, n$.

A $N = 2^n-$point inverse transform matrix T_N^{-1} can be represented as:

$$T_{2^n}^{-1} = T_{2^n}^T = (F_n F_{n-1} \cdots F_1)^T = F_1^T F_2^T \cdots F_n^T.$$

Thus, one can implement the transform $T_N f$ via the following consecutive computations

$$f \to F_1 f \to F_2(F_1 f) \to \cdots \to F_n(\cdots F_2(F_1 f) \cdots).$$

Based on this factorization the computational complexity is reduced from $O(N)$ to $O(N \log N)$. Since F_i contains only few nonzero terms per row, the transformation $T_N f$ can be efficiently accomplished be operating on f n times. For Fourier, Hadamard, slant transforms, F_i contains only two nonzero terms in each row. So an $N-$point one dimensional transform with above given decomposition can be implemented in $O(N \log N)$ operations, which is far fewer than N^2 operations. Since the Walsh-Hadamard transform functions assume only the value -1 and $+1$, their computation require only additions and subtractions.

The increasing importance of processing large vectors in many scientific and engineering applications requires new ideas for designing highly efficient algorithms for various transforms. The computation of unitary and invertible transforms is in general a complicated and time consuming task and it would not be possible to use these transforms in signal and image processing applications without effective algorithms for calculating them.

A coding method which reconstruct an original digital image without distortion is called ”reversible coding”. Note that in case that we use classical block transform coding (Cosine, Hadamard, Haar and etc.) we have to make the number of levels of the transform coefficient very large in order to reconstruct the input signal with no distortion. In this section we propose reversible Hadamard transform for image coding. We give a recursion method for generation of reversible Hadamard transform matrices of higher order and corresponding fast transform algorithms.

2 Reversible Walsh-Hadamard Transform

It is well known that the Hadamard transform, which is mostly known as the Walsh-Hadamard transform, is one of the widely used transforms in signal and image processing. Nevertheless, the Walsh-Hadamard transform is just a particular case of general class of transforms based on Hadamard matrices [2]. Recently, Hadamard transforms and their variations have found a widely usage in audio and video processing [3–6, 10, 11]. Fast algorithms have been developed [1, 3–16] for efficient computation of these transforms.

In this section we introduce the recursion formulas for generating the reversible Walsh-Hadamard transform matrices of order $N = 2^n$.

The Hadamard matrix of order n is the $(\pm 1)-$matrix H_n of size $n \times n$ satisfying the orthogonality condition

$$H_n H_n^T = H_n^T H_n = nI_n,$$

where T is a transposition sign, I_n is an identity matrix of order n.

One of the most known Hadamard matrices is the Sylvester matrix [12], which is probably, the oldest Hadamard matrix of order 2^k, and can be generated recursively as follows [2, 13]

$$H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix}, H_1 = (1), k = 1, 2, \ldots. \tag{1}$$

The forward Sylvester-Hadamard (or Walsh-Hadamard) transform of input column-vector $x = (x_0, x_1, \ldots, x_{N-1})$ (N is the power of 2) is defined as $y = H_N x$. For example for $N = 2$ we have

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \text{-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0 + x_1 \\ x_0 - x_1 \end{pmatrix}.$$

In [17,18] there is paid attention to the fact that $x_0 - x_1$ is even (or odd), if $x_0 + x_1$ is even (or odd) and is showed that reconstruction without distortion is possible in the following transform

$$\begin{pmatrix} \dfrac{x_0 + x_1}{2} \\ x_0 - x_1 \end{pmatrix},$$

where [c] is the largest integer which is not greater than c.

By analogy with (1) we can define recursively reversible Walsh-Hadamard transform matrices as

$$
\begin{aligned}
[RWH]_{2^{k+1}} &= \begin{pmatrix} \frac{1}{2}[RWH]_{2^k} & \frac{1}{2}[RWH]_{2^k} \\ [RWH]_{2^k} & -[RWH]_{2^k} \end{pmatrix}, \\
[RWH]^{-1}_{2^{k+1}} &= \begin{pmatrix} [RWH]^{-1}_{2^k} & \frac{1}{2}[RWH]^{-1}_{2^k} \\ [RWH]^{-1}_{2^k} & -\frac{1}{2}[RWH]^{-1}_{2^k} \end{pmatrix},
\end{aligned} \tag{2}
$$

where

$$[RWH]_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}, \quad [RWH]^{-1}_2 = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix}.$$

Note that the conventional Walsh-Hadamard forward and inverse transform matrices H_N and H_N^{-1} of order $N = 2^n$ can be factored by

$$
\begin{aligned}
H_N &= A_n A_{n-1} \cdots A_2 A_1, \\
H_N^{-1} &= \tfrac{1}{N} A_1 A_2 \cdots A_{n-1} A_n,
\end{aligned} \tag{3}
$$

where

$$A_k = I_{2^{k-1}} \otimes \left(H_2 \otimes I_{2^{n-k}}\right), \quad k = 1, 2 \ldots, n. \tag{4}$$

Here and trough this paper $\otimes$ is the sign of Kronecker product defined as $A \otimes B = \{(a_{i,j}B)\}$.

It can be shown that the forward and inverse reversible Walsh-Hadamard transform matrices $[RW]_N$ and $[RW]_N^{-1}$ (see (2)) can be factored as ($N = 2^n$)

$$
\begin{aligned}
[RWH]_N &= B_n B_{n-1} \cdots B_2 B_1, \\
[RWH]_N^{-1} &= B_1^{-1} B_2^{-1} \cdots B_{n-1}^{-1} B_n^{-1},
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
B_k &= I_{2^{k-1}} \otimes \left(Q \otimes I_{2^{n-k}}\right), \quad k = 1, 2 \ldots, n, \\
B_k^{-1} &= I_{2^{k-1}} \otimes \left(Q^{-1} \otimes I_{2^{n-k}}\right), \quad k = 1, 2 \ldots, n, \\
Q &= \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix}.
\end{aligned} \tag{6}
$$

Below there are given forward and inverse reversible Walsh-Hadamard transform matrices of order 4 and 8

$$[RWH]_4 = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad [RWH]_4^{-1} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 1 & -1/2 & 1/2 & -1/4 \\ 1 & 1/2 & -1/2 & -1/4 \\ 1 & -1/2 & -1/2 & 1/4 \end{pmatrix}.$$

$$[RWH]_8 = \begin{pmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix},$$

$$[RWH]_8^{-1} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 & 1/2 & 1/4 & 1/4 & 1/8 \\ 1 & -1/2 & 1/2 & -1/4 & 1/2 & -1/4 & 1/4 & -1/8 \\ 1 & 1/2 & -1/2 & -1/4 & 1/2 & 1/4 & -1/4 & -1/8 \\ 1 & -1/2 & -1/2 & 1/4 & 1/2 & -1/4 & -1/4 & 1/8 \\ 1 & 1/2 & 1/2 & 1/4 & -1/2 & -1/4 & -1/4 & -1/8 \\ 1 & -1/2 & 1/2 & -1/4 & -1/2 & 1/4 & -1/4 & 1/8 \\ 1 & 1/2 & -1/2 & -1/4 & -1/2 & -1/4 & 1/4 & 1/8 \\ 1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8 \end{pmatrix}.$$

From (5) and (6) we can see that $N-$point forward (and inverse) reversible Walsh-Hadamard transform need $Nlog_2N$ additions and $\frac{N}{2}log_2N$ shifts operations, in opposite to conventional Walsh-Hadamard transform, which need only $Nlog_2N$ additions.

Below we give detailed description of fast 8-point reversible Walsh-Hadamard forward transform.

Example 2.1 . *8-point reversible Walsh-Hadamard forward transform.*

As follows from (5) 8-point reversible Walsh-Hadamard forward transform can be calculated by

$$X = [RWH]_8 f = B_3 B_2 B_1 x,$$

where $x = (x_0, x_1, \ldots, x_7)$ is the input integer-valued column-vector.

Therefore 8-point reversible Walsh-Hadamard fast forward transform algorithm can be realized via the following 3 steps:

Step 1. *Calculate $B_1 x$*

$$B_1 x = (Q \otimes I_4) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} (x_0+x_4)/2 \\ (x_1+x_5)/2 \\ (x_2+x_6)/2 \\ (x_3+x_7)/2 \\ x_0-x_4 \\ x_1-x_5 \\ x_2-x_6 \\ x_3-x_7 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix}. \qquad (7)$$

Step 2. *Calculate $B_2 u$*

$$B_2 u = [I_2 \otimes (Q \otimes I_2)] \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} (u_0+u_2)/2 \\ (u_1+u_3)/2 \\ u_0-u_2 \\ u_1-u_3 \\ (u_4+u_6)/2 \\ (u_5+u_7)/2 \\ u_4-u_6 \\ u_5-u_7 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} \qquad (8)$$

Step 3. *Calculate B_3v*

$$B_3v = (I_4 \otimes Q)\begin{pmatrix} v_0\\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7 \end{pmatrix} = \begin{pmatrix} (v_0+v_1)/2\\ v_0-v_1\\ (v_2+v_3)/2\\ v_2-v_3\\ (v_4+v_5)/2\\ v_4-v_5\\ (v_6+v_7)/2\\ v_6-v_7 \end{pmatrix} = \begin{pmatrix} X_0\\ X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7 \end{pmatrix}. \tag{9}$$

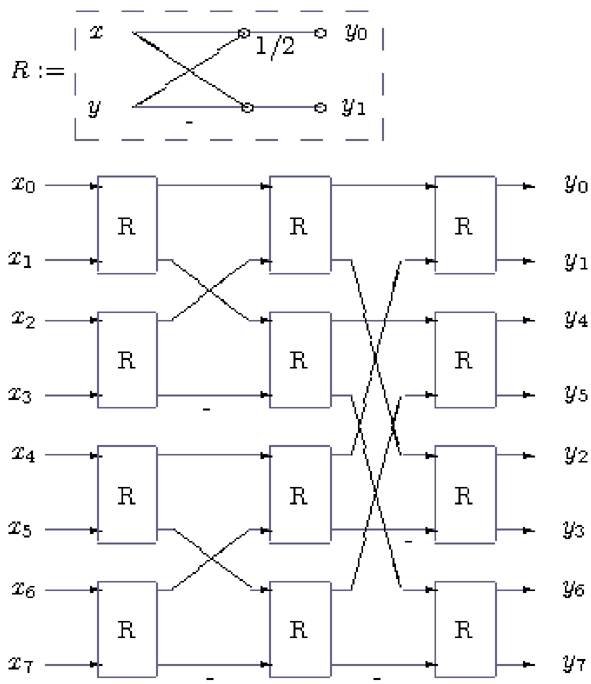

Fig. 1. Forward reversible Walsh-Hadamard transform of order 8.

It is easy to see that each of the forward and inverse reversible Walsh-Hadamard transforms need only 24 integer addition and 12 shift operations. Flow graph of the forward reversible Walsh-Hadamard transform is given in Fig. 1.

Let $[RH]_N$ and $[RH]_N^{-1}$ be the sequential ordered direct and inverse reversible Walsh-Hadamard matrices, and A_i is the $i-$th column and B_i is the $i-$th reverse column of $[RH]_N$. Then the following matrices

$$[RH]_{2N} = \left[Q\otimes A_1, Q_1\otimes A_2, \ldots, Q\otimes A_{N-1}, Q_1\otimes A_N\right],$$

$$[RH]_{2N}^{-1} = \begin{pmatrix} Q^{-1}\otimes B_1^T \\ -Q_1^{-1}\otimes B_2^T \\ \vdots \\ Q^{-1}\otimes B_{N-1}^T \\ -Q_1^{-1}\otimes B_N^T \end{pmatrix},$$

are direct and inverse sequential ordered reversible Walsh-Hadamard matrices of order 2N, where

$$Q = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1/2 & 1/2 \\ -1 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix}, \quad Q_1^{-1} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}.$$

The direct and inverse sequential ordered reversible Walsh-Hadamard matrices of order 4 and 8 are given below

$$[RH]_4 = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \tag{10a}$$

$$[RH]_4^{-1} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 1 & 1/2 & -1/2 & -1/4 \\ 1 & -1/2 & -1/2 & 1/4 \\ 1 & -1/2 & 1/2 & -1/4 \end{pmatrix}. \tag{10b}$$

$$[RH]_8 = \begin{pmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & -1/2 & -1/2 \\ 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & -1/4 & 1/4 \\ 1/2 & -1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & -1/2 & 1/2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix},$$

$$[RH]_8^{-1} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 & 1/2 & 1/4 & 1/4 & 1/8 \\ 1 & 1/2 & 1/2 & 1/4 & -1/2 & -1/4 & -1/4 & -1/8 \\ 1 & 1/2 & -1/2 & -1/4 & -1/2 & -1/4 & 1/4 & 1/8 \\ 1 & 1/2 & -1/2 & -1/4 & 1/2 & 1/4 & -1/4 & -1/8 \\ 1 & -1/2 & -1/2 & 1/4 & 1/2 & -1/4 & -1/4 & 1/8 \\ 1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8 \\ 1 & -1/2 & 1/2 & -1/4 & -1/2 & 1/4 & -1/4 & 1/8 \\ 1 & -1/2 & 1/2 & -1/4 & 1/2 & -1/4 & 1/4 & -1/8 \end{pmatrix}.$$

3 Reversible Walsh-Paley Transform

In this section we present factorizations of reversible Walsh-Paley transform matrices. The conventional Walsh-Paley transform matrix and its factorization are defined by

$$[WP]_{2^n} = \begin{bmatrix} [WP]_{2^{n-1}} \otimes \begin{pmatrix} + & + \end{pmatrix} \\ [WP]_{2^{n-1}} \otimes \begin{pmatrix} + & - \end{pmatrix} \end{bmatrix}, \qquad [WP]_{2^n} = W_0 W_1 \cdots W_{n-1}, \tag{11}$$

where $[WP]_1 = (1)$, and

$$W_m = I_{2^{n-1-m}} \otimes \begin{bmatrix} I_{2^m} \otimes \begin{pmatrix} + & + \end{pmatrix} \\ I_{2^m} \otimes \begin{pmatrix} + & - \end{pmatrix} \end{bmatrix}, \quad m = 0, 1, \ldots, n-1. \tag{12}$$

Similarly to equations (11) and (12) we define the reversible Walsh-Paleytransform matrix and its factoring version as

$$[RP]_{2^n} = \begin{bmatrix} [RP]_{2^{n-1}} \otimes \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \\ [RP]_{2^{n-1}} \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{bmatrix},$$

$$[RP]^{-1}_{2^n} = \begin{bmatrix} [RP]^{-1}_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, [RP]^{-1}_{2^{n-1}} \otimes \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \end{bmatrix},$$

$$[RP]_{2^n} = P_0 P_1 \cdots P_{n-1}, \quad [RP]^{-1}_{2^n} = P^{-1}_{n-1} P^{-1}_{n-2} \cdots P^{-1}_0,$$

where $[RP]_1 = (1)$, and

$$P_m = I_{2^{n-1-m}} \otimes \begin{bmatrix} I_{2^m} \otimes \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \\ I_{2^m} \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{bmatrix}, \quad m = \overline{0, n-1},$$
$$P^{-1}_m = I_{2^{n-1-m}} \otimes \begin{bmatrix} I_{2^m} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, I_{2^m} \otimes \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \end{bmatrix}, \quad m = \overline{0, n-1}. \tag{13}$$

Example 3.1 *The reversible Walsh-Paley matrices of order 2, 4, and 8 are given below*

$$[RP]_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}, \quad [RP]^{-1}_2 = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix},$$

$$[RP]_4 = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad [RP]^{-1}_4 = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 1 & 1/2 & -1/2 & -1/4 \\ 1 & -1/2 & 1/2 & -1/4 \\ 1 & -1/2 & -1/2 & 1/4 \end{pmatrix},$$

$$
[RP]_8 = \begin{pmatrix}
1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\
1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 \\
1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & -1/4 \\
1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 \\
1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 \\
1/2 & -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & -1/2 & 1/2 \\
1/2 & -1/2 & -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{pmatrix},
$$

$$
[RP]_8^{-1} = \begin{pmatrix}
1 & 1/2 & 1/2 & 1/4 & 1/2 & 1/4 & 1/4 & 1/8 \\
1 & 1/2 & 1/2 & 1/4 & -1/2 & -1/4 & -1/4 & -1/8 \\
1 & 1/2 & -1/2 & -1/4 & 1/2 & 1/4 & -1/4 & -1/8 \\
1 & 1/2 & -1/2 & -1/4 & -1/2 & -1/4 & 1/4 & 1/8 \\
1 & -1/2 & 1/2 & -1/4 & 1/2 & -1/4 & 1/4 & -1/8 \\
1 & -1/2 & 1/2 & -1/4 & -1/2 & 1/4 & -1/4 & 1/8 \\
1 & -1/2 & -1/2 & 1/4 & 1/2 & -1/4 & -1/4 & 1/8 \\
1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8
\end{pmatrix}.
$$

4 Reversible Complex Hadamard Transform

In this section we present the factorization of complex Hadamard matrices. The complex Hadamard matrix H of order N is an unitary matrix with elements $\pm 1, \pm j$, where $j = \sqrt{-1}$, i.e.

$$HH^* = H^*H = NI_N,$$

where H^* represents the complex conjugate transpose of the matrix H.

It can be proved that if H is a complex Hadamard matrix of order N then N is even [13].

The matrix $[CH]_2 = \begin{pmatrix} 1 & j \\ -j & -1 \end{pmatrix}$ is an example of a complex Hadamard matrix of order 2. Complex Hadamard matrices of higher orders can be generated recursively by using Kronecker product i.e.

$$[CH]_{2^n} = H_2 \otimes [CH]_{2^{n-1}}, \quad n = 2, 3, \ldots. \tag{14}$$

It can be shown, that the complex Hadamard matrix $[CH]_{2^n}$ of order 2^n (see equation (14)) can be factored as

$$[CH]_{2^n} = \left[\prod_{m=1}^{n-1} \left(I_{2^{m-1}} \otimes H_2 \otimes I_{2^{n-m}} \right) \right] \left(I_{2^{n-1}} \otimes [CH]_2 \right). \tag{15}$$

Similarly to (15) we define the direct and inverse reversible complex Hadamard matrices by

$$
\begin{aligned}
&[RC]_{2^n} = \Big[\prod_{m=1}^{n-1}\big(I_{2^{m-1}}\otimes Q\otimes I_{2^{n-m}}\big)\Big]\big(I_{2^{n-1}}\otimes [CH]_2\big),\\
&[RC]_{2^n}^{-1} = \big(I_{2^{n-1}}\otimes [CH]_2^{-1}\big)\Big[\prod_{m=n-1}^{1}\big(I_{2^{m-1}}\otimes Q^{-1}\otimes I_{2^{n-m}}\big)\Big],
\end{aligned}
\tag{16}
$$

where Q from (6), and

$$
[CH]_2^{-1} = \begin{pmatrix} 1/2 & j/2 \\ -j/2 & -1/2 \end{pmatrix}.
$$

Below there are given the direct and forward reversible complex Hadamard matrices of orders 2, 4, and 8.

$$
[RC]_2 = \begin{pmatrix} 1 & j \\ -j & -1 \end{pmatrix},
$$

$$
[RC]_2^{-1} = \begin{pmatrix} 1/2 & j/2 \\ -j/2 & -1/2 \end{pmatrix},
$$

$$
[RC]_4 = \begin{pmatrix} 1/2 & j/2 & 1/2 & j/2 \\ -j/2 & -1/2 & -j/2 & -1/2 \\ 1 & j & -1 & -j \\ -j & -1 & j & 1 \end{pmatrix},
$$

$$
[RC]_4^{-1} = \begin{pmatrix} 1/2 & j/2 & 1/4 & j/4 \\ -j/2 & -1/2 & -j/4 & -1/4 \\ 1/2 & j/2 & -1/4 & -j/4 \\ -j/2 & -1/2 & j/4 & 1/4 \end{pmatrix},
$$

$$
[CS]_8 = \begin{pmatrix}
1/4 & j/4 & 1/4 & j/4 & 1/4 & j/4 & 1/4 & j/4 \\
-j/4 & -1/4 & -j/4 & -1/4 & -j/4 & -1/4 & -j/4 & -1/4 \\
1/2 & j/2 & -1/2 & -j/2 & 1/2 & j/2 & -1/2 & -j/2 \\
-j/2 & -1/2 & j/2 & 1/2 & -j/2 & -1/2 & j/2 & 1/2 \\
1/2 & j/2 & 1/2 & j/2 & -1/2 & -j/2 & -1/2 & -j/2 \\
-j/2 & -1/2 & -j/2 & -1/2 & j/2 & 1/2 & j/2 & 1/2 \\
1 & j & -1 & -j & -1 & -j & 1 & j \\
-j & -1 & j & 1 & j & 1 & -j & -1
\end{pmatrix},
$$

$$[CS]_8^{-1} = \begin{pmatrix} 1/2 & j/2 & 1/4 & j/4 & 1/4 & j/4 & 1/8 & j/8 \\ -j/2 & -1/2 & -j/4 & -1/4 & -j/4 & -1/4 & -j/8 & -1/8 \\ 1/2 & j/2 & -1/4 & -j/4 & 1/4 & j/4 & -1/8 & -j/8 \\ -j/2 & -1/2 & j/4 & 1/4 & -j/4 & -1/4 & j/8 & 1/8 \\ 1/2 & j/2 & 1/4 & j/4 & -1/4 & -j/4 & -1/8 & -j/8 \\ -j/2 & -1/2 & -j/4 & -1/4 & j/4 & 1/4 & j/8 & 1/8 \\ 1/2 & j/2 & -1/4 & -j/4 & -1/4 & -j/4 & 1/8 & j/8 \\ -j/2 & -1/2 & j/4 & 1/4 & j/4 & 1/4 & -j/8 & -1/8 \end{pmatrix}.$$

Below we give detailed description of fast 8-point reversible complex Hadamard forward transform.

Example 4.1 . *8-point reversible complex Hadamard forward transform.*

As follows from (16) 8-point reversible complex Hadamard forward transform can be calculated by

$$X = [RC]_8 x = (Q \otimes I_4)(I_2 \otimes Q \otimes I_2)(I_4 \otimes [RC]_2)x,$$

where $x = (x_0, x_1, \ldots, x_7)^T$ *is the input integer-valued vector.*

Therefore 8-point reversible complex Hadamard fast forward transform algorithm can be realized via the following 3 steps:

Step 1. *(This step no need any arithmetical operations).*

$$(I_4 \otimes [RC]_2)x = z_1 + jz_2 = \begin{pmatrix} x_0 \\ -x_1 \\ x_2 \\ -x_3 \\ x_4 \\ -x_5 \\ x_6 \\ -x_7 \end{pmatrix} + j \begin{pmatrix} x_1 \\ -x_0 \\ x_3 \\ -x_2 \\ x_5 \\ -x_4 \\ x_7 \\ -x_6 \end{pmatrix}.$$

Step 2. *Compute* $(I_2 \otimes Q \otimes I_2)(z_1 + jz_2) = v + jw$, *where*

$$v = \begin{pmatrix} (x_0+x_2)/2 \\ -(x_1+x_3)/2 \\ (x_0-x_2) \\ -(x_1-x_3) \\ (x_4+x_6)/2 \\ -(x_5+x_7)/2 \\ (x_4-x_6) \\ -(x_5-x_7) \end{pmatrix}, \quad w = \begin{pmatrix} -v_1 \\ -v_0 \\ -v_3 \\ -v_2 \\ -v_5 \\ -v_4 \\ -v_7 \\ -v_6 \end{pmatrix}.$$

Step 3. *Compute* $(Q \otimes I_4)(v + jw) = X + jY$, *where*

$$X = \begin{pmatrix} (v_0+v_4)/2 \\ (v_1+v_5)/2 \\ (v_2+v_6)/2 \\ (v_3+v_7)/2 \\ (v_0-v_4) \\ (v_1-v_5) \\ (v_2-v_6) \\ (v_3-v_7) \end{pmatrix}, \quad Y = \begin{pmatrix} X_5/2 \\ X_4/2 \\ -X_3 \\ -X_2 \\ 2X_1 \\ 2X_0 \\ -X_7 \\ -X_6 \end{pmatrix}.$$

From (16) it follows that 1D $N-$point reversible complex Hadamard transform need $N\log_2 N - N$ addition and N shift operations.

5 Reversible Williamson-Hadamard Matrices

At first we briefly describe the Williamson's approach [19] to the Hadamard matrices construction.

The following matrix we call *parametric Williamson array*

$$W(a,b,c,d) = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}, \quad a,b,d,c \in \{1,-1\}. \qquad (17)$$

Theorem 5.1 *(Williamson [13, 19, 20]). Suppose there exist four* $(\pm 1)-$*matrices* A, B, C, D *of order n satisfying*

$$\begin{aligned} &PQ^T = QP^T, \quad P,Q \in \{A,B,C,D\}, \\ &AA^T + BB^T + CC^T + DD^T = 4nI_n. \end{aligned} \qquad (18)$$

then $W(A,B,C,D)$ *is Hadamard matrix of order* $4n$.

The matrices A, B, C, D with properties (18) are called *Williamson matrices*. The matrix $W(A,B,C,D)$ is called the *Williamson-Hadamard matrix*.

Let A,B,C,D be cyclic symmetric Williamson matrices of order n with first rows $(a_i)),(b_i),(c_i),(d_i)$, respectively. Note that $a_{n-i} = a_i$, $b_{n-i} = b_i$, $c_{n-i} = c_i$, $d_{n-i} = d_i$, $i = 1,2,\dots,n-1$. Now Williamson-Hadamard matrix $W(A,B,C,D)$ of order $4n$ can be represented as block cyclic block symmetric matrix by

$$W_{4n} = \sum_{i=0}^{n-1} U^i \otimes Q_i, \quad \text{where} \quad Q_i = \begin{pmatrix} a_i & b_i & c_i & d_i \\ -b_i & a_i & -d_i & c_i \\ -c_i & d_i & a_i & -b_i \\ -d_i & -c_i & b_i & a_i \end{pmatrix}, \qquad (19)$$

where U is the cyclic matrix of order n with first row $(0,1,0,\dots,0)$.

From (19) we can see that $Q_{n-i} = Q_i$, and all the blocks of matrix W_{4n} are Williamson-Hadamard matrices of order 4. In [21] it was proved that cyclic symmetric Williamson-Hadamard block matrices can be constructed using only 5 different blocks such as

$$Q_0 = \begin{pmatrix} + & + & + & + \\ - & + & - & + \\ - & + & + & - \\ - & - & + & + \end{pmatrix}, \quad Q_1 = \begin{pmatrix} + & + & + & - \\ - & + & + & + \\ - & - & + & - \\ + & - & + & + \end{pmatrix}, \quad Q_2 = \begin{pmatrix} + & + & - & + \\ - & + & - & - \\ + & + & + & - \\ - & + & + & + \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} + & - & + & + \\ + & + & - & + \\ - & + & + & + \\ - & - & - & + \end{pmatrix}, \quad Q_4 = \begin{pmatrix} + & - & - & - \\ + & + & + & - \\ + & - & + & + \\ + & + & - & + \end{pmatrix}. \tag{20}$$

For example, Williamson-Hadamard matrix of order 12 is given by

$$H_{12} = I_3 \otimes Q_0 + U \otimes Q_4 + U^2 \otimes Q_4.$$

Now we introduce the following parametric matrix of order 4 which we call *reversible Williamson (RW) array*

$$[RW](a,b,c,d) = \begin{pmatrix} a/4 & b/4 & c/4 & d/4 \\ -b/2 & a/2 & -d/2 & c/2 \\ -c/2 & d/2 & a/2 & -b/2 \\ -d & -c & b & a \end{pmatrix}. \tag{21}$$

The inverse reversible Williamson array is given by

$$[RW]^{-1}(a,b,c,d) = \begin{pmatrix} a & -b/2 & -c/2 & -d/4 \\ b & a/2 & d/2 & -c/4 \\ c & -d/2 & a/2 & b/4 \\ d & c/2 & -b/2 & a/4 \end{pmatrix}. \tag{22}$$

Now, the Theorem 5.1 for reversible Williamson-Hadamard matrices can be formulated as

Theorem 5.2 *(Generalized Williamson Theorem). Let A, B, C, and D be Williamson matrices of order n. Then the matrices $[RW]_{4n}$ and $[RW]_{4n}^{-1}$ are the direct and inverse reversible Williamson-Hadamard matrices of order $4n$, respectively*

$$[RW]_{4n} = \begin{pmatrix} \frac{1}{4}A & \frac{1}{4}B & \frac{1}{4}C & \frac{1}{4}D \\ -\frac{1}{2}B & \frac{1}{2}A & -\frac{1}{2}D & \frac{1}{2}C \\ -\frac{1}{2}C & \frac{1}{2}D & \frac{1}{2}A & -\frac{1}{2}B \\ -D & -C & B & A \end{pmatrix}. \tag{23}$$

$$[RW]_{4n}^{-1} = \frac{1}{n}\begin{pmatrix} A & -\frac{1}{2}B & -\frac{1}{2}C & -\frac{1}{4}D \\ B & \frac{1}{2}A & \frac{1}{2}D & -\frac{1}{4}C \\ C & -\frac{1}{2}D & \frac{1}{2}A & \frac{1}{4}B \\ D & \frac{1}{2}C & -\frac{1}{2}B & \frac{1}{4}A \end{pmatrix}. \tag{24}$$

Similar to (19) we can define block cyclic reversible matrices by following

$$\begin{aligned} [RW]_{4n} &= \sum_{i=0}^{n-1} U^i \otimes R_i, \\ [RW]_{4n}^{-1} &= \frac{1}{n}\sum_{i=0}^{n-1} U^{n-i} \otimes R_i^{-1}, \end{aligned} \tag{25}$$

where

$$R_i = \begin{pmatrix} a_i/4 & b_i/4 & c_i/4 & d_i/4 \\ -b_i/2 & a_i/2 & -d_i/2 & c_i/2 \\ -c_i/2 & d_i/2 & a_i/2 & -b_i/2 \\ -d_i & -c_i & b_i & a_i \end{pmatrix}. \tag{26}$$

$$R_i^{-1} = \begin{pmatrix} a_i & -b_i/2 & -c_i/2 & -d_i/4 \\ b_i & a_i/2 & d_i/2 & -c_i/4 \\ c_i & -d_i/2 & a_i/2 & b_i/4 \\ d_i & c_i/2 & -b_i/2 & a_i/4 \end{pmatrix}. \tag{27}$$

Below there are given direct and inverse reversible block cyclic Williamson-Hadamard matrices of order 12

$$[RW]_{12} = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1
\end{pmatrix}.$$

$$[RW]_{12}^{-1} = \frac{1}{3}\begin{pmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\
1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\
1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
-1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\
-1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\
-1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\
-1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\
-1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
-1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4}
\end{pmatrix}.$$

As in conventional case block cyclic reversible Williamson-Hadamard matrices can be constructed using only 5 different blocks which given in a table below

Table 1.

i	$[RW]_i$	$[RW]_i^{-i}$
0	$\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1 & -1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 & -1/2 & -1/4 \\ 1 & 1/2 & 1/2 & -1/4 \\ 1 & -1/2 & 1/2 & 1/4 \\ 1 & 1/2 & -1/2 & 1/4 \end{pmatrix}$
1	$\begin{pmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ 1 & -1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 & -1/2 & 1/4 \\ 1 & 1/2 & -1/2 & -1/4 \\ 1 & 1/2 & 1/2 & 1/4 \\ -1 & 1/2 & -1/2 & 1/4 \end{pmatrix}$
2	$\begin{pmatrix} 1/4 & 1/4 & -1/4 & 1/4 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ -1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 & 1/2 & -1/4 \\ 1 & 1/2 & 1/2 & 1/4 \\ -1 & -1/2 & 1/2 & 1/4 \\ 1 & -1/2 & -1/2 & 1/4 \end{pmatrix}$
3	$\begin{pmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & -1/2 & -1/4 \\ -1 & 1/2 & 1/2 & -1/4 \\ 1 & -1/2 & 1/2 & -1/4 \\ 1 & 1/2 & 1/2 & 1/4 \end{pmatrix}$
4	$\begin{pmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1 & 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 \\ -1 & 1/2 & -1/2 & 1/4 \\ -1 & 1/2 & 1/2 & -1/4 \\ -1 & -1/2 & 1/2 & 1/4 \end{pmatrix}$

Note that all of matrices from the Table 1 can be represented only by sequential ordered reversible Walsh-Hadamard matrix of order 4 $[RH]_4$ from (10) as

$$[RW]_0 = \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} [RH]_4 \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix},$$

$$[RW]_1 = \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix} [RH]_4 \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & - \end{pmatrix},$$

$$[RW]_2 = \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & 0 & - \end{pmatrix} [RH]_4 \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & + \end{pmatrix},$$

$$[RW]_3 = \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} [RH]_4 \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & + & 0 \\ 0 & - & 0 & 0 \end{pmatrix},$$

$$[RW]_4 = \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix} [RH]_4 \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix}.$$

Note that the Kronecker product of two reversible Hadamard matrices of orders m and n is the reversible Hadamard matrix of order mn. Below we present one design method which allows as construct the reversible Hadamard matrix of order $\frac{mn}{2}$.

Let H_{4n}, H_{4n}^{-1} and H_{4m}, H_{4m}^{-1} two pairs of reversible Hadamard matrices. Represent it as follows

$$H_{4n} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad H_{4n}^{-1} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}, \quad H_{4m} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad H_{4m}^{-1} = \begin{pmatrix} B_1 & B_2 \end{pmatrix}.$$

We can check that

$$\begin{aligned} &P_1Q_1 = P_2Q_2 = I_{2n}, \qquad Q_1P_1 + Q_2P_2 = I_{2n}, \qquad P_1Q_2 = P_2Q_1 = 0, \\ &A_1B_1 = A_2B_2 = I_{2m}, \qquad B_1A_1 + B_2A_2 = I_{2m}, \qquad A_1B_2 = B_2A_1 = 0. \end{aligned} \tag{28}$$

Introduce the following matrices

$$R_1 = \frac{2P_1 + P_2}{2}, \qquad R_2 = \frac{2P_1 - P_2}{2}, \quad \Phi_1 = \frac{Q_1 + 2Q_2}{2}, \qquad \Phi_2 = \frac{Q_1 - 2Q_2}{2}. \tag{29}$$

Now we can show that the following matrices are reversible Hadamard matrices of order $8mn$

$$\Gamma = R_1 \otimes B_1 + R_2 \otimes B_2, \quad \Gamma^{-1} = \Phi_1 \otimes A_1 + \Phi_2 \otimes A_2. \tag{30}$$

Therefore we can formulate the multiplicative theorem [7, 20] for reversible Hadamard matrices.

Theorem 5.3 *(Multiplicative theorem). If there exist reversible Hadamard matrices of order m and n, then there exists the reversible Hadamard matrix of order* $\frac{mn}{2}$.

Give an example. As initial reversible Hadamard matrices we use the reversible Walsh-Hadamard and Williamson-Hadamard matrices of order 4 from equation

(10) and Table 1, respectively. Using above given notations from (29) we obtain

$$R_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$
$$\Phi_1 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}. \tag{31}$$

Therefore, according to multiplicative theorem, we obtain reversible Hadamard matrix of order 8 given below

$$\Gamma = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix},$$
$$\Gamma^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

6 Golay Sequences and Reversible Transform Matrices

Give some definitions. The cyclic $(0,\pm1)-$matrices X_1, X_2, X_3, X_4 are called *$T-$matrices of order k* if the following conditions are satisfied

$$X_i \odot X_j = 0, \quad i \neq j, \quad i,j = 1,2,3,4,$$
$$X_1 + X_2 + X_3 + X_4 \quad \text{is } (\pm1)-\text{matrix},$$
$$X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = kI_k,$$

where $\odot$ is a Hadamard (pointwise) product.

Let $A = \{\{a_i\}_{i=0}^{N-1}$ be sequence of length N such that $a_\in\{-1,+1\}$. The following function is called *aperiodic auto-correlation function* of sequence A $[7,22]$

$$P_A(k) = \sum_{i=0}^{N-k-1} a_i a_{i+k}, \quad 0 \leq k \leq N-1.$$

Two $(\pm 1)-$sequences $A = \{a_i\}$ and $B = \{b_i\}$ with length N are called *Golay complementary sequences* [22] if

$$P_A(k) + P_B(k) = 0, \quad k = 1, 2, \ldots, N-1.$$

Note that there exist Golay sequences of length $2^a 10^b 26^c$, where a, b, c are nonzero positive integers [23]. Golay sequences of lengths 2, 10, and 26 are given below

$$\begin{array}{ll} n = 2: & ++ \\ & +-; \\ n = 10: & +--+-+---+ \\ & +-------++-; \\ n = 26: & +++--+++-+------+-++--+---- \\ & ---++---+-++-+-+-++--+----. \end{array}$$

Let now $A_1 = \{a_i\}_{i=1}^{k}$, $B_1 = \{b_i\}_{i=1}^{k}$ be Golay sequences of length k. We can check that the sequences $A = \{0, A_1\}$ and $B = \{0, B_1\}$ are complementary sequences of length $k+1$, and cyclic matrices with first rows A and B satisfy the condition

$$AB = BA, ARB^T = BRA^T,$$

$$AA^T + BB^T = 2kI_{k+1}$$

where R is a back identity matrix of order k.

It is possible to prove that matrices

$$X_1 = I_{k+1}, \quad X_2 = \frac{A+B}{2}, \quad X_3 = \frac{A-B}{2} \tag{32}$$

are cyclic $T-$matrices of order $k+1$.

Let v_1, v_2, v_3, v_4 and w_1, w_2, w_2, w_4 are four length vectors representing the rows and columns of sequential ordered direct and inverse reversible Hadamard matrices $[RH]_4$ and $[RH]_4^{-1}$, respectively. Consider the following matrices

$$\begin{aligned} P &= v_1 \otimes X_1 + v_2 \otimes X_2 + v_3 \otimes X_3, \\ P^{-1} &= w_1 \otimes X_1^T + w_2 \otimes X_2^T + w_3 \otimes X_3^T, \end{aligned} \tag{33}$$

where X_i from (32), and

$$\begin{aligned} &v_1 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}), \quad v_2 = (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), \quad v_3 = (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}), \\ &w_1^T = (1, 1, 1, 1), \quad w_2^T = (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), \quad w_3^T = (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}). \end{aligned} \tag{34}$$

Now using (33) and (34) we form the following matrices of order $k+1$ (k length of Golay sequences)

$$\begin{aligned} P_1 &= \tfrac{1}{4}X_1 - \tfrac{1}{2}X_2 - \tfrac{1}{2}X_3, \qquad & P_2 &= \tfrac{1}{4}X_1 + \tfrac{1}{2}X_2 + \tfrac{1}{2}X_3, \\ P_3 &= \tfrac{1}{4}X_1 - \tfrac{1}{2}X_2 + \tfrac{1}{2}X_3, \qquad & P_4 &= \tfrac{1}{4}X_1 + \tfrac{1}{2}X_2 - \tfrac{1}{2}X_3. \end{aligned} \tag{35}$$

It can be shown that

$$P_1P_1^T + P_2P_2^T + P_3P_3^T + P_4P_4^T = \frac{4k+1}{4}I_{k+1}.$$

Using the matrices from (35) we obtain some interesting matrices.

(i) The following matrix

$$G = \frac{2}{\sqrt{4k+1}} \begin{pmatrix} P_1 & P_2R & P_3R & P_4R \\ -P_2R & P_1 & -P_4^TR & P_3^TR \\ -P_3R & P_4^TR & P_1 & -P_2^TR \\ -P_4R & -P_3^TR & P_2^TR & P_1 \end{pmatrix}$$

is the orthonormal integer transform matrix of Geothals-Seidel type of order $4k+1$ with elements $\{\pm 1/4, \pm 1/2\}$.

(ii) The following matrices A and B of size 3×12 and 12×3, respectively, satisfy the condition: $AB = I_3, \quad AA^T = \frac{9}{4}I_3, \quad B^TB = \frac{2}{3}I_3$

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

$$B^T = \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(iii) The following four cyclic matrices $A_1 = (1/4, 1/2, 1/2)$, $A_2 = (1/4, 1/2, -1/2)$, $A_3 = (1/4, -1/2, -1/2)$, $A_4 = (1/4, -1/2, 1/2)$ satisfy the condition:

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = \frac{9}{4}.$$

(iv) Using the matrices from (35) we find

$$G_{4(k+1)} = \begin{pmatrix} \frac{1}{4}P_1 & \frac{1}{4}P_2R & \frac{1}{4}P_3R & \frac{1}{4}P_4R \\ -\frac{1}{2}P_3R & \frac{1}{2}P_4^TR & \frac{1}{2}P_1 & -\frac{1}{2}P_2^TR \\ -P_4R & -P_3^TR & P_2^TR & P_1 \end{pmatrix},$$

$$G_{4(k+1)}^{-1} = \begin{pmatrix} P_1^T & -\frac{1}{2}RP_2^T & -\frac{1}{2}RP_3^T & -\frac{1}{4}RP_4^T \\ RP_2^T & \frac{1}{2}P_1^T & \frac{1}{2}RP_4 & -\frac{1}{4}RP_3 \\ RP_3^T & -\frac{1}{2}RP_4 & \frac{1}{2}P_1^T & \frac{1}{4}RP_2 \\ RP_4^T & \frac{1}{2}RP_3 & -\frac{1}{2}RP_2 & \frac{1}{4}P_1^T \end{pmatrix}.$$

References

[1] A. Rao, *Orthogonal Transforms for Digital Signal Processing*. New York: Springer-Verlag, 1975.

[2] S. S. Agaian, “Hadamard matrices and their applications,” in *Lecture Notes in Mathematics*. Springer, 1985, vol. 1168.

[3] G. Reddy and P. Satyanarayana, “Interpolation algorithm using walsh-hadamard and discrete fourier/hartley transforms,” *IEEE*, pp. 545–547, 1991.

[4] C.-F. Chan, “Efficient implementation of a class of isotropic quadratic filters by using walsh-hadamard transform,” in *IEEE Int. Symposium on Circuits and Systems*, Hong Kong, June 9-12, 1997, pp. 2601–2604.

[5] B. K. Harms, J. B. Park, and S. A. Dyer, “Optimal measurement techniques utilizing hadamard transforms,” *IEEE Trans. on Instrumentation and Measurement*, vol. 43, no. 3, pp. 397–402, June 1994.

[6] C. Anshi, L. Di, and Z. Renzhong, “A research on fast hadamard transform (fht) digital systems,” in *IEEE TENCON 93*, Beijing, 1993, pp. 541–546.

[7] H. G. Sarukhanyan, “Hadamard matrices: Construction methods and applications,” in *Proc. of The Workshop on Transforms and Filter Banks*, Tampere, Finland, Feb.21–27, 1998, p. 35.

[8] Yarlagadda, R. K. Rao, and E. Hershey, *Hadamard Matrix Analysis and Synthesis with Applications and Signal/Image Processing*, 1997.

[9] S. Samadi, Y. Suzukake, and H. Iwakura, “On automatic derivation of fast hadamard transform using generic programming,” in *Proc. 1998 IEEE Asia-Pacific Conference on Circuit and Systems*, Thailand, 1998, pp. 327–330.

[10] R. Stasinski and J. Konrad, “A new class of fast shape-adaptive orthogonal transforms and their application to region-based image compression,” *IEEE Trans. on Circuits and systems for Video Technology*, vol. 9, pp. 16–34, 1999.

[11] M. Barazande-Pour and J. W. Mark, “Adaptive MHDCT,” *IEEE*, pp. 90–94, 1994.

[12] J. J. Sylvester, “Thoughts on inverse orthogonal matrices, simultaneous sign successions and tesselated pavements in two or more colours, with applications to newton’s rule, ornamental til-work, and the theory of numbers,” *Phil. Mag.*, vol. 34, pp. 461–475, 1867.

[13] J. Seberry and M. Yamada, "Hadamard matrices, sequences and block designs. surveys in contemporary design theory," in *Wiley-Interscience Series in Discrete Mathematics*. Jhon Wiley, New York, 1992.

[14] D. Coppersmith, E. Feig, and L. E., "Hadamard transforms on multiply/add architectures," *IEEE Trans. on Signal Processing*, vol. 42, no. 4, pp. 969–970, 1994.

[15] Z. Li, H. Sorensen, and C. .Burus, "Fft and convolution algotithms an dsp microprocessors," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, pp. 289–294, 1986.

[16] R. K. Montoye, E. Hokenek, and S. L. Runyon, "Design of the ibm risc system/6000 floating point execution unit," *IBM J. Res. Develop.*, vol. 34, pp. 5–50, 1990.

[17] K. Ire and R. Kishimoto, "A study on perfect reconstructive subband coding," *IEEE Trans. Circuits and Syst. Video Technol.*, pp. 42–48, Mar. 1991.

[18] ——, "Design of perfect reconstructive subband coding," *Trans. IEICEJ*, vol. J75-B-1, pp. 57–66, Jan. 1992.

[19] J. Williamson, "Hadamard determinant theorem and sum of four squares," *Duke Math. J.*, no. 11, pp. 65–81, 1944.

[20] S. Agaian and H. Sarukhanyan, "Recurrent formulas for construction of williamson type matrices," *Math. Notes*, vol. 30, no. 4, pp. 603–617, 1981.

[21] S. Agaian, "Spatial and high dimensional block hadamard matrices," *Mathematical Problems of Computer Science*, vol. 12, pp. 5–50, 1984, (In Russian, Yerevan).

[22] P. Robinson and J. Wallis, "A note on using sequences to construct orthogonal designs," in *Colloqua Mathematica Societatis Janos Bolyai. Combinatorics*, vol. 18, Kesthely, Hungary, 1976, pp. 911–932.

[23] R. J. Turyn, "Hadamard matrices, baumert-hall units, four-symbol sequences, pulls compression, and surface wave encoding," *J. Comb. Theory, Ser. A*, no. 16, pp. 313–333, 1974.