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Critical Curves and 2D Coupled Maps

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Abstract: The theory of critical curves for maps of the plane provides powerful tools for locating the chief characteristic features of a discrete dynamical system in two dimensions: the location of its chaotic attractors, its basin boundaries, and the mechanisms of its bifurcations. Nowadays one begins to recognize the role played by critical curves of maps in the analysis, in the understanding and description of the bifurcations, and transition to chaotic behavior in coupled maps. In this paper we consider some properties of such maps, which possess a chaotic attractor. Some examples are considered in this paper in which we can see the effective role played by such curves in bifurcation theory.

Keywords: Homoclinic points, critical curves, bifurcations in endomorphisms, two-dimensional maps.

1 Introduction

We deal with noninvertible two-dimensional maps, defined by continuous functions, piecewise continuously differentiable, which possesses a chaotic set or area, and whose dynamics are considered as a function of a real parameter. Such behaviors have been studied extensively, particularly in the applied dynamics literature.

The critical curve notion is an important mathematical tool [1] used to study bifurcations, that take place in invariant areas of two-dimensional endomorphisms, for either invariant absorbing areas or chaotic areas. The term of critical curve was first introduced in 1964 by Mira who provides an entry into certain areas of current research on noninvertible maps and the role of such curve in bifurcations

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basin. It is a natural generalization in $\mathbb{R}^2$ of the notion of critical points of one-dimensional endomorphisms. We define the critical curve LC of an endomorphism T in the plane $\mathbb{R}^2$ (in means of Mira [2]) as the geometrical locus of points x having at least two coincident preimages of first rank. One determines this locus denoted by LC_{-1}, when T is differentiable, by taking the Jacobian of T equal to zero ($J = \det(DT(x,y)) = 0$). A critical line LC is constituted of one or several branches. These branches separate the plane in open regions, where all points of a region have the same number of first rank antecedents.

Several autors have investigated and have shown the importance of critical curves in the bifurcations specially Gumowski and Mira [3] who have developed the role of critical curves in bifurcations, and Gardini [4, 5]. As in one-dimensional endomorphisms, where critical points define the boundary of absorbing intervals and invariant intervals, and characterize the global bifurcations leading to chaotic dynamics, also in two-dimensional endomorphisms critical curve play the useful role to determine the boundary of trapping areas or of invariant areas, at least in the simplest cases, and thus to characterize the global bifurcations of invariant sets.

We consider two examples, the first is a two coupled 1D maps and investigate critical curves. We also investigate the effect of asymmetry of coupling on the bifurcation mechanism for the loss of synchronous chaos. In the second example, a coupled chaotic systems with symmetry is considered, here absorbing areas surround the synchronized chaotic attractor on the diagonal $y = x$ after riddling bifurcations.

The main purpose of this work is that of stressing the crucial role of the critical curves in the characterization of the global properties of the map considered in this paper, and, in particular, their role in the occurrence of the different dynamical behaviors.

2 Preliminary Considerations

The endomorphism T considered here defines a discrete dynamical system in $\boldsymbol{R}^2$

$$\begin{aligned}(x_{n+1},\, y_{n+1}) &= T(x_n, y_n) \\ &= (f(x_n, y_n; \lambda), g(x_n, y_n; \lambda))\end{aligned} \tag{1}$$

where $f(x,y,\lambda)$ et $g(x,y,\lambda)$ are continuous functions with respect to real variables x, y and to the real parameter λ.

Definition 1 *An absorbing area E is a closed and bounded subset as:*

(i) $T(E) \subseteq E$

(ii) *its frontier, ∂E is made up of a finite or infinite number of critical arcs of LC, $LC_1, LC_2, \ldots, LC_k$, such that $LC = T(LC_{-1})$; $LC_i = T^i(LC)$ for $i \geq 1$.*

(iii) *A neighborhood $U(E)$ exists, such that images of finite rank of its points are in interior of E.*

Definition 2 *A chaotic area A, is an invariant absorbing area ($T(A) = A$)), the points of which give rise to iterated sequences having the property of sensitivity to initial conditions.*

Definition 3 *A closed invariant set A is said to be a weak attractor in Milnor sense (or simply Milnor attractor [6, 7]) if its basin B (A), i.e. the set of points whose ω-limit sets of x belongs to A has positive Lebesgue measure.*

Definition 4 *The basin of attraction $B(A)$ of an attractor A is riddled if its complement intersects every disk in a set of positive measure.*

Remark 1 *If A is a Milnor attractor, then its basin $B(A)$ is called riddled basin if it is such that any neighborhood of it contains points whose trajectory converge to another attractor. In other words, a riddled basin does not include any open subset, so it corresponds to an extreme form of uncertainty, we use the word "riddled" to denote a basin which is full of holes.*

To describe how riddling can possibly arise as a system parameter changes : the key point is that the chaotic attractor A in the invariant subspace has embedded within itself an infinite number of unstable periodic orbits, and they constitute the skeleton of the attractor. Depending on the parameter, these periodic orbits can be stable or unstable with respect to perturbations transverse to the invariant subspace. Riddling occurs when an unstable periodic orbit, typically of low period , first becomes transversely unstable. When this occurs, a set consisting of an infinite number of tonguelike structures is open at the location of the periodic orbit and the locations of all its preimages. The roots of these structures are thus dense in this subspace, and have a Lebesgue measure zero. The complement of the set of these roots thus assumes the full measure in this subspace. By continuity, in the vicinity of the subspace, the complement of the set of tongues, which is the basin of the attractor in the subspace.

Definition 5 *The basins of attraction $B(A)$ and $B(B)$ of the attractors A and B are intermingled if each disk which intersects one of the basins in a set of positive measure also intersects the other basin in a set of positive measure.*

3 Examples of Maps and Critical Curves

Example 1. In our first example we consider the map defined by the following pair of equations

$$T_1 : \begin{cases} x_{n+1} = 1 - a x_n^2 \\ y_{n+1} = 1 - a y_n^2 + b\,|x_n - y_n| + c \end{cases} \tag{1}$$

Here x, y represent dynamical variables, a is the control parameter of the uncoupled one-dimensional map, b the coupling parameter, and c a third real parameter.

First, we are concerned about global behavior in this two coupled chaotic systems without symmetry. By varying the coupling parameter, we investigate the stability of the synchronized chaotic attractor. When all periodic saddles embedded in the synchronized chaotic attractor are transversely stable, we have strong synchronization without any burstings from the diagonal.

The stability of fixed points is ruled by the equation $|J - \lambda I| = 0$, where λ is the eigenvalue, I is the identity matrix and J is the Jacobian matrix of the mapping, which is non constant $J = 2ax(2ay - b)$ if $x > y$ and $J = 2ax(2ay + b)$ if $x < y$. Since T is noninvertible, so global analysis which use the theory of critical lines cited above apply. Noninvertibility means that there exists a set in phase plane where the Jacobian determinant of the map vanishes. The forward image of this set is called line LC. The existence of such a set brings a specific character into bifurcation scenarios, shapes of attracting sets and their basins of attraction, etc., different from those known for invertible maps.

We also note that the Jacobian determinant J becomes zero on the critical curves: $LC_{-1} = \left\{(x;y) \in R^2 : x = 0\,; y = x \text{ and } y = \pm\frac{b}{2a}\;\; if\;\; x \gtrless y\right\}$.

A finite number of segments of images $LC_k = [T^k(LC_{-1})(k = 1;2;...)]$ of the critical curves LC_{-1} can be used to define the boundary of a compact absorbing area.

It is known that basins generated by two-dimensional noninvertible maps may be either simply connected, or multiply connected, or non connected, depending on the situation of their boundary with respect to the critical set LC. The critical curves have been used to obtain the boundary of a compact trapping region, called absorbing area following many authors (see [2, 8]). In particular, in Gardini [5], the concept of minimal invariant absorbing area is defined in order to give a global characterization of the different dynamical scenarios related to riddling bifurcation. The minimal invariant absorbing area is the smallest absorbing area that includes the Milnor attractor [6] on which the synchronized dynamics occur. Its delimitation is important in order to characterize the global properties which influence the qualitative effects of riddling bifurcation. In fact, a minimal invariant absorbing area that surrounds a Milnor attractor defines a compact region of the phase plane that acts as a trapping bounded set inside which the trajectories starting near are confined. Moreover, contacts between the portions of critical curves bounding the minimal absorbing area surrounding a Milnor attractor and the basin boundaries may mark the transition between local and global riddling phenomena, as it will be shown here.

As the control parameter a is increased, the coupled map T exhibits an infinite sequence of period-doubling bifurcations of attractors with period $2n$ $(n = 0;1;2,\ldots)$ on the invariant $y = x$ line, accumulating at a finite point $a_\infty (= 1.401155)$. Beyond the accumulation point a_∞, chaotic attractors exist on the $y = x$ line for the a values in a positive measure set. When crossing ”a critical” line in the (a,b) plane, a transition from periodic to chaotic synchronization occurs see figures 1and 2. Bifurcation effects appear in this coupled system on varying parameters; notably the route to creation of riddled basins via a riddling bifurcation is put in evidence.

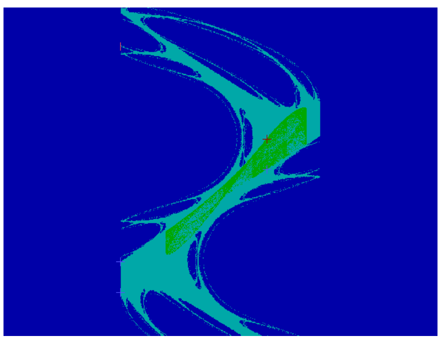

Fig. 1. A mixed absorbing area, surrounding the synchronized attractor in which the saddle (blue cross) and the repellor (red cross are embedded $a = 1.62, b = 3.635, c = 00$.

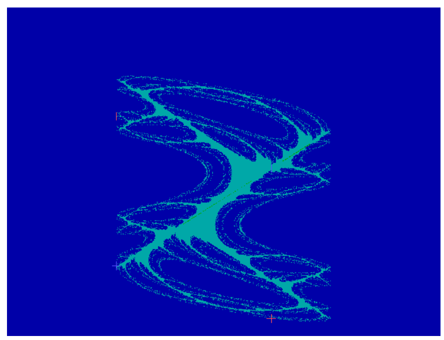

Fig. 2. The repeller approaches the saddle (saddle-node bifurcation), basin globally riddled. $a = 1.47, b = 1.4, c = 0$.

The synchronized chaotic attractor continues to contact its basin boundary at a new repelling fixed point giving other situation : since the synchronized chaotic attractor is touching its basin boundary at the saddle point, such a riddling bifurcation induces a contact bifurcation between the synchronized chaotic attractor and its basin boundary. Note also that an infinitely narrow tongues, emanating from the saddle point and its preimages, as shown in the inset of this Figure.

Here, we are concerned about a piecewise character of this map without symmetry. By varying a coupling parameter, the stability of the synchronized chaotic attractor changes with respect to a perturbation transverse to the synchronization subspace. As the coupling parameter varies and passes a threshold value, a saddle fixed point is found to first become transversely unstable through a transcritical bifurcation.

Then we have a weak synchronization. For this case, the basin of the synchronized chaotic attractor becomes riddled with a dense set of holes.

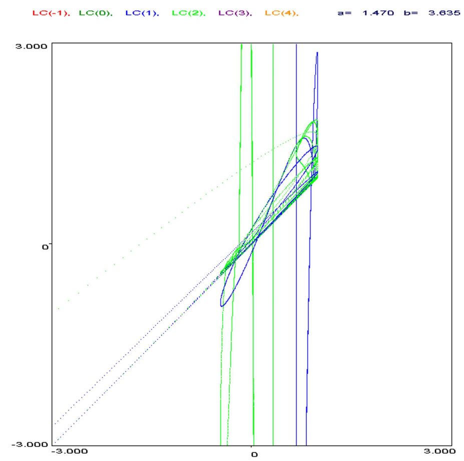

Fig. 3. LC_k curves in asymmetric case.

Example 2. The second example is inspired from works investigated by [9] to study the sudden destruction of hyperchaotic attractors, the authors have shown that an asynchronous hyperchaotic attractor may appear through a blowout bifurcation, where the synchronous chaotic attractor on the invariant synchronization line becomes unstable with respect to perturbations transverse to the synchronization line. It is a symmetrically coupled system T_2 consisting of two identical $1D$ maps, given by

$$T_2 : \begin{cases} x_{n+1} = 1 - ax_n^2 + b(y_n^2 - x_n^2) \\ y_{n+1} = 1 - ay_n^2 + b(x_n^2 - y_n^2) \end{cases} \tag{2}$$

Our inspired example is as follows

$$T_3 : \begin{cases} x_{n+1} = 1 - ax_n^2 + \dfrac{b'}{2\pi} \sin 2\pi(y_n^2 - x_n^2) \\ y_{n+1} = 1 - ay_n^2 + \dfrac{b'}{2\pi} \sin 2\pi(x_n^2 - y_n^2) \end{cases} \tag{3}$$

The map T_3 has a symmetry property that implies the invariance of the diagonal $x = y$, so that synchronized dynamics is possible. Interesting phenomena are observed in [9] such that this sudden destruction of the hyperchaotic attractor occurs without any contact with its basin boundary. In this work, a particular dynamic is considered for which the local study of the transverse stability, in a neighborhood of the invariant submanifold in which synchronized dynamics takes place, is combined with a study of the global behavior of the map T_3. This global behavior is investigated by studying of the critical manifolds of the map. Global bifurcations of the basins of attraction are evidenced through contacts between critical curves and basin boundaries.

In this example, some global bifurcations are described: one that changes the structure of the basins, one that causes the disappearance of the invariant area described and one causing the disappearance of any bounded attractor. Such global bifurcations are characterized as contact bifurcations, related to tangencies between critical curves and basin boundaries see Figures 4and 5.

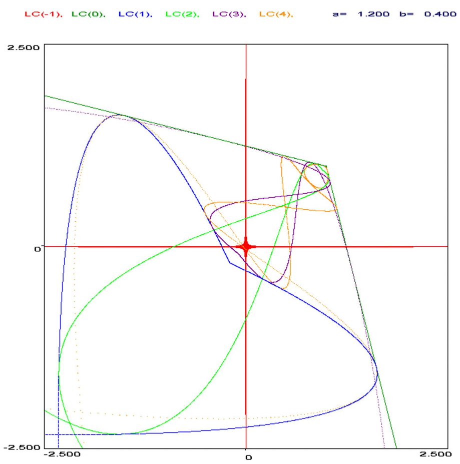

Fig. 4. Portions of critical curves of increasing rank of T_2.

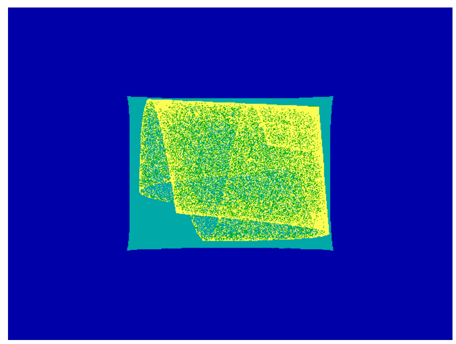

Fig. 5. Attraction basin and Absorbing area in yellow for T_2

From [9], concrete analysis is presented, For $a = 1.84$, a synchronous attractor with a single band exists on the synchronization diagonal. This attractor is strongly stable in the region of b $(= -1.406419 \leq b \leq -0.433579)$. A riddling bifurcation occurs when b passes the right endpoint of this region, and then the strongly-stable synchronous attractor becomes weakly stable. For this case, the period-1 saddle embedded in the attractor becomes transversely unstable via a supercritical period-doubling bifurcation, leading to the birth of a new asynchronous period-2 saddle.

Further increase in the coupling parameter, the synchronous attractor loses its transverse stability through a blowout bifurcation for $b = -0.35809$, and then an asynchronous attractor bounded to the mixed absorbing area appears. The symmetric period-2 saddle becomes stabilized by emitting a pair of asymmetric period-2 saddles through a subcritical pitchfork bifurcation. This stabilization of the symmetric period-2 attractor leads to the sudden destruction of the hyperchaotic attractor without any collision with its basin.

Global bifurcations, that change the structure of the basins of attraction and cause the destruction of the bounded attracting sets, are characterized as contact bifurcations due to tangencies between critical curves and basin boundaries. The effects of such global bifurcations are evidenced both for the changes induced in the basins structure and for the different cases, usually described in the literature, that characterize the possible behaviors of the locally repelled trajectories starting close to the invariant manifold which contains a Milnor attractor which is not asymptotically stable.

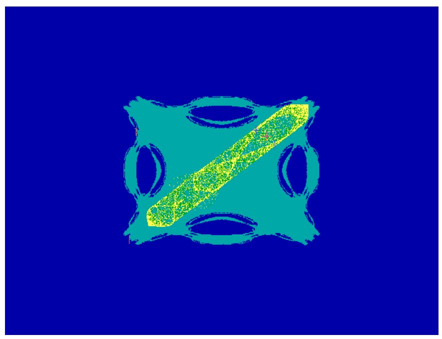

Fig. 6. Synchronous; attractor for $a = 1.84$, $b' = -2$.

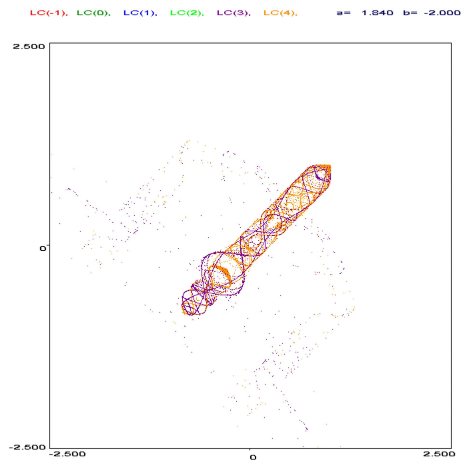

Fig. 7. Portions of critical curves of increasing rank bound an attractor of Milnor.

Weakly stable synchronous attractor on the main diagonal for $a = 1,84$ and $b' = -2$ (see Figure 6.) which is surrounded by a mixed absorbing area, bounded by the union of segments of the unstable manifolds of the symmetric period-2 saddle.We have the same behavior in Figure 8.

The models used to interpret the results on critical curves are rather complex to illustrate, they represent structures that provide both the asymmetry for the first (see Figure 3.) and the symmetry for the second example (with Figures 7and 9.) and the nonlinearity required to produce chaotic states.

4 Conclusion

The main results of the paper concern the global behavior of two maps. Such global behavior is characterized by the properties of the critical curves of the noninvertible

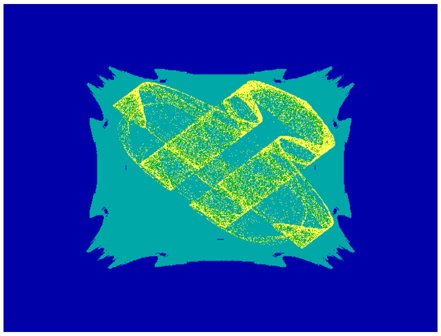

Fig. 8. Attraction basin and Absorbing area in yellow bounded by critical curves for T_3, $a = 1.2$, $b' = 2.05$.

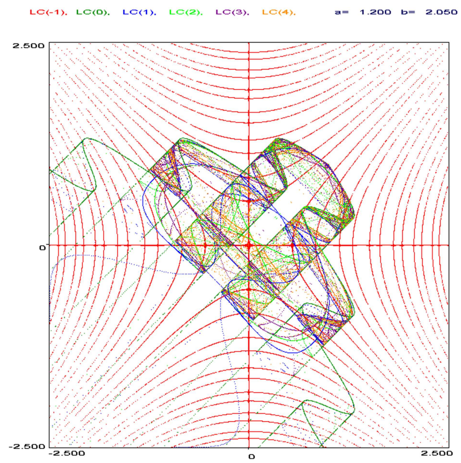

Fig. 9. Portions of critical curves of increasing rank bound an attractor of Milnor $a = 1.2$, $b' = 2.05$.

map. Portions of critical curves of increasing rank bound an invariant asymptotically attracting twodimensional set that includes a Milnor attractor to which the synchronized trajectories converge. Such a two-dimensional trapping region gives an upper bound to the intermittency phenomena and becomes the only attractor when the Milnor attractor is transformed into a chaotic saddle. We show that the folding action of the critical curves places an upper bound on how the trajectories starting near the invariant submanifold can get away from it. In other words, the study of the critical curves leads to an estimate of the amplitude of the 'bursts' transverse to the diagonal and synchronized behaviors.

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