FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 20, no. 1, April 2007, 85-92

Dynamics of a Three Parameters Family of Piecewise Maps

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Abstract: We study the behavior under iteration of a three parameters family of piecewise linear maps of the plane. Our purpose is to study a particular kind of bifurcation for this kind of maps. We wish to show that this family possesses interesting properties. Coexistence of several attractors, and characteristics of intermingled basins of different attractors are obtained.

Keywords: Piecewise linear map, attractor, nonconservative system, intermingled basins.

1 Introduction

The purpose of this article is to illustrate the beauty and complexity of a family of piecewise linear maps and describe its behavior for some choices of parameter values. In parameter space and phase space, we examine some simple but nonetheless typical cases of such maps. However, we find an interesting pattern of multiply Arnold tongues with a weak dependence on parameters. Since linerarity on intervals enables effective and simple calculations, piecewise linear maps remain the favorite maps to study and to apply various developments of the different notions of dynamical systems. The present work has as aim to make see another side of the former study done by [1] where the island structure has been finely detailed. In several papers Devaney chooses this map for its area-preserving property and showed a very rich and interesting structure associated with this map, which demonstrates

Manuscript received June 22, 2006. Revised December 2006.

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many aspects of this kind of maps. There are many intriguing open problems relating to these mappings, for instance with regard to the existence of global attractors and aperiodic solutions. We consider then this map with coexisting attractors, and we analyze the problem of the structure of the boundaries that separate their basins of attraction, this may become particularly challenging since the system is represented by the iteration of a piecewise map, because in this case nonconnected basins can be obtained, formed by several portions. Our system possesses a large number of coexisting periodic attractors and intermingled basins of attraction. Recently, it has been shown that, when restricted attractors are also attractors for the full system, under certain conditions, their basins of attraction are intermingled [4]. By this we mean that the basins are so finely mixed that the basin of one attractor has points from the basin of other attractor arbitrarily nearby, and vice versa. It is further pointed in [3] that when this type of basin structure occurs, the dynamics becomes qualitatively undecidable. It is the case where the system has invariant submanifolds of lower dimension with respect to the state space, where chaotic Milnor attractors are embedded. We also show how such particular dynamic situations arise in this model, and we analyze the related phenomena of riddled basins and intermingled basins.

In [1], the area preserving map

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

with

$$f(x,y) = 1 - y + |x|\,, g(x,y) = x$$

is described in details. This system contains an absolute value with

$$J(f,g) = (\frac{\partial f}{\partial x})(\frac{\partial g}{\partial y}) - (\frac{\partial f}{\partial y})(\frac{\partial g}{\partial x}) = 1$$

This is a conservative system with a very rich and interesting structure, which demonstrates many aspects of iterated maps.

In our work we replace this map by the discontinuous piecewise map T depending on three parameters:

$$T = \begin{cases} x' = 1 - ax - by, & y' = x \quad if\ x > 0 \\ x' = 1 + ax - by + c, & y' = x \quad if\ x < 0 \end{cases} \tag{1}$$

defined by linear functions, where a, b, c are real parameters, this map will be treated by numerical methods. In our work we choose $b \neq 1$ to complete the former study.

In this model with evolutive mechanism, a situation denoted as multistability is present, i.e. several attractors exist, each with its own basin of attraction. This leads to the question of the delimitation of the basins of attraction and their changes as the parameters vary. This issue cannot be studied by local methods but through a global study of the map, often requiring an interplay among geometric and numerical methods. Moreover the complexity related to the structure of the basins is not related, in general, to the existence of chaotic attracting sets, in the sense that simple attractors may have basins with a complicated topological structure whereas strange attractors, may exist whose basins have simple boundaries.

2 Presentation of Main Results

We consider the piecewise linear map (1), a meaningful characterization of T consists in the identification of its singularities, and the behavior of the latter as the parameters varies. First, we obtain the bifurcation diagram shown in Fig. 1, which presents information on stability region for the fixed point (blue domain), and the existence region for attracting cycles of order k exists $(k \leq 14)$. The black regions $(k = 15)$ corresponds to the existence of bounded iterated sequences. Consecutive narrow Arnold tongues of period 4, 6, 8, 10, 12, 14 appear (see for details $[7]$). We notice that we have a succession of cycles of even period.

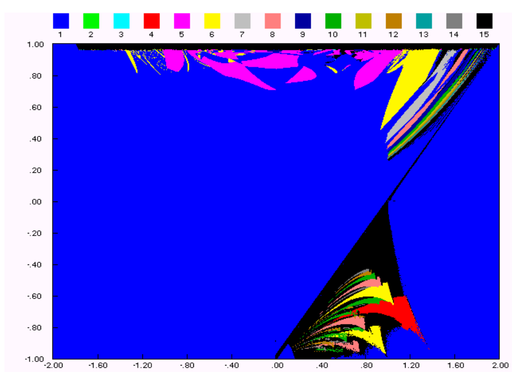

Fig. 1. Bifurcation diagram for map T in (a,b) plane with $c = -1$.

We identify existence of bifurcation organization called "boxes in files" a bifurcation structure which corresponds to an ordering of the farey sequence of fractions in their lowest terms. It was elaborated from Leonov's results on the bifurcations of one-dimensional piecewise continuous, piecewise linear maps (see $[5]$ and $[7]$). We consider an extension of the notion of attractor, known as Milnor attractor $[6]$, whose basin of attraction may assume a structure, called *riddled basin*, characterized by an extreme form of complexity, according to the following definitions (see [2]). The more general notion of Milnor attractor has been introduced to evidence the existence of invariant sets which attract many points even if they are not attractors in the usual topological sense. Bischi and Gardini give evidence that the change from hard to soft blowout is caused by a collision between the basin boundary and the absorbing area containing the attractor. More recently, they show that this can only be the case if A is an invariant absorbing area, as an absorbing area for which $T(A) \subseteq A$ is a proper containment will have transient points on the boundary. Moreover, they indicate that their arguments in $[3]$ go through if we consider absorbing area of mixed type (these have boundaries that consist not only of segments of critical curves). We would like to emphasize and indicate here that their results can be understood in the general setting of weak attractors; the precise form of the boundary of a weak attractor appears to be unimportant. The minimal invariant absorbing area is the smallest absorbing area that includes the Milnor attractor on which the synchronized dynamics occur. Its delimitation is important in order to characterize the global properties which influence the qualitative effects of riddling bifurcations. In fact, a minimal invariant absorbing area that surrounds a Milnor attractor defines a compact region of the phase plane that acts as a bounded set inside which the trajectories starting near are confined. Moreover, contacts between the portions of critical curves bounding the minimal absorbing area surrounding a Milnor attractor and the basin boundaries may mark the transition between local and global riddling phenomena, as it will be shown here. A closed invariant set A is said to be a weak attractor in Milnor sense (or simply Milnor attractor) if its basin $B\ (A)$, i.e. the set of points whose ω-limit sets of x belongs to A has positive Lebesgue measure.

Definition 1 *The basin of attraction $B(A)$ of an attractor A is riddled if its complement intersects every disk in a set of positive measure.*

If A is a Milnor attractor, then its basin $B(A)$ is called *riddled basin* if it is such that any neighborhood of it contains points whose trajectory converge to another attractor. In other words, a riddled basin does not include any open subset, so it corresponds to an extreme form of uncertainty, we use the word *"riddled"* to denote a basin which is full of holes, and we use the word *"intermingled"* to denote several basins which are dense in each other.

Definition 2 *The basins of attraction $B(A)$ and $B(B)$ of the attractors A and B are intermingled if each disk which intersects one of the basins in a set of positive measure also intersects the other basin in a set of positive measure.*

Note that an attractor in the usual (topological) sense is also a Milnor attractor, but the converse is not true. In fact, a topological attractor is such that its basin $B(A)$ contains an open neighborhood of A, whereas for a Milnor attractor initial conditions arbitrarily close to A can generate trajectories that are locally repelled out from A. In this case $B(A)$ is called realm of attraction reserving the term basin when $B(A)$ is an open set. However, since the term basin is more standard in the literature, we shall use such term even when A is a Milnor (but not topological) attractor, for which $B(A)$ is not, in general, an open set. Results on transverse stability have mainly been studied when the dynamics restricted to the invariant submanifold are chaotic. In this case the question of transverse stability is related to the phenomenon of chaos synchronization. The particular feature of the invariance of a submanifold of lower dimension is a standard occurrence if the map T has some symmetry property, a situation that often occurs in applications. Not stable in Lyapunov sense, attractors appear quite naturally in this context, together with new and striking phenomena, like riddled basins and intermingled basins. We choose parameter values $a = -1.3, b = 0.92$ and $c = -1$, numerically computed basins of attraction of cycles of period $1, 6, 11$ are shown in Fig. 2.

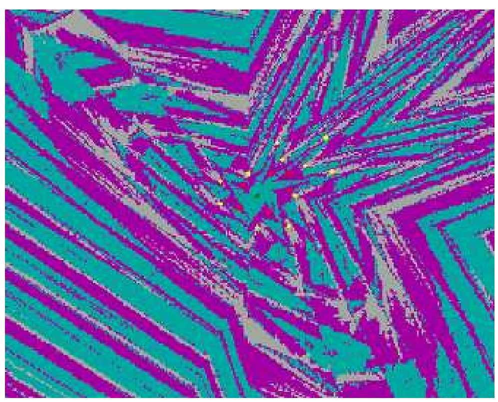

Fig. 2. The basin structures for the map (1) with the parameters $a = -1.3$, $b = 0.92$, $c = -1$: in green period-1 basin, in pink period-6 basin, in gray basin of cycle 11.

This figure show that the structure may be quite different as the values of parameters vary, and we try to understand the basic mechanisms that cause such qualitative changes. In order to understand how complex basin structures are obtained, we start from a situation in which periodic points of these cycles are represented in Fig. 2, their basins of attraction are shown, represented by different colors. Numerical basin results strongly suggests that basins of attraction are intermingled.

Again for the same value of c and $a = -0.5$, $b = 0.92$, we have plotted the different attractors and their basins (fig. 3). The basin structure in this case roughly resembles to Fig. 2.

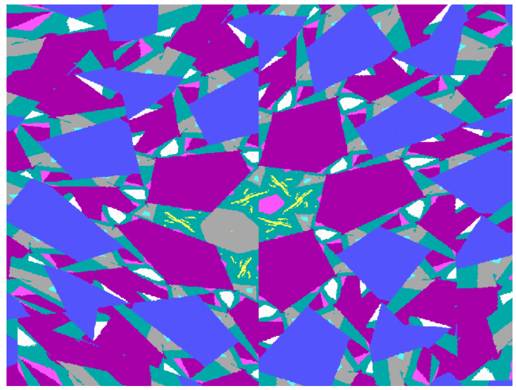

Fig. 3. Basins of attraction for the map (1) with the parameters $a = -0.5$, $b = 0.96$, $c = -5$: in green period-6 chaotic attractor basin, in pink period-5 cycle basin, in grey basin of fixed point.We have a multistability and coexistence of several attractors.

These numerical basin results strongly suggests that basins of attraction are intermingled. Further evidence of intermingling is provided by examining the dynamics of map (1).

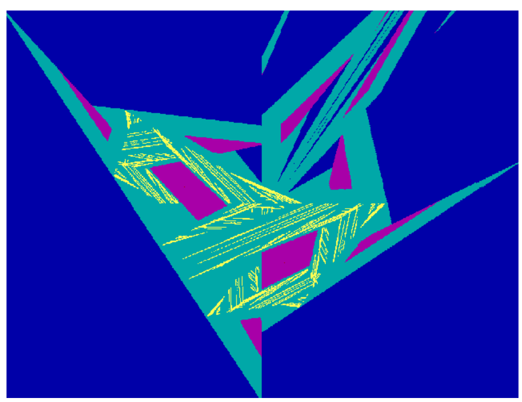

Fig. 4. Riddled Basins of attraction for the map (1). A chaotic attractor coexists with an attractive cycle of period 2.

For $c = 4$, and different values for a and b. Specifically, when $a = 1.28$, $b = -0.89$, we observe riddled basins of a chaotic attractor confined in a minimal absorbing area and a cycle of period 2 (Fig.4). When the value of the parameter

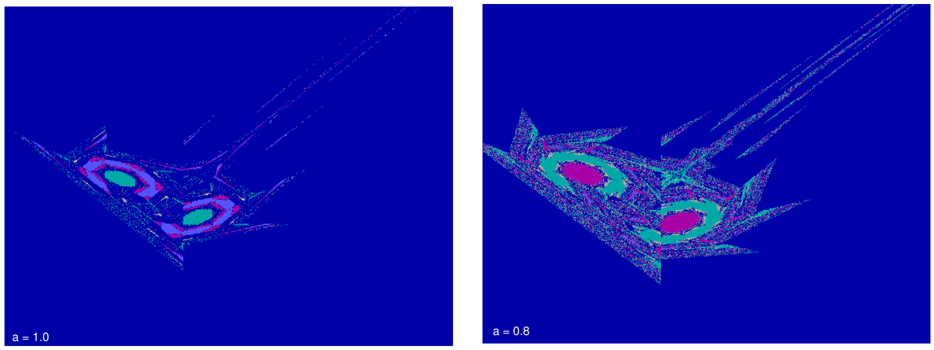

Fig. 5. The different changes for basins of attraction for $a = 1.0$ and $a = 0.8$.

a decreases the chaotic attractor disappears by a contact bifurcation with its basin boundary. For $a = 1.15$, $b = -0.89$ the map (1) has two cycles of periods 2, and 8, their basins are intermingled. Such contact bifurcations can only be revealed numerically, since the equations of the curves involved in the contact cannot be analytically expressed in terms of elementary functions. This happens frequently in nonlinear dynamical systems of dimension greater than one, where the study of global bifurcations is generally obtained through an interplay between theoretical and numerical methods, and the occurrence of these bifurcations is shown by computer-assisted proofs, based on the knowledge of the properties of the map.

In Fig. 5 and 6, we give different changes of attractor's basins by varying parameter a, where $b = -0.98$, and $c = 4.0$.

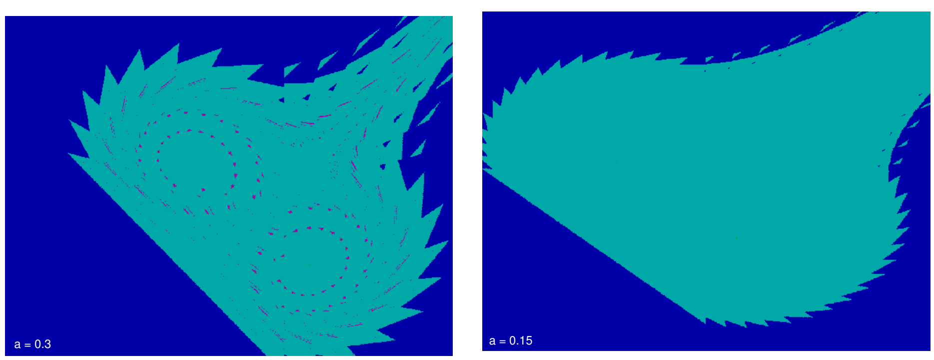

Fig. 6. The different changes for basins of attraction for $a = 0.3$ and $a = 0.15$.

The occurrence of the bifurcation which transforms the basins from simply connected to nonconnected causes a loss of predictability about the long-run evolution of the map.

The presence of the infinitely many components of both basins causes a sort of sensitivity with respect to these initial conditions.

3 Conclusion

In this paper we investigate bifurcations associated with two-dimensional piecewise linear map in the parameter space. Our principal results are on intermingled basins, it seems that they appear for a very large region of phase space.

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