

# Closed-Form Variance Formula of the RPHD Single-Tone Frequency Estimator

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**Abstract:** In this paper, we provide a very convenient, closed-form formula for the variance of the single tone frequency estimator in the reformed Pisarenko harmonic decomposer method. Several computer experiments show that calculated and measured variances are in excellent agreement. Our formula is useful when a small number of signal samples are used for frequency estimation, as asymptotic forms can lead to important errors.

**Keywords:** noisy real sinusoid, frequency estimation, RPHD method, variance analysis

## 1 Introduction

The problem of the frequency estimation of a single real sinusoid from a finite number of noisy data samples is relevant for various applications, such as speech analysis, radar, sonar, communication systems, measurements, and adaptive control [1–3]. To solve this problem, many frequency estimation techniques have been proposed and analyzed, for example the modified covariance (MC) method [4, 5] and the Pisarenko harmonic decomposer (PHD) method [6–8].

In a recently proposed method, called reformed Pisarenko harmonic decomposer (RPHD) [9–11], a closed-form, asymptotically unbiased frequency estimator is proposed and analyzed, based on the linear prediction (LP) property of sinusoidal signals, and on a modified least-squares (LS) cost function. The same frequency estimator is derived in [12] from a constrained notch-filter point of view.

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In this paper we derive a closed-form expression of the RPHD variance, based on the same variance analysis technique used in [11, 13]. We obtained a much more convenient formula as compared to [11].

In section 2, the RPHD method for single-tone frequency estimation is reviewed based on [9–11]. The variance analysis technique, the main steps and results of the closed-form formula derivation for RPHD variance are presented in Section 3. Details of this derivation are developed in the Appendix. Results of computer experiments, presented in Section 4, show that the measured RPHD variance agrees with the analytical calculations based on the derived formula. Conclusions are drawn in the last section.

## 2 RPHD method

In single frequency estimation, the following signal model is used [11]:

$$x(n) = s(n) + q(n) = \alpha \cos(\omega_0 n + \phi) + q(n), \quad n = 0, 1, 2, \dots, N \quad (1)$$

where  $\alpha > 0$ ,  $\omega_0 \in (0, \pi)$ , and  $\phi$  are the unknown amplitude, frequency and phase of the sinusoid, and  $q(n)$  is a zero-mean white noise, which we will suppose Gaussian. The sinusoid is linearly predictable from the past samples:

$$s(n) = 2 \cos(\omega_0) s(n-1) - s(n-2), \quad (2)$$

a fact that allows to define an error function

$$e(n) = x(n) - 2 \cos(\lambda) x(n-1) + x(n-2), \quad (3)$$

where  $\lambda$  is the parameter to be determined.

In order to obtain an asymptotically unbiased frequency estimator, a modified error function is defined in [11] as follows:

$$\varepsilon(n) = \frac{e(n)}{\sqrt{2(2 + \cos(2\lambda))}}. \quad (4)$$

The corresponding LS cost function is

$$J_N(\lambda) = \sum_{n=3}^N \frac{e^2(n)}{2(2 + \cos(2\lambda))}. \quad (5)$$

Solving  $\frac{dJ_N(\lambda)}{d\lambda} = 0$  yields [9–11]:

$$2A_N \cos^2(\lambda) - B_N \cos(\lambda) - A_N = 0 \quad (6)$$

where

$$A_N = \sum_{n=3}^N (x(n) + x(n-2))x(n-1) \quad (7)$$

and

$$B_N = \sum_{n=3}^N \left( (x(n) + x(n-2))^2 - 2x^2(n-1) \right). \quad (8)$$

The root of (6) that provides the estimate is

$$\rho^* = \frac{B_N + \sqrt{B_N^2 + 8A_N^2}}{4A_N}. \quad (9)$$

The frequency estimate, which is denoted by  $\hat{\omega}_0$ , is computed as

$$\hat{\omega}_0 = \cos^{-1}(\rho^*). \quad (10)$$

### 3 RPHD variance development

The variance analysis technique [11] is based on defining a second order polynomial

$$f(\rho) = 2A_N\rho^2 - B_N\rho - A_N \quad (11)$$

and utilizes the following formulas

$$\text{var}\{\rho^*\} \approx \frac{E\{f^2(\rho)\}}{\left(E\{f'(\rho)\}\right)^2} \Bigg|_{\rho=\cos(\omega_0)} \quad (12)$$

$$\text{var}(\hat{\omega}_0) \approx \frac{\text{var}(\rho^*)}{\sin^2(\omega_0)}. \quad (13)$$

By using

$$\rho = \cos(\omega_0) \quad (14)$$

and (11) there results

$$f^2(\rho) \Big|_{\rho=\cos(\omega_0)} = \cos^2(2\omega_0)A_N^2 - 2\cos(\omega_0)\cos(2\omega_0)A_NB_N + \cos^2(\omega_0)B_N^2 \quad (15)$$

$$f'(\rho) \Big|_{\rho=\cos(\omega_0)} = 4\cos(\omega_0)A_N - B_N. \quad (16)$$

In order to compute the variance of  $\hat{\omega}_0$  using (12) and (13), the values of  $E\{A_N\}$ ,  $E\{B_N\}$ ,  $E\{A_N^2\}$ ,  $E\{B_N^2\}$ , and  $E\{A_NB_N\}$  are required. The main steps that we followed in computing these terms are presented in the Appendix. We obtained:

$$E\{A_N\} = \alpha^2(N-2 + \beta(2, N-1)) \cos(\omega_0) \quad (17)$$

$$E\{B_N\} = \alpha^2(N-2 + \beta(2, N-1)) \cos(2\omega_0) \quad (18)$$

$$\begin{aligned} E\{A_N^2\} &= \alpha^4(N-2 + \beta(2, N-1))^2 \cos^2(\omega_0) \\ &\quad + \alpha^2 \sigma_q^2 [4N-10 + (4N-12) \cos(2\omega_0) + \beta(2, N-1)(2 + \cos(2\omega_0)) \\ &\quad + 2\beta(2, N-2) + 2\beta(3, N-1) + \beta(3, N-2)] + \sigma_q^4(4N-10) \end{aligned} \quad (19)$$

$$\begin{aligned} E\{B_N^2\} &= \alpha^4(N-2 + \beta(2, N-1))^2 \cos^2(2\omega_0) \\ &\quad + \alpha^2 \sigma_q^2 [4N + 4(N-4) \cos(4\omega_0) + 8\beta(2, N-1)(2 + \cos(2\omega_0)) \\ &\quad - 16\beta(2, N-2) - 16\beta(3, N-1) + 8\beta(3, N-2)(1 + \cos(2\omega_0))] \\ &\quad + \sigma_q^4(4N-8) \end{aligned} \quad (20)$$

$$\begin{aligned} E\{A_N B_N\} &= \alpha^4(N-2 + \beta(2, N-1))^2 \cos(\omega_0) \cos^2(2\omega_0) \\ &\quad + \alpha^2 \sigma_q^2 [(4N-14)(\cos(\omega_0) + \cos(3\omega_0)) \\ &\quad + 2(\beta(2, N-2) + \beta(3, N-1)) \frac{\cos(2\omega_0)}{\cos(\omega_0)} + 4\beta(3, N-2) \cos(\omega_0)] \end{aligned} \quad (21)$$

where

$$\begin{aligned} \beta(k_1, k_2) &= \sum_{n=k_1}^{k_2} \cos(2(\omega_0 n + \phi)) \\ &= \frac{\sin(\omega_0(k_2 - k_1 + 1)) \cos(\omega_0(k_2 + k_1) + 2\phi)}{\sin(\omega_0)} \end{aligned} \quad (22)$$

By using (12)...(21), and after denoting the signal-to-noise ratio

$$SNR = \frac{\alpha^2}{2\sigma_q^2}, \quad (23)$$

we obtained the closed-form variance formula of the RPHD single-tone frequency estimator as

$$\begin{aligned} \text{var}\{\hat{\omega}_0\} &\approx \frac{2 + 2(2 + \cos(2\omega_0))\beta(2, N-1) - 2\beta(2, N-2) - 2\beta(3, N-1) + \beta(3, N-2)}{2SNR(N-2 + \beta(2, N-1))^2 \sin^2(\omega_0)} \\ &\quad + \frac{(2N-5) \cos^2(2\omega_0) + (2N-4) \cos^2(\omega_0)}{2SNR^2(N-2 + \beta(2, N-1))^2 (2 + \cos(2\omega_0))^2 \sin^2(\omega_0)}. \end{aligned} \quad (24)$$

The shape of the variance in (24) is more convenient and it has a simpler structure when compared to the original results presented in [11]. In that paper, an asymptotic form of the variance is also considered. That form can be approached by making  $\beta = 0$  in (24):

$$\begin{aligned} \text{var}_{\text{asympt}}\{\hat{\omega}_0\} &\approx \frac{1}{SNR(N-2)^2 \sin^2(\omega_0)} \\ &+ \frac{(2N-5)\cos^2(2\omega_0) + (2N-4)\cos^2(\omega_0)}{2SNR^2(N-2)^2(2 + \cos(2\omega_0))^2 \sin^2(\omega_0)} \end{aligned} \quad (25)$$

Minor differences are probably caused by small calculation errors in the original work.

#### 4 Numerical examples

In order to confirm the expression (24) for the variance, and to evaluate its asymptotic form (25), we have performed some computer experiments. RPHD variances have been measured for data sequences we have generated using (1), with  $\alpha = \sqrt{2}$ , and several values for  $\sigma_q^2$ . In every experiment we computed the frequency estimate using (9) and (10) for 500 independent runs, and we evaluated the measured frequency variance of the RPHD method in terms of the mean square frequency errors.

We used for evaluation purposes the Cramer-Rao lower bound (CRLB) for the frequency estimator of a single sinusoid [3]

$$CRLB = \frac{24\sigma_q^2}{N(N^2 - 1)\alpha^2}. \quad (26)$$

The measured frequency variances with the RPHD method, the theoretical frequency variances calculated with (24), the asymptotic forms calculated with (25), and the CRLB are shown in every figure that illustrates the experimental results.

In Fig. 1 and Fig. 2, the frequency variances versus  $\omega_0$  are represented, for  $N = 20$ ,  $SNR = 20$  dB,  $\phi = 0$ , and  $\phi = \pi/4$  respectively. The measured variances are very close to the variances calculated with (24), and fluctuate in function of frequency and phase around the curve representing the asymptotic variance (they are not always symmetric around  $\omega_0 = \pi/2$ ;  $\beta(k_1, k_2)$  is symmetric around  $\omega_0 = \pi/2$  for  $\phi = 0$ , but it is not for  $\phi = \pi/4$ ).

In Fig. 3 and Fig. 4, the frequency variances versus SNR are represented, at  $\omega_0 = 0.2\pi$ ,  $N = 20$ ,  $\phi = 0$ , and  $\phi = \pi/4$  respectively. It can be noticed that, like for all methods, errors increase at low values of the SNR.

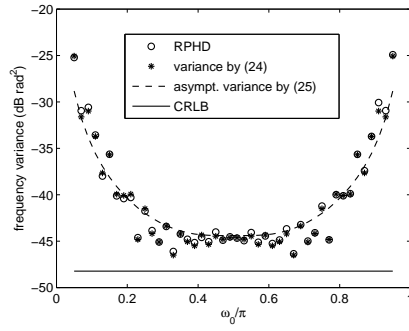


Fig. 1. Frequency variances versus  $\omega_0$  at  $SNR = 20$  dB,  $N = 20$  and  $\phi = 0$ .

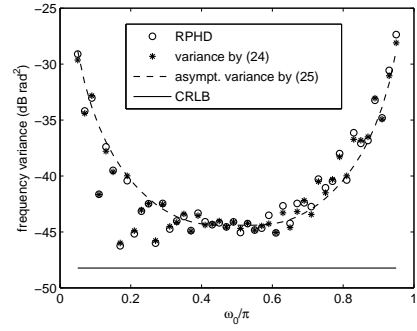


Fig. 2. Frequency variances versus  $\omega_0$  at  $SNR = 20$  dB,  $N = 20$  and  $\phi = \pi/4$ .

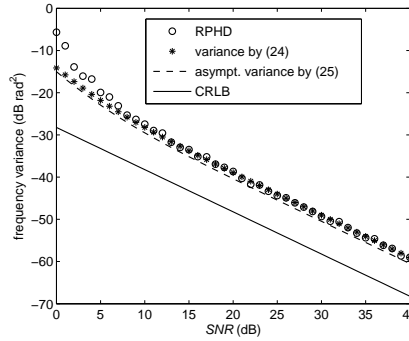


Fig. 3. Frequency variances versus  $SNR$  at  $\omega_0 = 0.2\pi$ ,  $N = 20$  and  $\phi = 0$ .

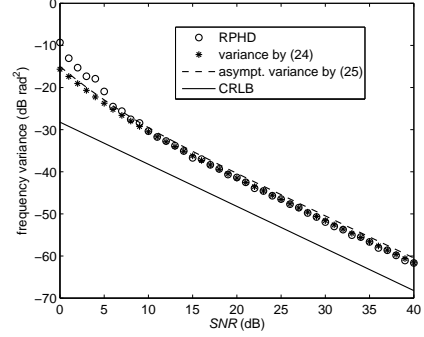


Fig. 4. Frequency variances versus  $SNR$  at  $\omega_0 = 0.2\pi$ ,  $N = 20$  and  $\phi = \pi/4$ .

Fig. 5 and Fig. 6 show the frequency variances in function of  $N$  for  $\omega_0 = 0.2\pi$ ,  $SNR = 20$  dB,  $\phi = 0$ , and  $\phi = \pi/4$  respectively. The coincidence between the measured and the calculated variances is remarkable again. It can be seen from (24) and (25) that, while the calculated variance depends on  $\phi$ , its asymptotic form does not. Figures 5 and 6 show that important differences between the two can occur in function of  $\phi$ , for the small values of  $N$  that are used in practice and in our experiments, and attenuate relatively when  $N$  becomes large.

## 5 Conclusions

In this contribution we provided a very convenient, closed-form expression for the variance of the frequency estimator in the RPHD method of single-tone frequency estimation from a finite set of data samples. The signal model consisted in a sam-

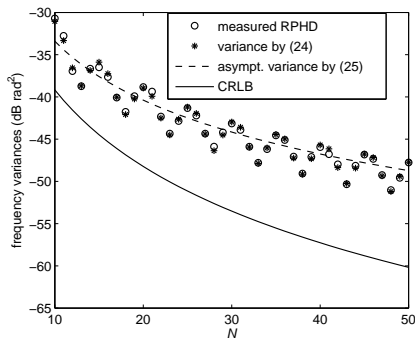


Fig. 5. Frequency variances versus  $N$  at  $\omega_0 = 0.2\pi$ ,  $SNR = 20$  dB and  $\phi = 0$ .

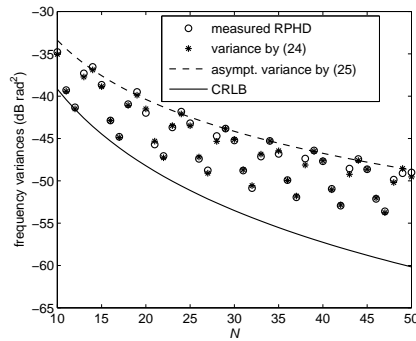


Fig. 6. Frequency variances versus  $N$  at  $\omega_0 = 0.2\pi$ ,  $SNR = 20$  dB and  $\phi = \pi/4$ .

pled sinusoid, embedded in white, gaussian noise. We also derived an asymptotic form of the variance, which agreed with previously published results. We validated our results by means of several computer experiments. Coincidence between the experimental variances and those calculated with our formula (24) was remarkable in all cases, except for very low signal-to-noise ratios. According to the experimental results, the fact that the asymptotic variance (25) does not depend on the phase of the sampled sinusoid, while (24) does, can be a serious drawback, so that quite large errors are possible in the case of a small number of samples. This is an argument for the usefulness of our closed-form formula (24) as, in common applications, a number as small as possible of signal samples is desired.

## Appendix

The main steps that are necessary for computing the values of  $E\{A_N\}$ ,  $E\{B_N\}$ ,  $E\{A_N^2\}$ ,  $E\{B_N^2\}$ , and  $E\{A_N B_N\}$  are presented in this Appendix.

We use (1), (2), (7), and (22) in order to get

$$A_N = T_{A1} + T_{A2} + T_{A3}, \quad (\text{A1})$$

where

$$\begin{aligned} T_{A1} &= \sum_{n=3}^N 2\alpha^2 \cos(\omega_0) \cos^2(\omega_0(n-1) + \phi) \\ &= \alpha^2 (N-2 + \beta(2, N-1)) \cos(\omega_0), \end{aligned} \quad (\text{A2})$$

$$T_{A2} = \sum_{n=3}^N \alpha \cos(\omega_0(n-1) + \phi) (2\cos(\omega_0)q(n-1) + q(n) + q(n-2)), \quad (\text{A3})$$

and

$$T_{A3} = \sum_{n=3}^N q(n-1)(q(n) + q(n-2)). \quad (\text{A4})$$

As  $q(n)$  is white, with zero mean, we have

$$E\{q(n)\} = 0; \quad E\{q(n)q(m)\} = \delta_m^n \sigma_q^2. \quad (\text{A5})$$

There results  $E\{T_{A2}\} = E\{T_{A3}\} = 0$ ,  $E\{A_N\} = E\{T_{A1}\} = T_{A1}$  and (17).

In order to calculate  $E\{A_N^2\}$ , we note that, due to (A5), we have  $E\{T_{Ai}T_{Aj}\} = 0$  for  $i \neq j$ . Therefore

$$E\{A_N^2\} = T_{A1}^2 + E\{T_{A2}^2\} + E\{T_{A3}^2\}. \quad (\text{A6})$$

We evaluate now the second and the third term in the RHS of (A6).

$$\begin{aligned} E\{T_{A2}^2\} &= \sum_{n=3}^N \alpha^2 \cos^2(\omega_0(n-1) + \phi) \\ &\quad \times [4 \cos^2(\omega_0) E\{(q(n-1))^2\} + E\{(q(n))^2\} + E\{(q(n-2))^2\}] \\ &\quad + 2 \sum_{n=3}^N \sum_{m=n+1}^N \alpha^2 \cos(\omega_0(n-1) + \phi) \cos(\omega_0(m-1) + \phi) \\ &\quad \times E\{(2 \cos(\omega_0)q(n-1) + q(n) + q(n-2)) \\ &\quad \times (2 \cos(\omega_0)q(m-1) + q(m) + q(m-2))\}. \end{aligned}$$

By performing the multiplications and by using (A5) there results

$$\begin{aligned} E\{T_{A2}^2\} &= \sum_{n=3}^N \alpha^2 \frac{1 + \cos(2\omega_0(n-1) + 2\phi)}{2} (4\sigma_q^2 \cos^2(\omega_0) + 2\sigma_q^2) \\ &\quad + \sum_{n=3}^N \sum_{m=n+1}^N \alpha^2 [\cos(\omega_0(n+m-2) + 2\phi) + \cos(\omega_0(m-n))] \\ &\quad \times [2 \cos(\omega_0) E\{q(n-1)q(m-2)\} \\ &\quad + 2 \cos(\omega_0) E\{q(n)q(m-1)\} + E\{q(n)q(m-2)\}]. \end{aligned}$$



Using again (A5) we get

$$\begin{aligned}
 E\{T_{A2}^2\} &= \alpha^2 \sigma_q^2 (N-2 + \beta(2, N-1)) (\cos(2\omega_0) + 2) \\
 &\quad + \alpha^2 \sigma_q^2 \sum_{n=3}^{N-1} 4 \cos(\omega_0) \left( \cos((2n-1)\omega_0 + 2\phi) + \cos(\omega_0) \right) \\
 &\quad + \alpha^2 \sigma_q^2 \sum_{n=3}^{N-2} (\cos(2\omega_0 n + 2\phi) + 2 \cos(2\omega_0)) \\
 &= \alpha^2 \sigma_q^2 [4N - 10 + (4N - 12) \cos(2\omega_0) \\
 &\quad + \beta(2, N-1)(2 + \cos(2\omega_0)) + 2\beta(2, N-2) \\
 &\quad + 2\beta(3, N-1) + \beta(3, N-2)]. \tag{A7}
 \end{aligned}$$

Then

$$\begin{aligned}
 E\{T_{A3}^2\} &= E\left\{ \sum_{n=3}^N [(q(n-1)q(n))^2 + 2q^2(n-1)q(n)q(n-2) \right. \\
 &\quad \left. + (q(n-1)q(n-2))^2] \right\} + 2E\left\{ \sum_{n=3}^N \sum_{m=n+1}^N q(n-1)q(m-1) \right. \\
 &\quad \left. \times (q(m) + q(m-2))(q(n) + q(n-2)) \right\} \\
 &= 2(N-2)\sigma_q^4 + 2 \sum_{n=3}^{N-1} E\{q^2(n)q^2(n-1)\} \\
 &= (4N-10)\sigma_q^4. \tag{A8}
 \end{aligned}$$

With the use of (A2), (A6), (A7), and (A8) we obtain (19).  
 From (1), (2), (8), and (22) we get

$$B_N = T_{B1} + T_{B2} + T_{B3} \tag{A9}$$

where

$$T_{B1} = \alpha^2 (N-2 + \beta(2, N-1)) \cos(2\omega_0), \tag{A10}$$

$$\begin{aligned}
 T_{B2} &= 4\alpha \sum_{n=3}^N \cos(\omega_0(n-1) + \phi) \\
 &\quad \times \left( \cos(\omega_0)(q(n) + q(n-2)) - q(n-1) \right), \tag{A11}
 \end{aligned}$$

and

$$T_{B3} = \sum_{n=3}^N (q^2(n) + q^2(n-2) + 2q(n)q(n-2) - 2q^2(n-1)). \tag{A12}$$

The application of (A5) yields as before  $E\{B_N\} = T_{B1}$ , (18) and

$$E\{B_N^2\} = T_{B1}^2 + E\{T_{B2}^2\} + E\{T_{B3}^2\}. \quad (\text{A13})$$

We have

$$\begin{aligned} E\{T_{B2}^2\} &= 16\alpha^2 \sum_{n=3}^N \cos^2(\omega_0(n-1) + \phi) (2\sigma_q^2 \cos^2(\omega_0) + \sigma_q^4) \\ &\quad + 32\alpha^2 \sum_{n=3}^N \sum_{m=n+1}^N \cos(\omega_0(n-1) + \phi) \cos(\omega_0(m-1) + \phi) \\ &\quad \times E\left\{ \left( \cos(\omega_0)(q(n) + q(n-2)) - q(n-1) \right) \right. \\ &\quad \left. \times \left( \cos(\omega_0)(q(m) + q(m-2)) - q(m-1) \right) \right\}. \end{aligned}$$

By performing the multiplications and by using again (A5) we get

$$\begin{aligned} E\{T_{B2}^2\} &= 16\alpha^2 \sigma_q^2 (2\cos^2(\omega_0) + 1) \sum_{n=3}^N \frac{1 + \cos(2\omega_0(n-1) + 2\phi)}{2} \\ &\quad + 16\alpha^2 \sum_{n=3}^N \sum_{m=n+1}^N \left( \cos(\omega_0(m+n-2) + 2\phi) + \cos(\omega_0(m-n)) \right) \\ &\quad \times \left[ \cos^2(\omega_0) E\{q(n)q(m-2)\} \right. \\ &\quad \left. - \cos(\omega_0) (E\{q(n)q(m-1)\} + E\{q(n-1)q(m-2)\}) \right] \\ &= 8\alpha^2 \sigma_q^2 (N-2 + \beta(2, N-1)) (\cos(2\omega_0) + 2) \\ &\quad + 16\alpha^2 \sigma_q^2 \sum_{n=3}^{N-2} (\cos(2\omega_0 n + 2\phi) + \cos(2\omega_0)) \cos^2(\omega_0) \\ &\quad - 32\alpha^2 \sigma_q^2 \sum_{n=3}^{N-1} \left( \cos(\omega_0(2n-1) + 2\phi) + \cos(\omega_0) \right) \cos(\omega_0). \end{aligned}$$

After some manipulations there results

$$\begin{aligned} E\{T_{B2}^2\} &= \alpha^2 \sigma_q^2 [4N + 4(N-4) \cos(4\omega_0) + 8\beta(2, N-1) (2 + \cos(2\omega_0)) \\ &\quad - 16\beta(2, N-2) - 16\beta(3, N-1) + 8\beta(3, N-2) (1 + \cos(2\omega_0))]. \end{aligned} \quad (\text{A14})$$

For the last term in (A9) we have

$$\begin{aligned} E\{T_{B3}^2\} &= 4(N-2) \sigma_q^4 \\ &\quad + 2 \sum_{n=3}^N \sum_{m=n+1}^N E\left\{ (q^2(n) + q^2(n-2) + 2q(n)q(n-2) - 2q^2(n-1)) \right. \\ &\quad \left. \times (q^2(m) + q^2(m-2) + 2q(m)q(m-2) - 2q^2(m-1)) \right\} \\ &= 4(N-2) \sigma_q^4. \end{aligned} \quad (\text{A15})$$

With the use of (A10), (A13), (A14), and (A15) we obtain (20).

We now start from (A1)...(A4) and A(9)...A(12) in order to obtain

$$E\{A_N B_N\} = T_{C1} + T_{C2} + T_{C3} \quad (\text{A16})$$

where

$$T_{C1} = \alpha^4 \cos(\omega_0) \cos(2\omega_0) (N - 2 + \beta(2, N - 1))^2, \quad (\text{A17})$$

$$\begin{aligned} T_{C2} &= \alpha^2 \sum_{n=3}^N \sum_{m=3}^N 4 \cos(\omega_0(n-1) + \phi) \cos(\omega_0(m-1) + \phi) \\ &\quad \times E \left\{ \left( 2 \cos(\omega_0) q(n-1) + q(n) + q(n-2) \right) \right. \\ &\quad \left. \times \left( \cos(\omega_0) (q(m) + q(m-2) - q(m-1)) \right) \right\}, \end{aligned} \quad (\text{A18})$$

and

$$\begin{aligned} T_{C3} &= E \left\{ \sum_{n=3}^N \sum_{m=3}^N q(n-1) (q(n) + q(n-2)) \right. \\ &\quad \left. \times (q^2(m) + q^2(m-2) + 2q(m)q(m-2) - 2q^2(n-1)) \right\} \\ &= 0. \end{aligned} \quad (\text{A19})$$

The last equality in (A19) follows from (A5).

The quantity defined in (A18) can be calculated as follows

$$\begin{aligned} T_{C2} &= 2\alpha^2 \sum_{n=3}^N \sum_{m=3}^N \left( \cos(\omega_0(m+n-2) + 2\phi) + \cos(\omega_0(m-n)) \right) \\ &\quad \times \left( (4\cos^2(\omega_0) - 2)q(n-1)q(m) + 2\cos(\omega_0)q(n-1)q(m) \right) \\ &= 4\alpha^2 \sigma_q^2 \sum_{m=3}^{N-1} \cos(2\omega_0) \left( \cos(\omega_0(2m-1) + 2\phi) + \cos(\omega_0) \right) \\ &\quad + 4\alpha^2 \sigma_q^2 \sum_{m=3}^{N-2} \cos(\omega_0) \left( \cos(2\omega_0 m + 2\phi) + \cos(2\omega_0) \right). \end{aligned}$$

After some straightforward transformations we get

$$\begin{aligned} T_{C2} &= \alpha^2 \sigma_q^2 [(4N - 14) (\cos(\omega_0) + \cos(3\omega_0)) \\ &\quad + 2(\beta(2, N - 2) + \beta(3, N - 1)) \frac{\cos(2\omega_0)}{\cos(\omega_0)} + 4\beta(3, N - 2) \cos(\omega_0)]. \end{aligned} \quad (\text{A20})$$

With the use of (A16), (A17), (A19), and (A20) we obtain (21).

## References

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