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Interval Criterion for Stability Analysis of Discrete-Time Nonlinear Systems With Partial State Saturation Nonlinearities

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Abstract: A generalization of sufficient conditions for global asymptotic stability of the equilibrium of discrete-time nonlinear systems with saturation nonlinearities on part of the states in the case of interval uncertainties is considered. When using quadratic form Lyapunov functions, sufficient conditions based on the positive definite interval matrices are presented. In order to check this, a recently proposed method for determining the outer bounds of eigenvalues ranges is used. A numerical example illustrating the applicability of the method suggested is solved in the end of the paper.

Keywords: Robust stability analysis, outer bounds on eigenvalues of interval matrices with independent coefficients.

1 Introduction

It is well known that the model of discrete-time dynamical nonlinear systems with partial state saturation is [1]

$$x(k+1) = g[Ax(k)], \quad k = 0,1,\dots \tag{1}$$

where $x(k) = \begin{bmatrix} x_I(k) \\ \dots \\ x_{II}(k) \end{bmatrix} \in X_n^{n_1} \triangleq \left\{ y = \begin{bmatrix} y_I \\ \dots \\ y_{II} \end{bmatrix} : Y_i \in R^{n_1}, y_{II} \in D^{n_2} \right\}$,
$A = [a_{ij}] \in R^{n \times n},\ n = n_1 + n_2,\ D^{n_2} = \{y \in R^{n_2} : -1 \le y_i \le 1, i = 1, \dots n_2\}$,

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$$g(x) = \begin{bmatrix} x_I \\ \dots \\ \mathrm{sat}(\mathrm{x_{II}}) \end{bmatrix} \text{ for } \mathrm{x} = \begin{bmatrix} x_I \\ \dots \\ x_{II} \end{bmatrix}, \quad \mathrm{x_I} \in \mathrm{R^{n_1}}, \quad \mathrm{x_{II}} \in \mathrm{R^{n_2}},$$

$$\mathrm{sat}(\mathrm{x_{II}}) = [\mathrm{sat}(\mathrm{x_1}), \dots, \mathrm{sat}(\mathrm{x_{II}})]^{\mathrm{T}},$$

$$\mathrm{sat}(\mathrm{x_i}) = \begin{cases} 1, & x_i > 1 \\ x_i, & -1 \leq x_i \leq 1 \\ -1, & x_i < -1 \end{cases}$$

In particular, this general model describes also the discrete-time neural networks [2] working on hypercube.

This type of nonlinear systems has been investigated by many researchers (see e.g. [2–11]). They are stable if $x_e = 0$ is the only equilibrium of system (1) and in this case it is globally asymptotically stable. The condition of stability of matrix A (i.e. every eigenvalue λ_i of A satisfy $|\lambda_i| < 1$) does not ensure that $x_0 = 0$ is a unique equilibrium, and hence, it does not ensure that $x_e = 0$ is asymptotically stable in the large. For this reason, necessary and sufficient conditions of asymptotically stability of system (1), are proposed in [1].

When the elements of matrix A are intervals there are publications concerning the ranges of its eigenvalues in the case of continuous- and discrete-time systems [12–14] as well as inner and outer estimates of its bounds [12–15]. In some cases, the outer estimates may be rather conservative (they overestimate the range considerably) and lead to inconclusive stability analysis results, but always they can be consider as sufficient conditions for stability of the systems considered.

The paper is organized as follows. The problem statement when the elements of matrix A are independent intervals is described in the next section. A method for obtaining the outer bounds on the studied eigenvalues is presented in Section 3. Numerical example illustrating the applicability of the new method is solved in Section 4. The paper ends with concluding remarks in the last Section 5.

2 Problem Statement

Examine system (1). In practice the elements of matrix A cannot be determined exactly. Hence we will consider them as independent intervals. (In general, they are dependent intervals, but in first case the outer bounds of the eigenvalue ranges are larger which guarantee the stability of the system studied). Let A be a real $n \times n$ matrix, $\boldsymbol{A}$ - an interval matrix containing A, and A^-, A^+, A^0 and R_A - the left end, the right end, the center and the radius of $\boldsymbol{A}$, respectively (throughout the paper, bold face letters will be used to denote interval quantities while ordinary letters will stand for their non-interval counterparts).

2.1 Stability of the central problem

Based on [1] we apply the Corollary 1 of Theorem 1 for the central matrix A^0, i.e. the equilibrium of time-discrete nonlinear systems described by system (1) is globally asymptotic stable if

$$||A^0||_p < 1, \text{ for some } p = 1, 2, \infty. \tag{2}$$

This result is obtained by choosing a Lyapunov function $V(x) = ||x||_{1,2 \text{ or } \infty}$.

Let $y_s = \text{sat(y)}$ for $y \in r^N$ and let H denote a positive define matrix. Assume that

$$y_s^T H y_s < y^T H y \tag{3}$$

for all $y \in R^N$, $y \notin D^N = \{y \in R^N : -1 \leq y_i \leq 1,\ i = 1, \ldots, N$. If quadratic form Lyapunov function, based on the Assumption $(A-2)$ from [1] is taken, then the necessary and sufficient condition for stability of central matrix are connected with a $N \times N$ positive define matrix H. This matrix satisfies the Assumption $(A-2)$ if and only if

$$h_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^{N} |h_{ij}|,\ i = 1, \ldots, N. \tag{4}$$

Then the Theorem 2 from [1] can be written in the form:

Theorem 1: The equilibrium $x_e = 0$ of system (1) for A^0 is globally asymptotically stable, if A^0 is stable and if there exist positive definite matrices $H_I \in R^{n_1 \times n_1}$ and $H_{II} \in R^{n_2 \times n_2}$ with H_{II} satisfying (4) (with $N + n_2$), such that is $Q^0 = H - (A^0)^T H A^0$ positive semidefinite, where

$$H = \begin{bmatrix} H_I & 0 \\ 0 & H_{II} \end{bmatrix} \tag{5}$$

2.2 Stability of the interval problem

When the elements of matrix A are independent intervals, i.e. $A \in \boldsymbol{A}$, in accordance with the approach [1] to investigating the asymptotic stability of (1), we consider the two ”perturbed” eigenvalue problems first, for stability of interval matrix $\mathbf{A}$

$$Ax = \lambda x, \quad A \in \boldsymbol{A} = [A^-,\ A^+] = A^0 + [-R_A,\ R_A] \tag{6}$$

and second-for positive definite interval matrix $\boldsymbol{Q}$, $Q \in \boldsymbol{Q}$,

$$Q = H - (A)^T H A, \quad A \in \boldsymbol{A}, \tag{7}$$

$$Qx = \lambda x, \quad A \in \boldsymbol{A} = [A^-,\ A^+] = A^0 + [-R_A,\ R_A], \tag{8}$$

where matrix H has the same structure defined by (5). It is seen from (8) that matrix Q is implicit function of A.

Based on [1] it can be formulated the following interval criterion for asymptotic stability of (1) if $A \in \boldsymbol{A}$:

Theorem 2: The equilibrium of the discrete nonlinear system described by (1) is asymptotic stable if

- nterval matrix $\mathbf{A}$ is stable, i.e. all of its eigenvalues satisfied the condition

$$|\lambda_i^A(A)| < 1, \quad A \in \boldsymbol{A}, \quad i = 1, \ldots, n \tag{9}$$

  and

- interval matrix $\mathbf{Q}$ is positive semidefinite, i.e. all of its eigenvalues satisfied the condition

$$\lambda_i^Q(A) \geq 0, \quad A \in \boldsymbol{A}, \quad i = 1, \ldots, n \tag{10}$$

For simplicity of the presentation in next sections we will note the ranges of eigenvalues of matrices $\boldsymbol{A}$ and matrix $\boldsymbol{Q}$, with $\boldsymbol{I}^*$ and its outer bounds - with $\boldsymbol{I}$.

3 Outer Bounds on the Ranges for the Eigenvalues of Interval Matrices

There are a variety of methods for obtaining the outer bounds on the exact ranges for the eigenvalues of interval matrices with independent elements ([12, 15, 16]). In this paper, the method proposed in [16] - for real case and [15] - for complex case, is used because it provides the tight and cheap outer bounds of the ranges considered.

3.1 Outer bounds of the ranges of the eigenvalues of interval matrix $\boldsymbol{A}$

Consider again (6). It is seen from (6) that both λ and x are functions of A, i.e. $\lambda = \lambda(A)$ and $x = x(A)$. Let $x^{(k)}(A) = \left(x1^{(k)}(A), x2^{(k)}(A), \ldots, x_n^{(k)}(A)\right)^T$ be the eigenvector, corresponding to $\lambda_k(A)$, $k = 1, \ldots, n$.

Now let the pair $(\lambda^0,\ x^0)$b e the solution of the nominal (centre) problem

$$A^0 x = \lambda x \tag{11}$$

Assume that $n' \leq n$ of the components λ_k of the eigenvalue vector λ^0 are real while the remaining $n - n'$ eigenvalues are complex. To simplify the presentation of the method for obtaining outer bounds, we start by first considering the case of real eigenvalues of $\boldsymbol{A}$.

A. Real eigenvalues of $\boldsymbol{A}$

We need the following assumption (ensuring structural stability of the problem).

Assumption A1: For any $k \in K' = \{1, \ldots, n'$, all $\lambda_k(A)$ and $x^{(k)}(A)$ remain real for all $A \in \boldsymbol{A}$.

On account of Assumption A1, the intervals $\boldsymbol{I}_k^*$ for $k \in K'$ will, in this case, be real intervals

$$\boldsymbol{I}_k^* = \{\lambda_k(A) : A \in \boldsymbol{A}\}, \quad k \in K' \tag{12}$$

Thus, $\boldsymbol{I}_k^*$ is the range of $\lambda_k(A)$ when $A \in \boldsymbol{A}$.

For notational simplicity, we shall henceforth drop the index k. In this subsection, we are interested in finding an outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. an interval $\boldsymbol{I}$ with the property

$$\boldsymbol{I}^* \subset \boldsymbol{I} \tag{13}$$

Thus, the problem at hand is the following

Problem P1: Find an outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. an estimation $\boldsymbol{I}$ having the inclusion property (13). We now suggest a method for finding a "tight" outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. a bound with a small overestimation. To simplify the presentation of the method (without loss of generality), we need an additional assumption concerning the real eigenvector x^0 related to the real eigenvalue λ_k^0 considered.

Assumption A2: We assume that the nth component x_n^0 ofx^0 has the largest absolute value, i.e. $| x_n^0 | \geq | x_i^0 |$, $i = 1, \ldots, n$

Remark 1: In the general case where the index of the largest component is s, we just substitute s for n in all the relationships involved.

Now x^0 is normalized (dividing x^0 by x_n^0) to have

$$x_n^0 = 1 \tag{14a}$$

Further, we require that (14a) be also valid for $x_n(A)$, i.e.

$$x_n^0(A) = 1, \quad A \in \boldsymbol{A} \tag{14b}$$

Condition (14b) simplifies the new method for computing $\boldsymbol{I}$ to be presented below. We first introduce the n-dimensional real vector

$$y = (y_1, y_2, \ldots y_n)^T \tag{15a}$$

with

$$\begin{aligned} y_i &= x_i(A), \quad i = 1, \ldots, n-1 \\ y_n &= \lambda(A) \end{aligned} \tag{15b}$$

Using (15) and (14), (6) is rewritten as

$$\begin{aligned}&\sum_{j=1}^{n-1} a_{ij}y_j - y_n y_i + a_{in} = 0, \quad i = 1,\ldots,n-1\\ &\sum_{j=1}^{n-1} a_{nj}y_j - y_n + a_{nn}\end{aligned}\quad , \tag{16a}$$

where

$$a_i j \in \boldsymbol{a}_i j = [a_{ij}^-,\ a_{ij}^+], \tag{16b}$$

a_{ij}^- and a_{ij}^+ being the elements of matrices A^- and A^+, respectively. System (16) is a nonlinear (more precisely, an incomplete quadratic) system because of the products $y_n y_i$ in the first $n-1$ equations in (16a).

Let y_i^* denote the range of the ith component $y_i(A)$, $A \in \boldsymbol{A}$ of the solution $\mathbf{y}$ to (16). Let $\boldsymbol{y}^*$ be the vector made up of $\boldsymbol{y}_i^*$. Consider the following problem.

Problem P2: Find an outer solution y to (16), i.e. a solution enclosing the range vector y^*:

$$y^* \subset y' \tag{17}$$

Obviously, the nth component of the solution y to Problem P2 is a solution to the original Problem P1.

We now proceed to solving Problem P2. The approach adopted is based on ideas suggested recently in [13, 14]. If $z = z_0 + u \in z$ and $t = t_0 + v \in \boldsymbol{t}$, with z and $\boldsymbol{t}$ being intervals whose centers are z_0 and t_0, respectively, then

$$zt \in -z_0 t_0 + t_0 z + z_0 t + [-r_z r_1,\ r_z r_t] \tag{18}$$

where r_z and r_t are the respective radii. After letting

$$a_i j = a_{ij}^0 + u_{ij}, \quad y_i = y_i^0 + v_i, \quad i, j = 1,\ldots,n \tag{19}$$

where a_{ij}^0 are the elements of the centre matrix A^0 and y_i^0 are computed from (15) with $A = A^0$ and $u_{ij} = [-R(a_{ij}),\ R(a_{ij})]$, $v_j = [-r(y_j),\ r(y_j)]$. We apply (18) to express the products in (16a). On substitution of (19) into (16a), having in mind that the centers a_{ij} and y_i^0 satisfy system (16a) and following the techniques of [8], we get the system

$$\begin{aligned}(a_{11}^0 - y_n^0)v_1 + a_{12}^0 v_2 + \cdots + a_{1,n-1}^0 v_{n-1} - y_1^0 v_n &= \boldsymbol{b}_1\\ a_{21}^0 v_1 + (a_{22}^0 - y_n^0)v_2 + \cdots + a_{2,n-1}^0 v_{n-1} - y_2^0 v_n &= \boldsymbol{b}_2\\ &\ldots\\ a_{n-1,1}^0 v_1 + a_{n-1,2}^0 v_2 + \cdots + (a_{n-1,n-1}^0 y_n^0)v_{n-1} - y_{n-1}^0 v_n &= \boldsymbol{b}_{n-1}\\ a_{n1}^0 v_1 + a_{n2}^0 v_2 + \cdots + a_{n,n-1}^0 v_{n-1} - v_n &= \boldsymbol{b}_n\end{aligned} \tag{20a}$$

where $\boldsymbol{b}_i$ are intervals. It can be easily checked that their radii are

$$R(\boldsymbol{b}_i) = \sum_{j=1}^{n-1} |y_j^0| R_{ij} + R_{in} + \sum_{j=1}^{n-1} R_{ij} r_j + r_n r_i, \quad i = 1, 2, \ldots, n-1 \tag{20b}$$

$$R(\boldsymbol{b}_n) = \sum_{j=1}^{n-1} |y_j^0| R_{nj} + R_{nn} + \sum_{j=1}^{n-1} R_{nj} r_j \tag{20c}$$

where $R_i j$ are the elements of R_A while $r_i = R(v_i)$ is the radius of the unknown interval v_i. Now system (20) can be written in compact form

$$v = \tilde{A}_0^{-1} b, \quad b \in b \tag{21}$$

where $v = \tilde{A}_0^{-1}$ is the real coefficient matrix in (20a). Assuming $v = \tilde{A}_0^{-1}$ is invertible, let $C = |\tilde{A}_0^{-1}|$. If $r = (r_1, r_2, \ldots, r_n)^T$ and r_b denotes a column vector with components from (20b) and (20c), then from (21)

$$r = C r_b \tag{22}$$

Now we introduce the matrix: $\breve{R}$ which is the same as R except for the last column whose elements are now zeros. Using (20b), (20c) and the new notation, (22) can be put in the form

$$r = CR|x^0| + C\breve{R}r + Cg(r) \tag{23}$$

where x^0 is the normalized eigenvector and $g(r)$ is a nonlinear function with components $g_i(r) = r_i r_n$, $i = 1, \ldots, n-1$, $g_n(r) = 0$. Thus, (23) becomes

$$r = d + Dr + Cg(r) \tag{24a}$$

with

$$d = CR|x^0|, \quad D = C\breve{R} \tag{24b}$$

The matrix equation (24a) is a nonlinear real-valued (non-interval) system of n equations in n unknowns $\substack{r \\ i}$

$$r_i = d_i + \sum_{j=1}^{n-1} d_{ij} r_j + r_n \sum_{j=1}^{n-1} c_{ij} r_j, \quad i = 1, \ldots, n \tag{25}$$

The smallest positive solutions r_i of (25) solve Problem P2. Indeed, if $r_i > 0$, we can introduce the intervals

$$y_i = y_i^0 + [-r_i, r_i], \quad i = 1., \ldots, n \tag{26}$$

It can be proved that

$$y_i^* \subset y_i, \quad i = 1, \dots, n \tag{27}$$

i.e. the intervals (26) are really outer bounds on the ranges y_i^* for all i. Hence, the interval

$$y_n = y_n^0 + [-r_n,\ r_n] \tag{28}$$

is the solution to the original Problem P1 since it is, in fact, a bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$ satisfying the inclusion (13). More precisely, we have the following theorem.

Theorem 3: If the nonlinear system (25) has a positive solution $r = (r_1, r_2, \dots, r_n)^T$ that can be attained by the simple iteration method with initial vector $r^0 = 0$, then the interval (28) is an outer bound on the range $\boldsymbol{I}^*$ of the real eigenvalue $\lambda_k(A)$ considered (for a given k from K'). The present method for solving the original Problem P1 will be referred to as method M1. As shown above, it comprises, essentially, the following computations. First, the "nominal" eigenvalue problem (11) is solved. Then, for each $k \in K'$, the nonlinear system (25) is set up and the simple iteration method is applied to find the solutions r_i, $i = 1, \dots, n$. If all r_i are positive, the outer bound $\boldsymbol{I}$ on the corresponding eigenvalue $\lambda_k(A), A \in A$, is obtained by the interval (28). Since, in practice, R_A are only small percentage of a_{ij}^0, system (25) is mildly nonlinear and its solution does not present any difficulties.

B. Complex eigenvalues of $\boldsymbol{A}$

In this subsection, we are interested in the complex eigenvalues of (6), i.e. in finding outer bounds on the range $(I_k^*)_a$, with $k \in K'' = \{n+1, \dots, n\}$. In order to enclose $(\boldsymbol{I}_k^*)_a$, we need to introduce additionally the ranges

$$(\boldsymbol{I}_k^*)_{R_e} = \{\Re[\lambda_k(A)] : A \in \boldsymbol{A}\} \tag{29a}$$

$$(\boldsymbol{I}_k^*)_{Im} = \{\Im[\lambda_k(A)] : A \in \boldsymbol{A}\} \tag{29b}$$

To simplify notation, we again drop the index k and consider the intervals $\boldsymbol{I}_k^*)_{Re}$, $(\boldsymbol{I}_k^*)_{Im}$ and $(\boldsymbol{I}_k^*)_a$. The corresponding outer bounds will be denoted $\boldsymbol{I}_{Re}$, $\boldsymbol{I}_{Im}$ and $\boldsymbol{I}_a$. So

$$\boldsymbol{I}_{Re}^* \subset \boldsymbol{I}_{Re}, \quad \boldsymbol{I}_{Im}^* \subset \boldsymbol{I}_{Im} \tag{30a}$$

$$\boldsymbol{I}_a^* \subset \boldsymbol{I}_a \tag{30b}$$

If $\boldsymbol{I}_{Re}$ and $\boldsymbol{I}_{Im}$ are found, then the bound $\boldsymbol{I}_a$ can be computed as

$$\boldsymbol{I}_a = \sqrt{\boldsymbol{I}_{Re}^2 + \boldsymbol{I}_{Im}^2} \tag{31}$$

Thus, if suffices to solve the following problem.

Problem P3: Find an outer bound $\boldsymbol{I}_{Re}$ on $\boldsymbol{I}_{Re}^*$ and an outer bound $\boldsymbol{I}_{Im}$ on $\boldsymbol{I}_{Im}^*$. In this subsection the method M1 will be extended to solve Problem P3. This general method will be referred to as method M2.

Let

$$\lambda = \lambda_{Re} + j\lambda_{Im}, \quad x_i = x_{i,Re} + jx_{i,Im}, \quad i = 1,\dots,n. \tag{32}$$

As in the case of method M1, we appeal to Assumption A2 and normalize the complex eigenvalue x^0 (corresponding to a fixed $k \in K''$) through dividing all components of x^0 by $x^0_{n,Re}$

$$x^0_{n,Re} = 1quadx^0_{n,Im} = \alpha. \tag{33}$$

Further, we require that (33) be also valid for all $A \in \boldsymbol{A}$, i.e.

$$x_{n,Re}(A) = 1, \quad x_{n,Im}(A) = \alpha, \quad A \in \boldsymbol{A} \tag{34}$$

We introduce the 2n-dimensional real vector y with components

$$\begin{aligned} y_i &= x_{i,Re}(a), \quad i = 1,\dots,n-1 \\ y_n &= \lambda_{Re}(A) \\ y_{n+i} &= x_{i,Im}(A), \quad i = 1,\dots,n-1 \\ y_{2n} &= \lambda_{Im}(A) \end{aligned} \tag{35}$$

On substitution of (32) into (6), using (35), (6) becomes

$$\begin{aligned} &\sum_{j=1}^{n-1} a_{ij}y_j - y_n y_i + a_{in} + y_{2n}y_{n+i} = 0, \quad i = 1,\dots,n-1 \\ &\sum_{j=1}^{n-1} a_{nj}y_j - y_n + a_{nn} + y_{2n}\alpha = 0 \\ &\sum_{j=1}^{n-1} a_{ij}y_{n+j} - y_n y_{n+i} + a_{in} - y_{2n}y_i = 0, \quad i = 1,\dots,n-1 \\ &\sum_{j=1}^{n-1} a_{nj}y_{n+j} - y_n\alpha + a_{nn} - y_{2n} = 0 \end{aligned} \tag{36a}$$

where

$$a_{ij} \in \boldsymbol{a}_{ij}. \tag{36b}$$

Let $\boldsymbol{y}_i^*$ denote the range of the ith component $y_i(A)$, $A \in \boldsymbol{A}$, of the solution to (36); let $\boldsymbol{y}^*$ be the vector made up of $\boldsymbol{y}_i^*$. Consider the following problem:

Problem P4: Find an outer solution y to (36), i.e. a solution enclosing the range vector $\boldsymbol{y}^*$:

$$\boldsymbol{y}^* \subset y \tag{37}$$

Obviously, the nth and $2n$th components of the outer solution y to (36) provide the solution to the original Problem P3.

To solve Problem P4, we put $\boldsymbol{a}_{ij}$ and $\boldsymbol{y}_i$ in the centred form (19), i.e.

$$a_{ij} = a_{ij}^0 + u_{ij}, \quad i, j = 1, \ldots, n \tag{38a}$$

$$y_i = y_i^0 + v_i, \quad v_i \in \boldsymbol{v}_i, \quad i = 1, \ldots, 2n \tag{38b}$$

and apply the same approach as in the real case (method M1). Now the system, corresponding to system (20), is

$$\begin{aligned}
&(a_{11}^0 - y_n^0)v_1 + a_{12}^0 v_2 + \cdots + a_{1,n-1}^0 v_{n-1} - y_1^0 v_n + y_{n+1}^0 v_{2n} = \boldsymbol{b}_1 \\
&a_{21}^0 v_1 + (a_{22}^0 - y_n^0) + \cdots + a_{2,n-1}^0 v_{n-1} - y_2^0 v_n + y_{n+2}^0 v_{2n} = \boldsymbol{b}_2 \\
&\ldots \\
&a_{n-1,1}^0 v_1 + a_{n-1,2}^0 v_2 + \cdots + (a_{n-1,n-1}^0 - y_n^0)v_{n-1} - y_{n-1}^0 v_n + y_{2n-1}^0 v_{2n} = \boldsymbol{b}_{n-1} \\
&a_{n1}^0 v_1 + a_{n2}^0 v_2 + \cdots + a_{n,n-1}^0 v_{n-1} - v_n + \alpha v_{2n} = \boldsymbol{b}_n \\
&(a_{11}^0 - y_n^0)v_{n+1} + a_{12} v_{n+2} + \cdots + a_{1,n-1}^0 v_{2n-1} - y_{n+1}^0 v_n - y_1^0 v_{2n} = \boldsymbol{b}_{n+1} \\
&a_{21}^0 v_{n+1} + (a_{22}^0 - y_n^0)v_{n+2} + \cdots + a^0 2, n-1 v_{2n-1} - y_{n+2}^0 v_n - y_2^0 v_2 n = \boldsymbol{b}_{n+2} \\
&\ldots \\
&a_{n-1,1}^0 v_{n+1} + a_{n-1,2}^0 v_{n+2} + \cdots + (a_{n-1,n-1}^0 - y_n^0)v_{2n-1} - y_{2n-1}^0 v_n - y_{2n-1}^0 v_{2n} = \boldsymbol{b}_{2n-1} \\
&a_{n1}^0 v_1 + a_{n2}^0 v_2 + \cdots + a_{n,n-1}^0 v_{n-1} - \alpha v_n - v_{2n} \boldsymbol{b}_{2n}
\end{aligned} \tag{39}$$

It can be easily checked that the radius of $\boldsymbol{b}_i$ is

$$r(\boldsymbol{b}_i) = \sum_{j=1}^{n-1} R_{ij}|y_j^0| + \sum_{j=1}^{n-1} R_{ij} r_j + R_{in} + r_n r_i + r_{2n} r_{n+1}, \quad i = 1, \ldots, n-1 \tag{40a}$$

$$r(\boldsymbol{b}_n) = \sum_{j=1}^{n-1} R_{nj}|y_j^0| + \sum_{j=1}^{n-1} R_{nj} r_j + R_{nn} \tag{40b}$$

$$r(\boldsymbol{b}_{n+i}) = \sum_{j=1}^{n-1} R_{ij}|y_{n+j}^0| + \sum_{j=1}^{n-1} R_{ij} r_{n+j} + R_{in} + r_n r_{n+i} + r_{2n} r_i, \quad i = 1, \ldots, n-1 \tag{40c}$$

$$r(\boldsymbol{b}_{2n}) = \sum_{j=1}^{n-1} R_{nj}|y_{n+j}^0| + \sum_{j=1}^{n-1} R_{nj} r_{n+j} + R_{nn} \tag{40d}$$

Now system (39) can be written in a compact form

$$\tilde{A}_0' y = \boldsymbol{b}' \tag{41}$$

where $\tilde{A}'$ is the real coefficient matrix in (39). Let $C' = |(\tilde{A}'_0)^{-1}|$ and $r = (r_1, r_2, \ldots, r_{2n})^T$ with $r_i = R_{v_i}$ where v_i are the increments of y_i in (38b) $(v_i \in \boldsymbol{v}_i)$. From (41)

$$r = C' r_{\boldsymbol{b}'} \tag{42}$$

Using exactly the same techniques as in method M1, on substitution of (40) into (42) we get the nonlinear system

$$r = d' + D'r + C'g(r) \tag{43}$$

which has a structure similar to system (24). If (43) has a positive solution $\boldsymbol{r}$ which can be attained by the simple iteration method, starting from $\boldsymbol{r}^0 = 0$, then this solution solves Problem P4. Indeed, we can introduce the intervals

$$y'_i = y_i^0 + [-r_i,\ r_i], \quad i = 1, \ldots, 2n. \tag{44}$$

Once again, similarly to Theorem 3, we have

$$y_i^* \subset y'_i, \quad i = 1, \ldots, 2n \tag{45}$$

Hence, the intervals

$$y'_n = y_n^0 + [-r_n,\ r_n] \tag{46a}$$

$$y'_{2n} = y_{2n}^0 + [-r_{2n},\ r_{2n}] \tag{46b}$$

provide the solution to the original Problem P3. More precisely, we have the following theorem:

Theorem 4: If the nonlinear system (43) has a positive solution $\boldsymbol{r}$ that can be attained by the simple iteration method with initial vector $r^0 = 0$, then the intervals (46a) and (46b) determine the outer bounds I_{Re} and I_{Im} on the ranges $\boldsymbol{I}^*_{Re}$ and $\boldsymbol{I}^*_{Im}$, respectively.

The proof of Theorem 4 is similar to that of Theorem 3. On account of Theorem 4, the outer bound $\boldsymbol{I}_a$ can be computed as

$$I_a = \sqrt{\boldsymbol{I}^2_{Re} + \boldsymbol{I}^2_{Im}} \tag{47}$$

Conclusion: If all the outer bounds $|\lambda_i^A(A)| < 1$, $a \in A$, $i = 1, \ldots, n$, then interval matrix $\boldsymbol{A}$ is stable.

3.2 Outer bounds on the ranges for the eigenvalues of interval matrix $\boldsymbol{Q}$

Based on (7) we form the interval matrix $\boldsymbol{Q}$ with independent elements:

$$\boldsymbol{Q} = Q^0 + [-R_Q,\ R_Q], \tag{48a}$$

where

$$Q^0 = H - (A^0)^T HA^0 \tag{48b}$$

$$R_Q = (R_A)^T |H| R_A \tag{48c}$$

The application of the new method for obtaining the outer bounds of the ranges of $\boldsymbol{Q}$, described in the previous subsection lead to the following conclusion.

Conclusion: If all the outer bounds $\lambda_i^Q(A) > 0, A \in A$, $i = 1, \ldots, n$, then interval matrix $\boldsymbol{Q}$ is positive definite.

Final conclusion: If the interval matrix $\boldsymbol{A}$ is stable and interval matrix $\boldsymbol{Q}$ is positive definite, then the discrete-time nonlinear system considered is asymptotically stable.

4 Numerical Example

To demonstrate the applicability of the present method, we will solve the following problem. Let the interval matrix $\boldsymbol{A}$ of the discrete-time nonlinear system, in particular neural network, studied is $\boldsymbol{A} = A^0 + [-R_A,\ R_A]$ with

$$A^o = \begin{bmatrix} 0.4 & -0.2 & \vdots & 0 & -0.2 & 0.5 \\ -0.5 & -0.5 & \vdots & 0.2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -0.3 & \vdots & -0.2 & 0.3 & 0.4 \\ 0.1 & 0 & \vdots & -0.1 & 0.8 & 0.1 \\ 0 & 0.1 & \vdots & 0.05 & 0.3 & -0.8 \end{bmatrix} \tag{49a}$$

$$R_A = 0.01 * A^0 \tag{49b}$$

where $n_1 = 2$, $n_2 = 3$. Hence $n = n_1 + n_2 = 5$

4.1 Stability of the central problem

Stability analysis of system (1) when $A = A^0$ is described in detail in [1]. The main points of the investigation will be briefly presented here again. First, we check condition (2), i.e. if $\| A^0 \| < 1$for some $p = 1, 2, \infty$. The results show that (2) fails for $p = 1, 2, \infty$ as a global asymptotical stability test. Then we apply Theorem 1 and choose the matrices

$$H_I = \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 1.3 \end{bmatrix}, \quad H_{II} = \begin{bmatrix} 1.6 & -0.9 & 0.6 \\ -0.9 & 1.6 & -0.6 \\ 0.6 & -0.6 & 2.1 \end{bmatrix} \tag{50}$$

where matrix H_{II} satisfies (4). We compute matrices H (according to (5)) and $Q^0 = H - (A^0)^T HA^0$. Since Q^0 is positive definite, the equilibrium $x_e = 0$ of system (1) is globally asymptotically stable for $A = A^0$.

4.2 Stability of the interval problem

4.2.1 Stability of interval matrix A

The corresponding eigenvalue problem is

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4} + a_{i5}x_5 - \lambda x_i = 0 \\ a_{ij} \in a_{ij}, \quad i, j = 1, \ldots, 5 \end{aligned} \tag{51}$$

First, we solve (51) where $a_{ij} = a_{ij}^0$ to find the pair $(\lambda^0,\ x^0)$. In this case

$$\lambda^0 = (0.7497\ 0.5052\ -0.8780\ -0.3385 + j0.0605\ -0.3385 - j0.0605)^T \quad \text{(52a)}$$

and we will confine ourselves to finding an outer bound $\boldsymbol{I}$ on the range $\boldsymbol{I}^*$ for the third $(k = 3)$ real eigenvalue $\lambda^0 = \lambda_1^0 = -0.8780$ (since its absolute value is closer to 1). Therefore the bound $\boldsymbol{I}$ is computed using the method M1. Supporting that Assumption A1 holds, we normalize the eigenvector $x^0 = (x_1^0,\ x_2^0,\ x_3^0,\ x_4^0,\ x_5^0)$. Since $|x_5^0| = max(x_i^0)$, $i = 1, \ldots, 5$, Assumption A2 does not hold. In accordance with Remark 1, we have to change index n with the index corresponding to the maximum value component (in this instance, with 1). So

$$x^0 = (-0.4388\ -0.2312\ -0.6601\ -0.728\ 1)^T \quad \text{(52b)}$$

Thus, the vector y^0 is

$$y^0 = (-0.4388\ -0.2312\ -0.6601\ -0.728\ 0.8780)^T \quad \text{(53a)}$$

and

$$y = (y_1\ y_2\ y_3\ y_4\ y_5)^T = (x_1\ x_2\ x_3\ x_4 \lambda)^T \quad \text{(53b)}$$

Thus for the example considered system (16a) becomes

$$\begin{aligned} & a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3 + a_{i4}y_4 + a_{i5} - y_i y_5 = 0, \quad i = 1, \ldots, 4 \\ & a_{51}y_1 + a_{52}y_2 + a_{53}y_3 + a_{54}y_4 + a_{55} - y_5 = 0 \end{aligned} \tag{54}$$

The solution of (25) for the example considered, obtained by the simple iteration method, has the components

$$r = \begin{pmatrix} 0.0442 & 0.0248 & 0.0589 & 0.1093 & 0.0456 \end{pmatrix}^T \tag{55}$$

As all radii are positive, by Theorem 3 and (52a), the outer bound I is

$$\boldsymbol{I} = y_5 = y_5^0 + [-r_5,\ r_5] = [0.8324 \quad 039235] \tag{56}$$

From (56), it follows that the interval matrix $\boldsymbol{A}$, is stable.

4.2.2 Positive definiteness of interval matrix Q

We substitute the interval matrix $\boldsymbol{A}$, defined by (49), in (7) and get the matrix Q in interval form $\boldsymbol{Q}$, where

$$Q^0 = \begin{bmatrix} 0.319 & 0.3540 & \vdots & 0.0830 & -0.1030 & 0.0220 \\ 0.3540 & 0.7060 & \vdots & 0.0895 & -0.1310 & 0.3710 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.0830 & 0.0895 & \vdots & 1.5048 & -0.8215 & 0.6570 \\ -0.1030 & 0.1310 & \vdots & -0.8215 & 0.8350 & -0.3450 \\ 0.0220 & 0.3710 & \vdots & 0.6570 & 0.3450 & 0.7190 \end{bmatrix} \tag{57a}$$

$$R_q = 10^{-3} \begin{bmatrix} 0.0661 & 0.0578 & \vdots & 0.0215 & 0.0273 & 0.0350 \\ 0.0578 & 0.0666 & \vdots & 0.315 & 0.0623 & 0.0716 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.0215 & 0.0315 & \vdots & 0.0191 & 0.0538 & 0.0501 \\ 0.0273 & 0.0623 & \vdots & 0.0537 & 0.02205 & 0.1807 \\ 0.0350 & 0.0761 & \vdots & 0.0501 & 0.1807 & 0.2293 \end{bmatrix} \tag{57b}$$

The eigenvalues of central matrix Q^0 are:

$$\lambda^0 = \begin{pmatrix} 2.4480 & 0.9481 & 0.0098 & 0.4399 & 0.2379 \end{pmatrix}^T \tag{58}$$

It is seen that all the eigenvalues from (58) are real. Hence, we apply the method $M1$ fifth time for all of them to obtain the outer bounds of their ranges, when $A \in \boldsymbol{A}$.

The results of computations are the following:

$$I = \begin{bmatrix} [2.4408 & 2.4483] \\ [0.9479 & 0.9483] \\ [0.0095 & 0.0100] \\ [0.4395 & 0.4403] \\ [0.2376 & 0.2382] \end{bmatrix} \tag{59}$$

It is seen from (59) that the left bounds of all the components of interval vector $\boldsymbol{I}$ are positive. Therefore the ranges of all eigenvalues of interval matrix $\boldsymbol{Q}$ are positive and finally it follows that matrix Q, when $A \in \boldsymbol{A}$, is positive semidefine.

Since interval matrix $\boldsymbol{A}$ is stable and interval matrix $\boldsymbol{Q}$ is positive semidefine, the nonlinear system, in particular neural network, studied (1), with $A \in \boldsymbol{A}$ consider (49), is asymptotically stable.

5 Conclusions

The problem of stability analysis of discrete-time nonlinear system considered reduces to two tasks for assessing the intervals of the eigenvalues of interval matrices when $A \in \boldsymbol{A}$. First, the interval matrix $\boldsymbol{A}$ has to be stable, i.e. $|\lambda^A(a)| < 1$, $A \in \boldsymbol{A}$, $i = 1,\dots,n$. Second, the interval matrix $\boldsymbol{Q}$ has to be positive semidefine, i.e. $\lambda_i^Q(A) \geq 0$, $A \in \boldsymbol{A}$, $i = 1, /ldots, n$. Both tasks use the same technique. It consists of obtaining the outer bounds on the ranges for the eigenvalues of matrices A and $Q(A)$, when $A \in \boldsymbol{A}$. A recently proposed method for determining these outer bounds has been applied. It requires the evaluation of the eigenvalues and the corresponding eigenvectors from (11) for the center matrix A^0. Two versions of the method (for real and for complex eigenvalues, which are named $M1$ and $M2$) are discussed. The method $M1$ essentially consists of setting up and solving the system of n non-linear equations (25) for the positive solutions r_i, $i = 1,\dots,n$. The solution of the original problem $P1$ is then found by the radius r_n according to formula (28). The method $M2$ essentially consists of setting up and solving the system of $2n$ non-linear equations (43) for the positive solutions r_i, $i = 1,\dots,2n$. The solution (47) of the original problem $P3$ uses the radii r_n and r_2n according to (46). The conclusions for both $\boldsymbol{Q}$ and $\boldsymbol{A}$ are similar.

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