FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 19, April 2006, 1-11

Extending the Frequency Measurement of a Single Sinusoid Above the Nyquist Limit Based on Zero Crossings Method

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Abstract: This paper presents the measurement of the frequency of a single sinusoid by counting the zero crossings and extends it above the Nyquist limit. For this purpose, two sets of samples are acquired, at two different but close sampling frequencies. Based on the count of zero crossings obtained in each case and on the two sampling frequencies, four coefficients are computed. Then, the most appropriate coefficient from those four is used for calculating the frequency. The less the difference between the two sampling frequencies, the higher the maximum measurable frequency. The relative error is of the order of 10^{-4}, and the measurement time is about 100 ms. The method was experimentally tested using a National Instruments PCI 6023 data acquisition board.

Keywords: Frequency, sampling frequency, Nyquist limit, zero crossings, maximum measurable frequency.

1 Introduction

Usually, frequency or period measurement of periodic waveforms is achieved by means of counter-timer instruments [1], [2], [3] by counting the pulses within a specific time interval. The hardware of these instruments is not simple. Other methods are based on digital signal processing (DSP) techniques, [4] - [5]. They require a common data acquisition board and a software algorithm. Among DSP techniques, some allow the measurement of frequency in a narrow range, for instance the frequency of a power system [6], [5]. Other techniques allow the measurement in a

Manuscript received April 13, 2004.

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wide range [4], [7], but the range is limited at half the sampling frequency (Nyquist limit). Counting the zero crossings is a simple and well known technique [4], [7]. This paper also uses the method of counting the zero crossings. Even if, in comparison with [7], the error is a little higher, the measurement range is much larger than the Nyquist limit. In this section the basic principle of frequency measurement based on counting the zero crossings is presented, when the sampling theorem is satisfied (signal frequency below the Nyquist limit). Section 2 presents a detailed description of the used principle when this theorem is not satisfied and section 3 presents the experimental results.

Let a periodical signal with no dc component, whose zero crossings are equally spaced at half the period. A particular case is that of a single sinusoid. In order to measure its frequency by counting the zero crossings, N data samples must be acquired, Fig. 1. Let f_s and $1/T_s$ the sampling frequency and period, respectively. Then, the acquisition time T_{ac} can be computed by

$$T_{ac} = (N-1)T_s. \tag{1}$$

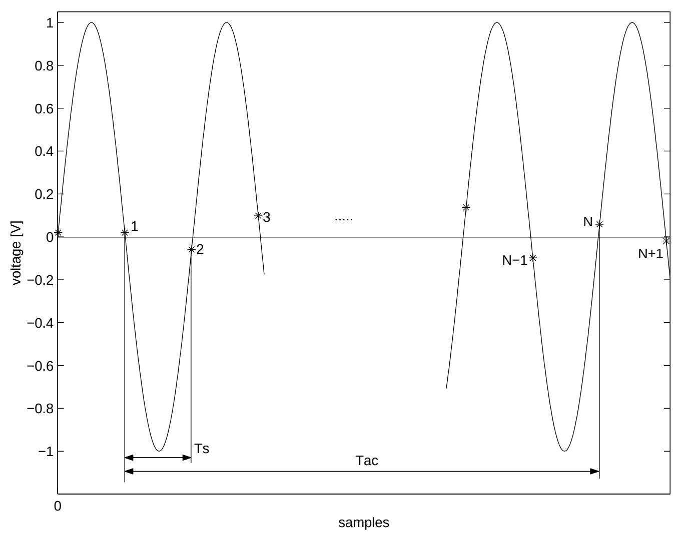

Fig. 1. The samples of the acquired signal.

Let $f_z(=1/T_z)$ the zero-crossings frequency. If N_z is the number of zero crossings corresponding to the N data samples, then

$$T_z = \frac{1}{f_z} = \frac{T_{ac}}{N_z/2} = \frac{2(N-1)}{f_s N_z}. \tag{2}$$

If the sampling theorem is satisfied, that is $f_s \geq 2f$, where $f = 1/T$ is the signal frequency, then

$$f = f_z = \frac{f_s N_z}{2(N-1)}. \tag{3}$$

Signal acquisition starts with the first zero crossing and ends after the detection of a zero crossing of the same type as the first one that occurs after a pre-established number of samples. Thus, in Fig. 1 the first zero crossing is from a positive sample, denoted by 1, to a negative sample denoted by 2. The last zero crossing, that ends the acquisition, is that from the N-th sample to the $(N+1)$-th sample, and is not counted in N_z. It follows that the number of acquired samples can slightly differ from one measurement to another. From Fig. 1 it can also be seen that T_{ac} is less than a multiple of T_z by at most T_s. It is easy to show that there is a similar case when T_{ac} is higher by at most T_s than a multiple of T_z. Therefore, T_z calculated with equation (2) can be affected by a maximum relative error of

$$\varepsilon_{T_z} = \varepsilon_{T_{ac}} = \frac{T_s}{(N-1)T_s} = \frac{1}{N-1}, \tag{4}$$

if the error of detecting zero crossings is zero. In the following section there will be a come-back to this problem. Because f is computed by (3) it follows that it is affected by the same error, that is $\varepsilon_f = 1/(N-1)$. It can be observed that this error does not depend on the value of measured frequency f.

In comparison with this principle, the method from [7] is as follows. The period of the signal in Fig. 1 is computed as the number of sampling intervals between two consecutive zero crossings of the same type (for instance, $2T_s$, between samples 1 and 3) and then the time interval between sample 3 and the next real zero crossing of waveform is added and the time interval between sample 1 and the next real crossing is subtracted. These time intervals are determined by linear interpolation. In this mode, a smaller error with comparison to equation (4) is obtained, but it is necessary that $f_s \gg 2f$ and further computations are required. The method presented in this paper has however the advantage of extending the measurement range above the Nyquist limit.

If the sampling theorem is not satisfied, that is frequency f is above the Nyquist limit, $f > f_s/2$, then f can be computed by one of the following equations [8]

$$f = k_1 f_s + f_z, \tag{5}$$

$$f = k_2 f_s - f_z. \tag{6}$$

Coefficients k_1 and k_2 are positive integers. In order to compute the frequency f, the required equation, either (5) or (6), and the value of the corresponding coefficient, (k_1 or k_2), must be determined.

2 Zero Crossings Method for Frequency Measurement

In this section the method for frequency measurement when the sampling theorem is not satisfied is presented.

The equations (3), (5) and (6) can be written as follows, depending on the ratio between f and f_s

$$f = \begin{cases} f_z & \text{for } f \in [0, f_s/2] \\ f_s - f_z & \text{for } f \in [f_s/2, f_s] \\ f_s + f_z & \text{for } f \in [f_s, 3f_s/2] \\ 2f_s - f_z & \text{for } f \in [3f_s/2, 2f_s] \\ 2f_s + f_z & \text{for } f \in [2f_s, 3f_s/2] \\ \dots & . \end{cases} \tag{7}$$

Based on (7), a similar expression for f_z can be obtained, that is graphically represented in Fig. 2, for a certain frequency f.

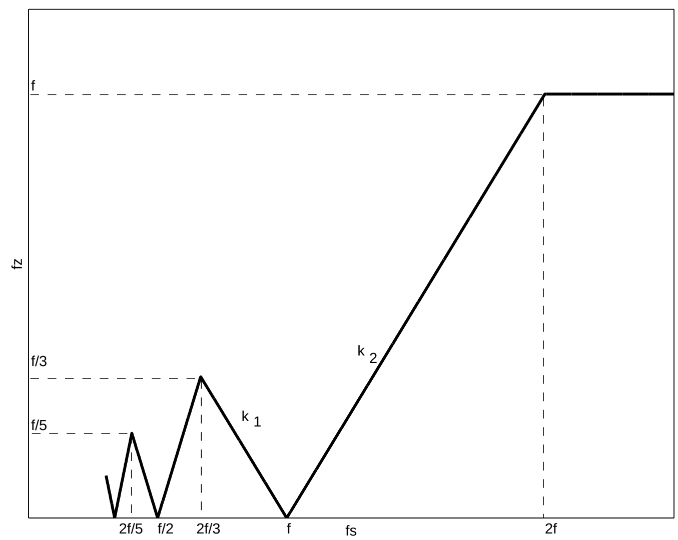

Fig. 2. The variation of f_z versus f_s.

To calculate the frequency of the signal, equation (5) is used when the sampling frequency is placed on a falling segment and equation (6) is used when the sampling frequency is placed on a rising segment. Therefore, falling segments will be named of k_1 type and rising segments will be named of k_2 type.

It is easy to observe that the minimums of the graph are 0 and are obtained for sampling frequencies that satisfy the following condition:

$$f_s = f/k_1 = f/k_2, \tag{8}$$

with $k_1 = k_2$.

The maximums are obtained for sampling frequencies that satisfy the following condition:

$$f_s = 2f/(2k_1 + 1) = 2f/(2k_2 - 1), \tag{9}$$

with $k_2 = k_1 + 1$. The values of f_z that correspond to maximums in the graph can be obtained by comparing equations (5) and (6):

$$f_z + k_1 f_s = k_2 f_s - f_z, \tag{10}$$

with $k_2 = k_1 + 1$.

Using (10) and (9), the values for each maximum of the graph from Fig. 2 are:

$$f_{zmax} = \frac{f_s}{2} = \frac{f}{2k_1 + 1} = \frac{f}{2k_2 - 1}, \tag{11}$$

where $k_2 = k_1 + 1$.

In order to compute the frequency f, the proposed method requires two sets of samples with sampling frequencies f_{s1} and f_{s2}, $f_{s2} > f_{s1}$. The two sampling frequencies must be on the same segment, or on two adjacent segments, in Fig. 2. This condition imposes an upper limit for the measured frequency, and four cases for the computation of k_1 or k_2. These cases are presented in the following:

1. The sampling frequencies f_{s1} and f_{s2} are placed on a segment of k_1 type.
2. The sampling frequencies f_{s1} and f_{s2} are placed on a segment of k_2 type.
3. The sampling frequency f_{s1} is placed on a segment of k_1 type, the sampling frequency f_{s2} is placed on a segment of k_2 type and $k_1 = k_2$. In this case, the two adjacent segments have a common point for $f_z = 0$.
4. The sampling frequency f_{s1} is placed on a segment of k_2 type, the sampling frequency f_{s2} is placed on a segment of k_1 type and $k_1 = k_2 - 1$. In this case, the two adjacent segments have a common point for $f_z = f/(2k_1 + 1)$.

For the four cases above, coefficients k_1 and k_2 can be determined as follows. Let f_{z1} and f_{z2} the zero crossings frequencies obtained for sampling frequencies f_{s1} and f_{s2}, respectively.

For case 1:

$$f = f_{z1} + k_1 f_{s1} = f_{z2} + k_1 f_{s2}. \tag{12}$$

It follows that

$$k_1 = \frac{f_{z1} - f_{z2}}{f_{s2} - f_{s1}}. \tag{13}$$

For case 2:

$$f = k_2 f_{s1} - f_{z1} = k_2 f_{s2} - f_{z2}. \tag{14}$$

It follows that

$$k_2 = \frac{f_{z2} - f_{z1}}{f_{s2} - f_{s1}}. \tag{15}$$

For case 3:

$$f = f_{z1} + k_1 f_{s1} = k_1 f_{s2} - f_{z2}. \tag{16}$$

It follows that

$$k_1 = k_2 = \frac{f_{z1} + f_{z2}}{f_{s2} - f_{s1}}. \tag{17}$$

For case 4:

$$f = (k_1 + 1) f_{s1} - f_{z1} = f_{z2} + k_1 f_{s2}. \tag{18}$$

It follows that

$$k_1 = \frac{f_{s1} - f_{z1} - f_{z2}}{f_{s2} - f_{s1}}, k_2 = k_1 + 1. \tag{19}$$

Thus, in order to obtain the value of k_1 or k_2, the expressions given by (13), (15), (17) and (19) must be computed. Then, only one of these four values must be a positive integer, because only one case from the four above is true for a certain frequency f, and that value will be the right value of k_1 or k_2. When the sampling theorem is verified, f_{z1}=f_{z2} and then $k_1 = k_2 = 0$. Finally, the measured frequency can be computed by one of the equations (12), (14), (16) and (18), depending on f_{z1} and f_{s1}, or on f_{z2} and f_{s2}.

In the following, the maximum measurable frequency f_{max}, depending on the two sampling frequencies will be computed. For this purpose we impose that the sampling frequencies correspond to a single segment of k_1 type or of k_2 type in graph from Fig. 2, for the maximum measurable frequency. In this case, the two sampling frequencies will be on the same segment, or on the two adjacent segments, if the frequency is less than the maximum measurable frequency.

1. The case of segment of k_1 type.

The condition that the sampling frequencies correspond to a single segment is obtained based on (8) and (9) for $f = f_{max}$:

$$f_{s1} \geq \frac{2 f_{max}}{2k_1 + 1}, \tag{20}$$

$$f_{s2} \leq \frac{f_{max}}{k_1}. \tag{21}$$

Then, from (20) and (21) it follows that

$$k_1 \leq \frac{f_{s1}}{2(f_{s2} - f_{s1})}. \tag{22}$$

The practical value of k_1 will be

$$k_{1max} = \left[\frac{f_{s1}}{2(f_{s2} - f_{s1})} \right], \tag{23}$$

where $[x]$ represents the largest integer smaller than x.

¿From (20), for $k_1 = k_{1max}$, it follows that the maximum measurable frequency is:

$$f_{max} \leq \frac{(2k_{1max} + 1) f_{s1}}{2}. \tag{24}$$

2. The case of segment of k_2 type.

The condition that the sampling frequencies correspond to a single segment is obtained based on (8) and (9) for $f = f_{max}$:

$$f_{s1} \geq \frac{f_{max}}{k_2}, \tag{25}$$

$$f_{s2} \leq \frac{2 f_{max}}{2k_2 - 1}. \tag{26}$$

Then, from (25) and (26) it follows that

$$k_2 \leq \frac{f_{s2}}{2(f_{s2} - f_{s1})}. \tag{27}$$

The practical value of k_2 will be

$$k_{2max} = \left[\frac{f_{s2}}{2(f_{s2} - f_{s1})} \right]. \tag{28}$$

From (25), for $k_2 = k_{2max}$, it follows that the maximum measurable frequency is:

$$f_{max} \leq k_{2max} f_{s1}. \tag{29}$$

Finally, the maximum measurable frequency will be the minimum value from those obtained by (24) and (29).

In the following, an error analysis for measuring the frequency above the Nyquist limit is presented. From equations (5) and (6), it follows that signal frequency will be determined with a maximum relative error of

$$\varepsilon_f = \frac{f_z}{f} \varepsilon_{f_z} + (1 - \frac{f_z}{f}) \varepsilon_{f_s} + f_s \frac{k}{f} \varepsilon_k, \tag{30}$$

where k can be k_1 or k_2, f_z can be f_{z1} when f_s is f_{s1}, and f_{z2} when f_s is f_{s2}.

The term ε_{f_z}, that affects f_z, has two components: the error of $1/(N-1)$, and the error of detecting zero crossings, ε_{N_z}, which can be higher for frequency values close to kf_{s1}, kf_{s2}, $(k-0.5)f_{s1}$ or $(k-0.5)f_{s2}$. The term f_{z1} (or f_{z2}) is contained in (12), (14), (16) or (18) but also can affect the calculations of k_1 or k_2 by (13), (15), (17) and (19). However, the value of k will be chosen by rounding the closest k to the corresponding integer. Therefore, it can be said that ε_{N_z} affects only ε_{f_z} in (30), and the third term in (30) will be 0 because $\varepsilon_k = 0$.

The error of detecting zero crossings occurs for signal frequencies close to one of the four values presented above, because there are many consecutive samples with very low values. In this case the presence of noise, including quantization noise, can cause more or less zero crossings in comparison with the real case.

In order to reduce these errors, a single zero crossing is considered even if there are more consecutive zero samples. Furthermore, when frequency values calculated depending on f_{s1} and N_{z1} or on f_{s2} and N_{z2} differ by more than 20 ppm, only the most accurate value should be kept. To determine the right value, the following should be considered.

If the signal frequency is close to kf_{s1} or to kf_{s2}, both N_{z1} and N_{z2} are small (under 100), and the smaller is affected by zero crossing detection errors. In this case, the frequency value corresponding to the larger N_z should be kept.

If the signal frequency is close to $(k-0.5)f_{s1}$ or to $(k-0.5)f_{s2}$, both N_{z1} and N_{z2} will be large (close to $N-1$), the larger being affected by zero crossing detection errors. In this case, the frequency value corresponding to the smaller N_z should be kept.

Finally, if the error of detecting zero crossings could be eliminated, it follows that ε_{f_z} in (30) would be $1/(N-1)$ and then ε_f would be less than $1/(N-1)$. However, as will be shown in the following section, this reduction is not so great as expected because the error of sampling frequency ε_{f_s} is not zero, and the error of detecting zero crossings are not completely eliminated.

3 Experiments

The method presented in Section 2 has been used to implement a frequency meter based on a National Instruments PCI 6023 data acquisition board. The corresponding program was written in C-language and run in real time.

The two sampling frequencies were: $f_{s1} = 1/(25\times 200\,\text{ns})$=200 kHz, and $f_{s2} = 1/(24\times 200\,\text{ns})$= 208.3333 kHz. Thus, from (23), (24), (28) and (29) it follows that the maximum measurable frequency is 2400 kHz. The number of data samples was $N = 10000$, thus the relative error is 10^{-4}. The measurement time is roughly the time required for the acquisition of the two sets of data samples, that is $2T_{ac}$, and is about 100 ms.

Table 1 presents several experimental results. The signal to be measured was taken from a HM8130 signal generator and, for a better accuracy, was also measured with a HM8122 counter-timer. The values in the first column of table 1 have been measured with HM8122. The next four columns present the computed values of coefficients k_1 or k_2. For each frequency, the value in italics was rounded to the nearest integer and then used to compute the result of the measurement, with the corresponding equations from (12), (14), (16), or (18). The value f_1 was obtained with the first member of one from previous equations, depending on f_{s1}, and value f_2 was obtained with the second member of one from previous equations, depending on f_{s2}. The values marked with the symbol * in table 1 are the right frequency values that should be kept (as shown in the previous section). It can be seen that in each case the right value is in f_2 column and is more accurate, because f_{s2} is not close to f/k or $f/(k-0.5)$. It can also be seen that the errors are less than 10^{-4}.

Table 1. Some experimental results for measuring different frequencies.

f [kHz]	k_1 with(11)	k_2 with(13)	k_1 with(15)	k_2 with(17)	f_1 [kHz]	f_2 [kHz]	Error [ppm]
499.968	*1.9928*	-1.9928	21.9874	3.0126	499.917	499.977*	18
600.007	-2.9566	2.9566	*3.0393*	21.9606	600.344	600.016*	15
600.029	-2.9397	2.9397	*3.0508*	21.9491	600.462	600.039*	16
750.002	-4.0002	*4.0002*	15.9964	9.0035	750.015	750.013	14
764.918	-3.9995	*3.9995*	12.4171	12.5828	764.926	764.930	15
883.557	*4.0001*	-4.0001	16.0559	8.9440	883.567	883.565	9
900.005	3.9676	-3.9676	19.9729	*5.0270*	900.247	900.021*	17
964.248	-4.9995	*4.9995*	13.5764	11.4235	964.260	964.264	12
1700.027	7.477	-7.477	15.9878	*9.0121*	1700.142	1700.043*	9
1800.049	-8.9479	8.9479	*9.0413*	15.9586	1800.382	1800.040*	5
1852.520	3.6131	-3.6131	*9.0007*	15.9992	1852.557	1852.551	16
2014.442	-6.5313	6.5313	*10.0005*	14.9994	2014.455	2014.450	14
2200.230	-10.866	10.8660	*11.0641*	13.9358	2200.822	2200.238*	3
2300.037	10.8640	-10.8641	12.8781	*12.1218*	2301.072	2300.050*	5
2394.538	-11.9581	11.9581	12.9582	*12.0417*	2394.561	2394.567	9

In the following, two examples with the acquired signals that show the error of detecting zero crossings are presented.

Fig. 3 shows the acquired signal when f is 900.005 kHz, that is close to $(k-0.5)f_{s1}$. In this case there should be a zero crossing at each sample starting with the second, and each sample should have the same amplitude (as absolute value). However, because the above relation is met approximately, the amplitude decreases slowly and at a time it will be very low, and errors in detecting the zero crossings can occur. Further, if the signal has a small dc component, many zero crossings can

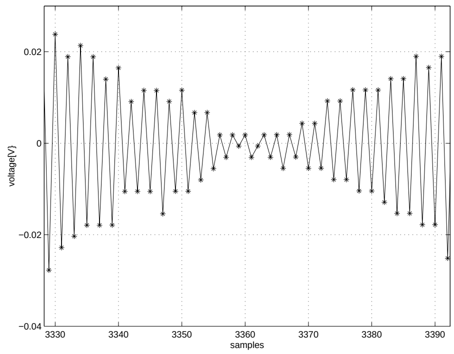

Fig. 3. The acquired signal when $f =$ 900.005 kHz.

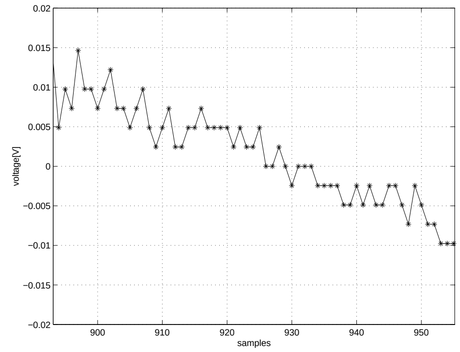

Fig. 4. The acquired signal when $f =$ 600.007 kHz.

be lost.

Fig. 4 shows the acquired signal when f is 600.007 kHz, that is close to $k f_{s1}$. In this case there should be no zero crossings, and each sample should have the same amplitude. Again, because the above relation is met approximately, a signal with a slow slope, that can have further zero crossings will be obtained.

4 Conclusions

The possibility to extend the range of frequency measurement of a single sinusoid based on counting the zero crossings, much beyond half the sampling frequency, has been presented. For this purpose, two sets of data samples are acquired at two different but close sampling frequencies. Then, simple calculations lead to the measurement result. The main advantage of this method is that an instrument with maximum measurable frequency much higher than the Nyquist limit of the data acquisition board can be implemented.

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