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On Regularly Varying Distributions Generated by Birth-Death Process

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Abstract: Skewed distributions generated by birth-death process with different particular forms of intensivities' moderate growth are used in biomolecular systems and various non-mathematical fields. Based on datasets of biomolecular systems such distributions have to exhibit the power law like behavior at infinity, *i.e.* regular variation.

In the present paper for the standard birth-death process with most general than before assumptions on moderate growth of intensivities the following problems are solved.

1. The stationary distribution varies regularly if the sequence of intensivities varies regularly.
2. The slowly varying component and the exponent of regular variation of stationary distribution are found.

Keywords: Standard birth-death process, moderate growth, skewed distribution, regular variation, biomolecular systems.

1 Introduction

1.1 Moderate growth

The standard birth-death process and it's stationary distributions are *well-known* (see, for instance [1]). In [2] for coefficients of the process the *moderate* growth assumptions which include earlier known ones have been made. Then, from the class of *all* stationary distributions a corresponding subclass named a class of *Distributions* with *Moderate Skewness* (DMS) has been extracted.

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Let $C \in R^{+} = (0, +\infty)$ be *a* parameter, $\{\varepsilon_n\}$ and $\{\delta_n\}$ be sequences of positive numbers with properties: $\{\delta_n\}$ increases,

$$\lim_{n\to+\infty} \delta_n = +\infty, \quad \lim_{n\to+\infty} \frac{\delta_{n+1}}{\delta_n} = 1, \quad \lim_{n\to+\infty} \frac{\varepsilon_n}{\delta_n} = 1. \tag{1.1}$$

DMS are of types $\{p_n^{\pm}\}$, $\{p_n\}$, where:

$$\begin{cases} p_0^{\pm} = \left(1 + C \cdot \sum_{n\geq 1} \frac{\theta^n}{\varepsilon_n} \prod_{m=1}^{n-1} \left(1 \pm \frac{b}{\delta_m}\right)\right)^{-1}, \\ p_k^{\pm} = p_0^{\pm} \cdot C \cdot \frac{\theta^k}{\varepsilon_k} \cdot \prod_{m=1}^{k-1} \left(1 \pm \frac{b}{\delta_m}\right), \qquad k = 1, 2, \cdots. \end{cases} \tag{1.2}$$

Here:

$0 < \theta < 1, 0 \leq b < +\infty$ for $\{p_n^{+}\}$;

$0 < \theta \leq 1, 0 < b < \delta_1$ for $\{p_n^{-}\}$.

Next, $\{p_n\}$ has the form of $\{p_n^{+}\}$ with $\theta = 1, 0 \leq b < +\infty$ and $\sum_{n\geq 1} \frac{1}{\delta_n} < +\infty$.

The described distributions have a *skew* to the right.

The mechanism of biomolecular large-scale systems dynamic can be explained with the help of birth-death models. Their stationary distributions generate skewed distributions. The number of skewed distributions being used in genetic systems and in other non-mathematical fields (distributions of words in the text, city sizes in country, citation of an author by other author, etc.), and generated by various birth-death processes has been increased over time.We have to indicate papers of Yule [3] (1924) and Simon [4] (1955), and some recent ones (see,for instance, Granzel and Shubert [5],Bornholdt and Ebel [6], Oluić-Vicović [7], Kuznetsov [8], etc.). All distributions suggested in [3]-[8] have *moderate* growth and a *skew* to the right.

1.2 Regular variation

Based on datasets for various large-scale biomolecular systems several authors declared that the frequency distribution $\{p_n\}$ of events in such systems exhibits *power* law distribution

$$p_n = c(\rho) \cdot n^{-\rho}, 1 < \rho < +\infty, n = 1, 2, \cdots, \tag{1.3}$$

where

$$c(\rho) = \left(\sum_{n\geq 1} n^{-\rho}\right)^{-1} (see, [9] - [14]).$$

The power law distribution is used for estimation of the connectivity number of metabolic networks [15], of the rates of protein synthesis in protein sets of prokaryotic organisms [16], of the number of expressed genes in eukaryotic cells [17]-[18], of DNA sequencing structures [18] *etc*.

But this distribution *not always* may be used. In log-log plot a power law asymptotically is represented by a straight line. But (see, [18]) the log-log plot of the most distributions, even in [9]-[14], *systematically* deviated from the straight line.Therefore, *new* statistical frequency distributions have been proposed (see, [3]-[4],[16],[19]).Some of them are particular cases of Pareto distribution

$$p_n = c(\rho,b)\cdot(n+b)^{-\rho}, -1 < b < +\infty, 1 < \rho < +\infty, n = 1,2,\cdots, \qquad (1.4)$$

where

$$c(\rho,b) = \left(\sum_{n\geq 1}(n+b)^{-\rho}\right)^{-1} \text{ (see, [19])},$$

which shows power law like behavior for large values of n.

Based on datasets the following conclusion have been made: the frequency distribution $\{p_n\}$ exhibits power laws-like behavior for large values of n (see, [9]-[12],[14]-[15]).

Below, we interpret this empirical fact as a *regular variation* of frequency distribution.

1.3 The goals

1. To extract from the class of DMS a subclass of distributions which may become regularly varying under some additional assumptions on $\{\delta_n\}$.
2. To find necessary and sufficient conditions on $\{\delta_n\}$ for the regular variation of the distributions from extracted subclass.
3. To get the slowly varying component and the exponent of regular variation of distributions from the extracted subclass.

2 Narrowing the Class of DMS

2.1 Definitions

Let us introduce two definitions

Definition 1.(see, [20]). The measurable on R^+ function $R(t) > 0$ *varies regularly* (at infinity) with *exponent* $\rho \in R^1 = (-\infty,+\infty)$ if for any $x \in R^+$ the limit exists $\lim_{t\to+\infty}(R(xt)/R(t)) = x^\rho$.

If $\rho = 0$, then we say that $L(t) = R(t), t \in R^+$, *varies slowly.*

Thus, a function $R > 0$, measurable on R^+, varies regularly with exponent ρ iff $R(t) = t^\rho \cdot L(t), t \in R^+$.

The identity

$$\frac{p_{sn}}{p_n} = \frac{1}{s^\rho}, \quad -1 < \rho < +\infty, s = 2, 3, \cdots, \quad n = 1, 2, \cdots \tag{2.1}$$

is a *characteristic* property of power law (1.3). Therefore, the limit relation

$$\lim_{n \to +\infty} \frac{p_{sn}}{p_n} = \frac{1}{s^\rho}, \quad -1 < \rho < +\infty, s = 2, 3, \cdots, \tag{2.2}$$

represents in mathematical sense the "power law like behavior" of frequency distribution $\{p_n\}$ for large values of n,which is the definition of regular variation.

For the regular variation of distributions (1.2) additional assumptions on $\{\varepsilon_n\}$ and $\{\delta_n\}$ are needed. Indeed,putting $b = 0$ and $\theta = 1$ in (1.2) we obtain

$$\frac{p_{sn}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}}, \quad s = 2, 3, \cdots, \quad n = 1, 2, \cdots. \tag{2.3}$$

Thus, the assumption of regular variation of $\{\varepsilon_n\}$ is necessary.

Definition 2. Functions $f > 0, g > 0$ defined on R^+ are *asymptotically equivalent* (at infinity) if $\lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1$. Then, we write $f(t) \approx g(t), t \to +\infty$.

Due to (1.1) sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ are asymptotically equivalent. Therefore, they vary regularly or not simultaneously and conditions may be put only on $\{\delta_n\}$.

2.2 The Result

Denote

$$\bar{A} = \overline{\lim_{n \to +\infty}} \frac{n}{\delta_n}. \tag{2.4}$$

Theorem 1 *Distributions* $\{p_n^\pm\}$ *with* $0 < \theta < 1$, *and* $\{p_n\}$ *with* $\bar{A} = +\infty$ *cannot vary regularly.*

In proof of **Theorem 1** we use

Lemma 1 *Let us denote for* $s = 2, 3, \cdots$

$$\bar{B}(s) = \overline{\lim_{n \to +\infty}} \sum_{m=n}^{sn-1} \frac{1}{\delta_m}. \tag{2.5}$$

Then: $\bar{A} < +\infty$ *implies* $\bar{B}(s) < +\infty$ *for all* $s = 2, 3, \cdots$;

$\bar{B}(s) < +\infty$ *for some* $s = 2, 3, \cdots$ *implies* $\bar{A} < +\infty$.

Proof. Since $0 < \overline{B}(2) \leq \overline{B}(3) \leq \cdots$, therefore, if $\overline{B}(s) < +\infty$ for *some* s, then $\overline{B}(l) < +\infty, l = 2,3,\cdots,s$.

Let us assume that $\overline{A} < +\infty$. For $s = 2,3,\cdots$ and $n = 1,2,\cdots$ we have

$$\frac{(s-1)n}{\delta_{sn}} < \sum_{m=n}^{sn-1} \frac{1}{\delta_m} < \frac{(s-1)n}{\delta_n} \tag{2.6}$$

because $\{\delta_m\}$ increases.From the second inequality (2.6) we obtain $\overline{B}(s) < +\infty$ for *all* $s = 2,3,\cdots$.

Reverse, let $\overline{B}(2) < +\infty$. From the first inequality (2.6), we get $(n/\delta_{2n}) < \sum_{m=n}^{2n-1} \delta_m^{-1}$, or letting $n \to +\infty$

$$\overline{\lim_{n\to+\infty}} \frac{2n}{\delta_{2n}} < 2\overline{B}(2) < +\infty. \tag{2.7}$$

Since $\{\delta_n\}$ increases and $\lim_{n\to+\infty} \delta_n^{-1} = 0$, so, from (2.7) we obtain

$$\overline{\lim_{n\to+\infty}} \frac{2n+1}{\delta_{2n+1}} = 2\overline{\lim_{n\to+\infty}} \frac{n}{\delta_{2n+1}} \leq 2\overline{\lim_{n\to+\infty}} \frac{n}{\delta_{2n}} < +\infty,$$

which together with (2.7) lead to $\overline{A} < +\infty$.

If $\overline{B}(s) < +\infty$ for *some* s, then $\overline{B}(2) < +\infty$, and we come to the previous case.

2.3 The Proof of Theorem 1

1. For integers $s > 1, n \geq 1$ and for $0 < \theta < 1$ we have

$$\frac{p_{sn}^-}{p_n^-} = \theta^{(s-1)n} \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left(1 - \frac{b}{\delta_m}\right), 0 < b < \delta_1.$$

Since

$\lim_{n\to+\infty} \theta^{(s-1)n} = 0$, $\left(1 - \frac{b}{\delta_m}\right) < 1$ for $m = n, n+1, \cdots$, $\frac{\varepsilon_n}{\varepsilon_{sn}} = \left(\frac{\varepsilon_n}{\delta_n}\right) \cdot \left(\frac{\delta_{sn}}{\varepsilon_{sn}}\right) \cdot \left(\frac{\delta_n}{\delta_{sn}}\right)$

(the last equality implies together with (1.1) that for $\varepsilon > 0$ starting from some index n_0 we have

$$\frac{\varepsilon_n}{\varepsilon_{sn}} < 1 + \varepsilon, n = n_0, n_0 + 1, \cdots), \tag{2.8}$$

therefore,

$$\lim_{n\to+\infty} \frac{p_{sn}^-}{p_n^-} = 0, s = 2,3\cdots. \tag{2.9}$$

2. For integers $s > 1, n \geq 1$ and for $0 < \theta < 1$ we have

$$\frac{p_{sn}^{+}}{p_{n}^{+}} = \theta^{(s-1)n} \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left(1 + \frac{b}{\delta_m}\right), \quad 0 < b < +\infty.$$

For $\varepsilon > 0$ satisfying condition $\theta \cdot (1+\varepsilon) < 1$ starting from some index n_0 (2.8) holds and $\frac{b}{\delta_m} < \varepsilon$ for $m = n_0, n_0+1, \cdots$ simultaneously. Therefore, for $n = n_0, n_0 + 1, \cdots$ and $s = 2, 3, \cdots$ we have

$$0 \leq \frac{p_{sn}^{+}}{p_{n}^{+}} < (\theta \cdot (1+\varepsilon))^{(s-1)n} \cdot (1+\varepsilon) \to 0 \quad \text{as} \quad n \to +\infty.$$

3. Let $\theta = 1$ and $\overline{A} = +\infty$ (for $\{p_n^-\}$ and $\{p_n\}$). Due to **Lemma 1**, for $s = 2, 3, \cdots$

$$+\infty = \overline{\lim}_{n\to+\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = \overline{B}(s) < \sum_{n\geq 1} \frac{1}{\delta_n}, \tag{2.10}$$

which excludes the case of $\{p_n\}$ from further consideration. So, we deal with $\{p_n^-\}$.

For integers $s \geq 2, n \geq n_0$ we have

$$0 < \frac{p_{sn}^{-}}{p_{n}^{-}} = \frac{\varepsilon_n}{\varepsilon_{sn}} \prod_{m=n}^{sn-1} \left(1 - \frac{b}{\delta_m}\right) < (1+\varepsilon) exp \left\{ \sum_{m=n}^{sn-1} \ln\left(1 - \frac{b}{\delta_m}\right) \right\}, 0 < b < \delta_1, \tag{2.11}$$

where the inequality (2.8) was used. Since, by **Lemma 1**, $\overline{B}(2) = \infty$, therefore there is a sequence $\{n_k\}$ of integers, $0 < n_1 < n_2 < \cdots$, such that

$$\lim_{k\to+\infty} \sum_{m=n_k}^{2n_k-1} \frac{1}{\delta_m} = +\infty. \tag{2.12}$$

By (2.11)-(2.12),

$$0 \leq \lim_{k\to+\infty} \frac{p_{2n_k}^{-}}{p_{n_k}^{-}} \leq (1+\varepsilon) \lim_{k\to+\infty} \exp\left\{ -b \cdot \sum_{m=n_k}^{2n_k-1} \frac{1}{\delta_m} \right\} = 0.$$

2.4 Narrowing the Class of DMS

Let us exclude DMS mentioned in **Theorem 1.** The remainder is a *subclass* of DMS,which is described as follows.

Let $C \in R^+$ be a *parameter*, $\{\varepsilon_n\}$ and $\{\delta_n\}$ be sequences of positive numbers such that

$$\begin{cases} \{\delta_n\} \quad \text{increases}, \quad \lim_{n\to+\infty}\delta_n = +\infty, \lim_{n\to+\infty}\frac{\delta_{n+1}}{\delta_n} = 1, \\ \\ \lim_{n\to+\infty}\frac{\varepsilon_n}{\delta_n} = 1, \overline{A} \stackrel{def}{=} \overline{\lim}_{n\to+\infty}\frac{n}{\delta_n} < +\infty. \end{cases} \tag{2.13}$$

The remaining DMS $\{p_n\} = \{p_n(C,b)\}$ take the forms

$$\begin{cases} p_0 = \left(1 + C\cdot\sum_{n\geq1}\frac{1}{\varepsilon_n}\prod_{m=1}^{n-1}\left(1-\frac{b}{\delta_m}\right)\right)^{-1}, \\ p_k = \frac{Cp_0}{\varepsilon_k}\cdot\prod_{m=1}^{k-1}\left(1-\frac{b}{\delta_m}\right), \quad k = 1,2,\cdots, \end{cases} 0 < b < \delta_1 \tag{2.14}$$

with $\sum_{n\geq1}\frac{1}{\delta_n} = +\infty$.

$$\begin{cases} p_0 = \left(1 + C\cdot\sum_{n\geq1}\frac{1}{\varepsilon_n}\prod_{m=1}^{n-1}\left(1+\frac{b}{\delta_m}\right)\right)^{-1}, -1 < b < +\infty, \\ p_k = \frac{Cp_0}{\varepsilon_k}\cdot\prod_{m=1}^{k-1}\left(1+\frac{b}{\delta_m}\right), \quad k = 1,2,\cdots, \sum_{n\geq1}\frac{1}{\delta_n} < +\infty. \end{cases} \tag{2.15}$$

Putting without loss of generality $\delta_1 = 1$ we may include the part of DMS of form (2.15) with $-1 < b < +\infty$ in (2.14).

$$\begin{cases} p_0 = \left(1 + C\cdot\sum_{n\geq1}\frac{1}{\varepsilon_n}\prod_{m=1}^{n-1}\left(1-\frac{|b|}{\delta_m}\right)\right)^{-1}, \\ \\ p_k = \frac{Cp_0}{\varepsilon_k}\cdot\prod_{m=1}^{k-1}\left(1-\frac{|b|}{\delta_m}\right), \quad k = 1,2,\cdots,. \end{cases} 0 < |b| < 1 \tag{2.14$'$}$$

DMS described above are *suspected* to vary regularly for *some* $\{\delta_n\}$.

3 Regularly Varying DMS. 1

3.1 The Result

Let $C \in R^+$ be a *parameter*, $\{\varepsilon_n\}$ and $\{\delta_n\}$ be sequences of positive numbers satisfying (2.13). Any *collection* $(C, \{\varepsilon_n\}, \{\delta_n\})$ generates a *one-parametric* family (with parameter b) of DMS of type (2.14) if $I = +\infty$ and of type (2.15) if $I < +\infty$, where

$$I = \sum_{n\geq1}\frac{1}{\delta_n}.$$

Our *goal* consists in discovering conditions on $\{\delta_n\}$ which lead to regular variation of $\{p_n\}$.

In the present *Section* we solve the problem with *additional* assumption: the limit exists

$$0 \le \lim_{n\to+\infty} \frac{n}{\delta_n} \stackrel{def}{=} A < +\infty. \tag{3.1}$$

The result is as follows.

Theorem 2 *1.* $\{p_n\}$ *varies regularly iff* $\{\delta_n\}$ *varies regularly.*
2.If $(-\rho)$ *and* α *are exponents of* $\{p_n\}$*'s and* $\{\delta_n\}$*'s regular variation, respectively, then*

$$\rho = \alpha + (|b|)\cdot A, \rho \in [1,+\infty), \alpha \in [1,+\infty). \tag{3.2}$$

Note that the relation $\alpha \in [1,+\infty)$ (see,(3.2)) is a *consequence* of (3.1).

Indeed, let us assume the opposite, *i.e.* $\alpha \in [0,1)$. Then, $\delta_n = 1 + n^{\alpha}\cdot L(n), n = 0,1,2,\cdots$ and, by known property on regular variation [20], for $\varepsilon \in (0, 1-\alpha)$ starting from some index

$$1 + n^{\alpha} L(n) < n^{\alpha+\varepsilon}.$$

Therefore, $A > \lim_{n\to+\infty} \frac{n}{n^{\alpha+\varepsilon}} = +\infty$, which contradicts (3.1).

Remark 1. For regularly varying $\delta_n$with exponent α the relation $\alpha \in [1,+\infty)$ holds even if (3.1) doesn't take place and only (2.13) holds.

The proof is similar to above given.

Theorem 2 is based on following *auxiliary*

Lemma 2 *If* (3.1) *holds, then for* $s = 2,3,\cdots$ *the limit exits*

$$B(s) \stackrel{def}{=} \lim_{n\to+\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = A\cdot \ln s. \tag{3.3}$$

Proof. Let $A = 0$. By (3.1),

$$\frac{1}{\delta_n} = o\left(\frac{1}{n}\right), n \to +\infty,$$

or for $s = 2,3,\cdots$

$$\sum_{m=n}^{sn-1} \frac{1}{\delta_m} = o\left(\sum_{m=n}^{sn-1} \frac{1}{m}\right), n \to +\infty.$$

Since for $s = 2,3,\cdots$

$$\sum_{m=n}^{sn-1} \frac{1}{m} = \ln s, \tag{3.4}$$

therefore,for $s = 2, 3, \cdots$

$$\sum_{m=n}^{sn-1} \frac{1}{\delta_m} = o(1), n \to +\infty.$$

Let $0 < A < +\infty$. For $\varepsilon \in (0,1)$ starting from some index $n \geq 1$ the inequalities hold

$$\frac{A\cdot(1-\varepsilon)}{m} < \frac{1}{\delta_m} < \frac{A\cdot(1+\varepsilon)}{m}, m = n, n+1, \cdots. \tag{3.5}$$

By (3.5) we obtain (3.3).

Theorem 2 has a *final* form if $\overline{A} = 0$.Then $\overline{A} = A (= 0)$.

Corollary 1. Let $\overline{A} = 0$. Then:

1'. $\{p_n\}$ varies regularly iff $\{\delta_n\}$ varies regularly.

2'. $\rho = \alpha \in [1, +\infty)$.

3.2 Proof of Theorem 2

Note that for $s = 2, 3, \cdots$

$$\lim_{n\to+\infty} \frac{\varepsilon_n}{\varepsilon_{sn}} = \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}} \quad \text{if limits exist.} \tag{3.6}$$

Indeed,

$$\lim_{n\to+\infty} \frac{\varepsilon_n}{\varepsilon_{sn}} = \lim_{n\to+\infty} \frac{\varepsilon_n}{\delta_n} \lim_{n\to+\infty} \frac{\delta_{sn}}{\varepsilon_{sn}} \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}} = \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}}, s = 2, 3, \cdots,$$

where (2.13) is used.

For $\{p_n\}$ of type (2.14) and $s = 2, 3, \cdots$, due to **Lemma 2** and (3.6),

$$\lim_{n\to+\infty} \frac{p_{sn}}{p_n} = \lim_{n\to+\infty} \frac{\varepsilon_n}{\varepsilon_{sn}} \exp\left\{-b\cdot \lim_{n\to+\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m}\right\} = \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}} \cdot \exp\{-b\cdot A\cdot \ln s\} =$$

$$= \frac{1}{s^{bA}} \cdot \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}}, 0 < b < 1, \tag{3.7}$$

if limits exist. From (3.7) we conclude that $\{p_n\}$ varies regularly iff $\{\delta_n\}$ varies regularly and $\rho = \alpha + b\cdot A$.

For $\{p_n\}$ of type (2.15) and $s = 2, 3, \cdots$, similarly to the previous case, we obtain

$$\lim_{n\to+\infty} \frac{p_{sn}}{p_n} = \frac{1}{s^{-bA}} \cdot \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}}, -1 < b < +\infty, \tag{3.8}$$

if limits exist. Since $I < +\infty$ in this case, so, $A = 0$, and in (3.8) we may replace the multiplier $\frac{1}{s^{-bA}}$ at the right-hand-side (3.8) by $\frac{1}{s^{|b|A}}$. Now, as in (3.7), formula (3.8) proves **Theorem 2** in this case.

3.3 On one regularity

Let us introduce a *new* approach to the problem. We want in future to weaken the assumptions in **Theorem 2**, based on this *approach*.

For $n = 1, 2, \cdots$ denote

$$q_n^{'} = \sum_{k \geq n} \frac{\varepsilon_k}{\delta_k} p_k, \quad D = q_1^{'}, \quad q_n = \sum_{k \geq n} p_k. \tag{3.9}$$

Let us show that

$$0 < D < +\infty. \tag{3.10}$$

Indeed, for $\varepsilon \in (0,1)$ starting from some index $n \geq 1$ the inequalities hold

$$1 - \varepsilon < \frac{\varepsilon_k}{\delta_k} < 1 + \varepsilon, \quad k = n, n+1, \cdots.$$

Therefore, for $m = n, n+1, \cdots$

$$(1-\varepsilon) q_m < q_m^{'} < (1+\varepsilon) q_m \tag{3.11}$$

which, due to $\sum_{k \geq 0} p_k = 1$, proves (3.10).

The *approach* consists in the existence of reverse to (2.14)-(2.15) equalities.

From (2.14) we have

$$\frac{p_{n+1}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{n+1}} \left(1 - \frac{b}{\delta_n}\right), n = 1, 2, \cdots, \quad \text{and} \quad \frac{p_1}{p_0} = C,$$

where without loss of generality we put $\varepsilon_0 = 1$, or, if we define

$$a_{n+1} = \varepsilon_{n+1} \cdot p_{n+1} \quad \text{and} \quad a_1 = p_1,$$

then:

$$a_{n+1} = a_n - b \frac{\varepsilon_n}{\delta_n} p_n = \cdots = a_1 - b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k} p_k = C p_0 - b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k} p_k.$$

Therefore, for $n = 2, 3, \cdots$ we have

$$\varepsilon_n = \frac{C p_0 - b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k} p_k}{p_n} = \frac{C p_0 - b \cdot D + b \cdot q_n^{'}}{p_n}, \quad 0 < b < 1. \tag{3.12}$$

Similarly, from (2.15) for $n = 1, 2, \cdots$ we obtain

$$a_{n+1} = a_n + b\frac{\varepsilon_n}{\delta_n}p_n = \cdots = a_1 + b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k}p_k = Cp_0 + b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k}p_k.$$

Therefore, for $n = 2, 3, \cdots$ the equality holds

$$\varepsilon_n = \frac{Cp_0 + b \cdot \sum_{k=1}^{n} \frac{\varepsilon_k}{\delta_k}p_k}{p_n} = \frac{Cp_0 + b \cdot D - b \cdot q_n^{'}}{p_n}, -1 < b < +\infty. \qquad (3.13)$$

The formulas (3.12)-(3.13) represent *reverse* equalities.

Let us exclude the case $b = 0$ in (2.15). The following result is *unexpected* because in particular case $\varepsilon_n = \delta_n, n = 1, 2, \cdots$, for DMS it gives simple expression for p_0:

$$p_0 = \frac{b}{C}(1 - p_0), \quad \text{or} \quad p_0 = \frac{b}{b + C}.$$

Theorem 3 *(a) For DMS of type (2.14)*

$$p_0 = \frac{bD}{C}, 0 < b < 1. \qquad (3.14)$$

(b) For DMS of type (2.15) with $b \neq 0$

$$\prod_{n \geq 1}\left(1 + \frac{b}{\delta_n}\right) \stackrel{def}{=} f \in \begin{cases} (0,1) & if \quad -1 < b < 0, \\ (1, +\infty) & if \quad 0 < b < +\infty, \end{cases}$$

and

$$p_0 = \frac{bD}{C \cdot (f - 1)}, -1 < b < +\infty, b \neq 0. \qquad (3.15)$$

Proof.(a) Forming the ratio (p_n/p_n), where p_n is taken from (3.12) and (2.14), respectively, for $n = 1, 2, \cdots$ we obtain

$$1 = \frac{Cp_0 - b \cdot \sum_{k=1}^{n-1} \frac{\varepsilon_k}{\delta_k}p_k}{Cp_0 \cdot \prod_{m=1}^{n-1}\left(1 - \frac{b}{\delta_m}\right)}, 0 < b < 1. \qquad (3.16)$$

Let us show that

$$\lim_{n \to +\infty} \prod_{m=1}^{n}\left(1 - \frac{b}{\delta_m}\right) = \prod_{n \geq 1}\left(1 - \frac{b}{\delta_n}\right) = f = 0. \qquad (3.17)$$

Indeed, for $n = 1, 2, \cdots$ we proceed

$$0 < \prod_{m=1}^{n}\left(1 - \frac{b}{\delta_m}\right) = \exp\left\{-\sum_{m=1}^{n} |\ln\left(1 - \frac{b}{\delta_m}\right)|\right\} <$$

$$< \exp\left\{-\sum_{m=1}^{n}\frac{b}{\delta_m} + \frac{1}{2}\sum_{m=1}^{n}\left(\frac{b}{\delta_m}\right)^2\right\} < \exp\left\{\frac{b^2}{2}\sum_{n\geq 1}\frac{1}{\delta_n^2}\right\}\cdot\exp\left\{-\sum_{m=1}^{n}\frac{b}{\delta_m}\right\}. \tag{3.18}$$

Since

$$\sum_{n\geq 1}\frac{1}{\delta_n^2} < \sum_{n\geq 2}\frac{1}{n(n-1)} + 1 = 1 + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2,$$

and $I = +\infty$, therefore, letting $n \to +\infty$ in (3.18) we prove (3.17).

Letting $n \to +\infty$ in (3.16) we conclude: since the limit of denominator at the right-hand-side of (3.16) as $n \to +\infty$ equals to zero and the left-hand-side of (3.16) is a finite positive number, therefore, necessarily numerator in the right-hand-side of (3.16) tends to zero as $n \to +\infty$.Thus,

$$\frac{Cp_0}{b} = \lim_{n\to+\infty}\sum_{k=1}^{n-1}\frac{\varepsilon_k}{\delta_k}p_k = D, \text{which coincides with (3.14).}$$

(b) From (3.13) and (2.15) for $n = 1, 2, \cdots$ and $b \neq 0$ we get the following equality

$$Cp_0\cdot\prod_{m=1}^{n}\left(1 + \frac{b}{\delta_m}\right) = Cp_0 + b\cdot\sum_{k=1}^{n}\frac{\varepsilon_k}{\delta_k}p_k, -1 < b < +\infty,$$

or tending $n \to +\infty$

$$p_0\cdot(f-1) = b\cdot\frac{D}{C}\begin{cases} > 0 & \text{if} \quad 0 < b < +\infty, \\ < 0 & \text{if} \quad -1 < b < 0. \end{cases} \tag{3.19}$$

Since

$$0 < p < 1, 0 < |\frac{bD}{C}| < +\infty,$$

so, from (3.19) we come to statement (b).

4 Regularly Varying DMS. 2

In the present *Section* by using the new approach **Theorem 2** is improved. The improvement, obviously, has to be made for the case

$$0 < \overline{A} \overset{def}{=} \overline{\lim_{n\to+\infty}}\frac{n}{\delta_n} < +\infty. \tag{4.1}$$

4.1 The Example

First of all, we must be convinced that *there is* a sequence $\{\delta_n\}$ satisfying conditions (2.13) (and (4.1)) such that

$$\underline{\lim}_{n\to+\infty}\frac{n}{\delta_n} < \overline{\lim}_{n\to+\infty}\frac{n}{\delta_n}(=\overline{A}) < +\infty. \tag{4.2}$$

Let us construct the required *example*.

Let us draw a ”broken” line whose pieces of straight lines are of two types: with slope $1-\varepsilon$ and with slope $1+\varepsilon$, where $\varepsilon\in(0,1)$ is some *fixed* number. The pieces of two types alternate each other. The curve begins from the point $(0,1)$ on the plane.

Denote by

$$(t_0,y_0)=(0,1),(t_1,y_1),(t_2,y_2),\cdots,(t_n,y_n),\cdots$$

the successive points of the curve's non-differentiability on the (t,y) plane.

Thus,the pieces of this curve $y=\delta(t)$ (broken line) in intervals

$$(t_{2n},t_{2n+1}),n=0,1,2,\cdots,$$

have a slope $1-\varepsilon$, and in intervals

$$(t_{2n-1},t_{2n}),n=1,2,\cdots,$$

have a slope $1+\varepsilon$.

We choose numbers $\{t_k\}$ satisfying conditions

$$\frac{y_{2n-1}-1}{t}=1-\varepsilon,\quad \frac{y_{2n}-1}{t}=1+\varepsilon,n=1,2,\cdots.$$

It is clear that the function $y=\delta(t)$ defined on R^+ is positive, $\delta(0)=1,\delta(t)$ increases as t increases,

$$\underline{\lim}_{t\to+\infty}\frac{t}{\delta(t)}=\frac{1}{1+\varepsilon},\quad \overline{\lim}_{t\to+\infty}\frac{1}{\delta(t)}=\frac{1}{1-\varepsilon}$$

and

$$\lim_{t\to+\infty}\frac{\delta(t+1)}{\delta(t)}=1.$$

The same properties has the sequence $\{\delta_n\}$ of positive numbers, where we put

$$\delta_n=\delta(n),n=0,1,2,\cdots.$$

Note that in our construction necessarily $t_{n+1}-t_n\to+\infty$ as $n\to+\infty$.

Thus, we built a sequence $\{\delta_n\}$ satisfying conditions (2.13) and (4.2).

4.2 The Result

The improvement of **Theorem 2**, uses the *forward* and *reverse* equalities (2.14) and (3.12), respectively. Let us make *comments*. The remaining case in **Theorem 2**, which requires an improvement, is related with DMS of type (2.14) with $\overline{A} \in R^{+}$. For this case the *reverse* equalities take the form (3.12).

The result is as follows.

Theorem 4 *Let us consider DMS of type* (2.14) *with* $\overline{A} \in R^{+}$. *Then:*

1. $\{p_n\}$ *varies regularly iff the limit* (3.1) *exists;*
2. *The exponent* $(-\rho)$ *of* $\{p_n\}$ *equals to* $(-(1+b\cdot A))$.

Proof. First of all we have to note that the existence of limit (3.1) with $A \in R^{+}$ implies the regular variation of sequence $\{\delta_n\}$ with exponent 1.

Indeed, for $s = 2,3,\cdots$ from (3.1) we obtain

$$\lim_{n\to+\infty} \frac{\delta_{sn}}{\delta_n} = \lim_{n\to+\infty} \frac{\delta_{sn}}{sn} \cdot \lim_{n\to+\infty} \frac{n}{\delta_n} \cdot s = \frac{A}{A} s = s.$$

For $s = 2,3,\cdots$ and $n = 1,2,\cdots$ from (3.12) we obtain (see, notations (3.9))

$$\frac{p_{sn}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \frac{Cp_0 - b\cdot D + b\cdot q'_{sn}}{Cp_0 - b\cdot D + b\cdot q'_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \frac{q'_{sn}}{q'_n}, \tag{4.3}$$

where the equality (3.14) is used.

¿From (3.11) it follows that the sequences $\{q_n\}$ and $\{q'_n\}$ are asymptotically equivalent. Since, at the same time, the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ also are asymptotically equivalent, therefore, (4.3) may be written in the form of asymptotical equivalence

$$\frac{p_{sn}}{p_n} \approx \frac{\delta_n}{\delta_{sn}} \frac{q_{sn}}{q_n}, n \to +\infty, s = 2,3,\cdots. \tag{4.4}$$

The *forward* equalities (2.14) for $s = 2,3,\cdots$ and $n = 1,2,\cdots$ give

$$\frac{p_{sn}}{p_n} = \frac{\varepsilon_n}{\varepsilon_{sn}} \exp\left\{\sum_{m=n}^{sn-1} \ln\left(1 - \frac{b}{\delta_m}\right)\right\},$$

which, due to asymptotical equivalence of sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$, in accordance with (3.7), implies

$$\frac{p_{sn}}{p_n} \approx \frac{\delta_n}{\delta_{sn}} \exp\left\{-b\cdot \sum_{m=n}^{sn-1} \frac{1}{\delta_m}\right\}, n \to +\infty, s = 2,3,\cdots. \tag{4.5}$$

The preparatory work is over.

Let us prove **Theorem 4.**

1. *The necessity.* Let $\{p_n\}$ varies regularly and, as we already know, it's exponent $(-\rho)$ satisfies condition $\rho \in [1, +\infty)$. By **Theorem 1** (a), p.281 [21], the sequence $\{q_n\}$ varies regularly with exponent $(-\rho+1)$. Therefore, by (4.4),

$$s^{-\rho} = \lim_{n\to+\infty} \frac{p_{sn}}{p_n} = \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}} \cdot \lim_{n\to+\infty} \frac{q_{sn}}{q_n} = s^{-(\rho-1)} \cdot \lim_{n\to+\infty} \frac{\delta_n}{\delta_{sn}},$$

or

$$\lim_{n\to+\infty} \frac{\delta_{sn}}{\delta_n} = s, s = 2, 3, \cdots,$$

which means that $\{\delta_n\}$ varies regularly with exponent $\alpha = 1$.

Next, from (4.5) for $s = 2, 3, \cdots$ we obtain

$$\lim_{n\to+\infty} \exp\left\{-b \cdot \sum_{m=n}^{sn-1} \frac{1}{\delta_m}\right\} = \lim_{t\to+\infty} \frac{p_{sn}}{p_n} \cdot \lim_{t\to+\infty} \frac{\delta_{sn}}{\delta_n} = s^{-(\rho-1)},$$

or for $s = 2, 3, \cdots$ the limit (3.3) exists

$$B(s) = \lim_{n\to+\infty} \sum_{m=n}^{sn-1} \frac{1}{\delta_m} = \frac{\rho-1}{b} \ln s.$$

Here $\rho > 1$, otherwise, by **Lemma 1**, if $\rho = 1$, then $\overline{A} = 0$, which contradicts the assumption $\overline{A} \in R^+$ of **Theorem 4**

Denote

$$A = \frac{\rho-1}{b}, \quad \text{so,} \quad B(s) = A \cdot \ln s.$$

Further, the following inequalities for $s = 2, 3, \cdots$ and $n = 1, 2, \cdots$ hold

$$\frac{\delta_{sn}}{\delta_n} \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m} < \frac{n}{\delta_n} < \frac{\delta_{(s+1)n}}{\delta_n} \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m}. \tag{4.6}$$

For $\varepsilon \in (0,1)$ starting from some index $k \geq 1$ the inequalities holds:

$$(1-\varepsilon)A \ln \frac{s+1}{s} = (1-\varepsilon)\left(B(s+1) - B(s)\right) < \sum_{m=sn}^{(s+1)n-1} \frac{1}{\delta_m} < (1+\varepsilon)\left(B(s+1) - B(s)\right) =$$

$$= (1+\varepsilon)A \ln \frac{s+1}{s}, s = 2, 3, \cdots, n = k, k+1, \cdots. \quad \text{Here} \quad k = k(s).$$

Then, for a fixed $s = 2, 3, \cdots$ and given ε the inequalities (4.6) may be rewritten in the form

$$(1-\varepsilon)A\ln\frac{s+1}{s}\cdot\frac{\delta_{sn}}{\delta_n} < \frac{n}{\delta_n} < (1+\varepsilon)A\ln\frac{s+1}{s}\cdot\frac{\delta_{(s+1)n}}{\delta_n}, n = k, k+1, \cdots.$$

In the last inequalities letting $n \to +\infty$ we get the following inequalities

$$(1-\varepsilon)A\cdot\ln\left(1+\frac{1}{s}\right)^s \le \underline{\lim}_{n\to+\infty}\frac{n}{\delta_n} \le \overline{\lim}_{n\to+\infty}\frac{n}{\delta_n} \le (1+\varepsilon)A\cdot\ln\left(1+\frac{1}{s}\right)^s, s = 2, 3, \cdots.$$

Letting $s \to +\infty$ and after that $\varepsilon \downarrow 0$ we obtain the *necessity* of statement 1.

2. *The sufficiency.* Since, the existence of limit (3.1) implies the regular variation of $\{\delta_n\}$, therefore, we are in conditions of statement 1. of **Theorem 2**, which implies the $\{p_n\}$'s regular variation. Now, statement 2. of **Theorem 4** follows from statement 2. of **Theorem 2.**

Theorem 4 is proved.

5 On the Slowly Varying Component

5.1 The Regular Class

In accordance with **Theorem 2** and **Theorem 4** in order to describe regularly varying DMS we have to change conditions on $\{\delta_n\}$, *i.e.* conditions (2.13). Namely, now we assume that: $\delta_0 = 1, \{\delta_n\}$ increases and varies regularly.

The limit exists

$$0 \le \lim_{n\to+\infty}\frac{n}{\delta_n} = A < +\infty. \tag{5.1}$$

The condition for positive sequence $\{\varepsilon_n\}$ is conserved

$$\lim_{n\to+\infty}\frac{\varepsilon_n}{\delta_n} = 1. \tag{5.2}$$

Then, DMS of types (2.14)-(2.15) together form *some* class of regularly varying distributions which we call a *Regular Class*, and DMS from this class we call *Regular Distributions*(RD).

RD are presented in the form

$$p_n \approx n^{-\rho}L_1(n), n \to +\infty, \rho \in [1, +\infty), \tag{5.3}$$

where $(-\rho)$ is the exponent of $\{p_n\}$'s regular variation, and $\{L_1(n)\}$ is some *slowly varying* sequence, which we call for $\{p_n\}$ a *slowly varying component* (SVC).

Knowledge of $\{L_1(n)\}$ is needed for applications to the large-scale biomolecular systems.

The form of presentation (5.3) gives possibility to choose $\{L_1(n)\}$ *suitable* for us and having *simple* expression (*asymptotocally*).

Below, some preliminary information on SVC of $\{p_n\}$ is given.

We assume that

$$\delta_n = 1 + n^{\alpha} L(n), n = 0, 1, 2, \cdots, \alpha \in [1, +\infty), \tag{5.4}$$

where α is the exponent of $\{\delta_n\}$'s regular variation, $\{L(n)\}$ is *explicit* SVC of $\{\delta_n\}$, and consider three possible cases of $\{L(n)\}$'s behavior.

The most simple is a case of RD of type (2.15).Indeed, due to (5.4),

$$\delta_n \approx n^{\alpha} L(n), n \to +\infty,$$

and because of (3.13) we have

$$p_n \approx \frac{1}{n^{\alpha} L(n)} \left(C p_0 + bD - b q_n' \right) \approx n^{-\alpha} \cdot \frac{C p_0 + bD}{L(n)}, n \to +\infty \tag{5.5}$$

with $-1 < b < +\infty$. Here we use

$$\lim_{n \to +\infty} q_n' = 0,$$

and, due to **Theorem 3**,

$$C p_0 + bD \neq 0.$$

Thus, in accordance with (5.3) and (5.5), in this case the SVC may be chosen in the form

$$L_1(n) = \frac{C p_0 + bD}{L(n)}, n = 1, 2, \cdots. \tag{5.6}$$

5.2 SVC for $\{p_n\}$ of Type (2.14)

For RD of type (2.14) with $A = 0$, as we know from **Theorem 2**, $\rho = \alpha = 1$. Due to (2.14), we have

$$p_n \approx \frac{bD}{nL(n)} \prod_{m=1}^{n-1} \left(1 - \frac{b}{mL(m)} \right).$$

Here also **Theorem 3** is used ($C p_0 = bD$).

It is clear that in this case the SVC may be taken in the form

$$L_1(n) = b \cdot D \cdot \prod_{m=1}^{n-1} \left(1 - \frac{b}{mL(m)} \right), 0 < b < 1. \tag{5.7}$$

It leads to *a new* conclusion.

Corollary 2. If $\{L(n)\}$ varies slowly and $\lim_{n+\infty} L(n) = +\infty$, then for any $b \in (0,1)$ $\{L_1(n)\}$ given by (5.7) varies slowly.

Let us show that it is possible to choose *another*, may be, more "pleasant" form of SVC.

Denote

$$\lambda_n = \frac{\varepsilon_n p_n}{b \cdot D \cdot \exp\left\{b \sum_{m=1}^{n-1} \frac{1}{\delta_m}\right\}} = \exp\left\{\sum_{m=1}^{n-1}\left[\ln\left(1-\frac{b}{\delta_m}\right)+\frac{b}{\delta_m}\right]\right\}. \tag{5.8}$$

For $0 < b < 1$ the function

$$f(x) = \ln\left(1-\frac{b}{x}\right)+\frac{b}{x}, x \in [1,+\infty),$$

is positive and increases as x increases. The positivity follows from the inequality $f(x) \geq \left(\frac{b}{x}\right)^2$ (it is the second term in $\ln(1-x)$'s expansion).

Further,

$$\frac{df(x)}{dx} = \frac{d}{dx}\left\{\ln(x-b) - \ln x + \frac{b}{x}\right\} = \frac{1}{x-b} - \frac{1}{x} - \frac{b}{x^2} = \frac{b}{x}\left(\frac{1}{x-b}-\frac{1}{x}\right) > 0.$$

Next,

$$\sum_{m=1}^{n-1}\left[\ln\left(1-\frac{b}{\delta_m}\right)+\frac{b}{\delta_m}\right] < \sum_{m=1}^{n-1}\left(\frac{b}{\delta_m}\right)^2 < b^2\left(1+\sum_{m\geq 2}\frac{1}{m(m-1)}\right) = 2b^2.$$

Therefore,

$$0 \leq \lim_{n\to+\infty}\sum_{m=1}^{n}\left[\ln\left(1-\frac{b}{\delta_m}\right)+\frac{b}{\delta_m}\right] = \sum_{n\geq 1}\left[\ln\left(1-\frac{b}{\delta_n}\right)+\frac{b}{\delta_n}\right] \stackrel{def}{=} P_b < +\infty.$$

It means that,due to (5.8),

$$\lambda_n = \exp\{P_b + \theta_n\}, n = 1,2,\cdots,$$

where $\lim_{n\to+\infty} \theta_n = 0$. Therefore, by (5.8), $\{L_1(n)\}$ may be chosen as follows

$$L_1(n) = \frac{b \cdot D \cdot \exp\{P_b\}\exp\left\{b \cdot \sum_{m=1}^{n-1}\frac{1}{1+mL(m)}\right\}}{L(n)}, n = 1,2,\cdots. \tag{5.9}$$

Note that in (5.9)

$$\sum_{m\geq 1}\frac{1}{1+mL(m)}=+\infty.$$

For RD of type (2.14) with $0<A<+\infty$, operating by the same manner, we may only claim in general that, as $\{L_1(n)\}$ may be chosen the sequence

$$L_1(n)=ACp_0n^{bA}\cdot\prod_{m=1}^{n-1}\left(1-\frac{b}{\delta_m}\right),n=1,2,\cdots, \tag{5.10}$$

where $\prod_{m=1}^{0}=1$.

With regard to (5.10) below we consider an important particular case.

5.3 The Linear $\{\delta_n\}$

Let us assume that $\{\delta_n\}$ is *linear, i.e.*

$$\delta_n=1+\frac{n}{A},n=0,1,2,\cdots,A\in R^{+}. \tag{5.11}$$

This case includes the well-known *family* of Waring Distributions (WD) with the following *traditional* form

$$\begin{cases} p_0=\left(1+p\cdot\sum_{n\geq 1}\frac{1}{q+n}\prod_{m=1}^{n-1}\frac{p+m}{q+m}\right)^{-1}, \\ p_k=\frac{p_0p}{q+k}\cdot\prod_{m=1}^{k-1}\frac{p+m}{q+m},\quad k=1,2,\cdots. \end{cases} \quad 0<p<q<+\infty. \tag{5.12}$$

The family (5.12) is equivalent to the generated by (2.14) family with $\{\delta_n\}$ of form (5.7), $\varepsilon_n=\delta_n, C=(1-b)A$.

There is a correspondence among parameters, which leads to the families' equivalence:

$$p=(1-b)A,\quad q=A.$$

Due to (5.11),(5.2),and (2.14), in accordance with (5.10), we have

$$L_1(n)=ACp_0\cdot n^{bA}\cdot\prod_{m=1}^{n-1}\left(1-\frac{bA}{A+m}\right),n=1,2,\cdots. \tag{5.13}$$

Now, we are going to show that there is some *positive* constant $P_{A,b}$ such that $\{L_1(n)\}$ may be chosen as $L_1(n)=P_{A,b}$.

Since

$$T_A\overset{def}{=}\sum_{n\geq 1}\frac{A}{n(A+n)}<+\infty,$$

therefore

$$\sum_{m=1}^{n}\frac{1}{A+m}=\sum_{m=1}^{n}\frac{1}{m}-T_A-\zeta_n, n=1,2,\cdots, \tag{5.14}$$

where $\lim_{n\to+\infty}\zeta_n=0$. At the same time (see,[22]),

$$\sum_{m=1}^{n}\frac{1}{m}=\ln n+E+\chi_n, n=1,2,\cdots, \tag{5.15}$$

where E is the *famous* Euler's constant and $\lim_{n\to+\infty}\chi_n=0$. From (5.14)-(5.15) we obtain

$$n^{bA}=\exp\{bA\ln n\}=\exp\left\{bA\cdot\sum_{m=1}^{n}\frac{1}{m}-bA\cdot E-\chi_n\cdot bA\right\}=$$

$$=\exp\left\{bA\cdot\sum_{m=1}^{n-1}\frac{1}{A+m}-bA\cdot E+\frac{1}{A+n}+T_AbA+\delta_n-\chi_n\cdot bA\right\}=$$

$$=e^{C_A}\cdot\pi_n\cdot\exp\left\{b\cdot A\cdot\sum_{m=1}^{n-1}\frac{1}{A+m}\right\}, n=1,2,\cdots, \tag{5.16}$$

where C_A is some constant and $\lim_{n\to+\infty}\pi_n=1$.

With the help of (5.16)we transform (5.13)

$$L_1(n)\approx r_A\cdot\exp\left\{\sum_{m=1}^{n-1}\left[\ln\left(1-\frac{bA}{A+m}\right)+\frac{bA}{A+m}\right]\right\}, n\to+\infty, \tag{5.17}$$

where r_A is some positive constant.

Now, we may do the same with the expression at the right-hand-side in (5.17),as it was done to $\{\lambda_n\}$ and was shown that $\lambda_n\approx const, n\to+\infty$.

Thus, as $L_1(n)$ may be chosen *some* positive constant.

More precisely,

$$L(n)=ACp_0\cdot\exp\{bA\cdot(T_A-E-M_A)\}\, n=1,2,\cdots,$$

where E is Eiler's constant, and

$$T_A=\sum_{n\geq1}\frac{A}{m(m+A)}, M_A=\sum_{n\geq1}\left\{\ln\left(1-\frac{bA}{m+A}\right)+\frac{bA}{m+A}\right\}.$$

6 Appendix

In [2] a *special* birth-death model, being a particular case of our model, for biomolecular applications has been presented. The *part* of stationary distributions of the *special* model, which are of interest, takes the form

$$\begin{cases} p_0 = \left(1 + (1-b)\cdot \sum_{n\geq 1} \frac{1}{\delta_n} \prod_{m=1}^{n-1} \left(1 - \frac{b}{\delta_m}\right)\right)^{-1}, \\ p_k = (1-b)p_0 \cdot \frac{1}{\delta_k} \cdot \prod_{m=1}^{k-1}\left(1 - \frac{b}{\delta_m}\right), \quad k = 1, 2, \cdots, \end{cases} \tag{6.1}$$

with

$$0 < b < 1, \quad \sum_{n\geq 1} \frac{1}{\delta_n} = +\infty,$$

where $\{\delta_n\}$ satisfies *all* conditions presented at the beginning of *Section 5*.

For regularly varying distributions (6.1) **Theorem 3** has a very simple form.

Corollary 3.

$$p_0 = b. \tag{6.2}$$

Indeed, since (compare to (2.14)) in case (6.1) we have

$$C = 1 - b \quad \text{and} \quad D = \sum_{n\geq 1} \frac{\varepsilon_n}{\delta_n} p_n = \sum_{n\geq 1} p_n = 1 - p_0,$$

therefore, from (3.14) we obtain

$$p_0 = \frac{bD}{C} = \frac{b}{1-b}(1 - p_0),$$

which implies (6.2).

The class of distributions (6.1) includes a family of WD in particular case

$$\delta_n = 1 + \frac{n}{A}, n = 0, 1, 2, \cdots, A \in R^+.$$

It is clear that (6.2) is true also for WD.

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