

## Al-Alaoui Operator and the $\alpha$ -Approximation for Discretization of Analog Systems

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**Abstract:** The  $\alpha$ -approximation for discretization of analog systems was recently introduced. In this paper it is shown that the  $\alpha$ -approximation is exactly the same as the parameterized Al-Alaoui operator.

**Keywords:** Al-Alaoui operator, discretization, digital filters.

### 1 Introduction

A popular method for designing IIR digital filters is to map the transfer function of a corresponding analog filter using an  $s$ -to- $z$  transformation [1]-[6]. It is desirable that the mapping procedures have the following two properties: 1) they should map the left half of the  $s$ -plane to the interior of the unit circle in the  $z$ -plane which would insure that stable analog filters map into stable digital filters, and 2) the imaginary axis of the  $s$ -plane should be mapped onto the unit circle circumference in the  $z$ -plane.

The bilinear transform meets the above requirements. However, it introduces a warping effect due to its nonlinearity, albeit it can be ameliorated somewhat by a pre-warping technique. The backward difference transform satisfies the first condition, but the second condition is not completely satisfied, since the imaginary axis of the  $s$ -plane maps onto the circumference in the  $z$ -plane centered at  $z = 1/2$  and having a radius of  $1/2$ . The mapping meets condition 2 rather closely for low frequencies [1]-[3], [4]-[6].

Other transforms were introduced in attempts to obtain better approximations [7]-[10]. In particular, in [7, 8] the approach interpolates the rectangular integration rules and the trapezoidal integration rule. The resulting digital integrator transfer

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function in the  $z$ -domain was equated to the analog ideal integrator,  $1/s$ , in the  $s$ -domain to obtain the  $s$ -to- $z$  transformation. In [7] a fixed weighting of 0.75 was assigned to the rectangular rule and 0.25 for the trapezoidal rule, while in [8] the interpolation was parameterized with an  $\alpha$ -parameter. The resulting operator in [7] was designated Al-Alaoui operator and applied in fractional order discretization schemes by Chen and Moore in [11]. The operator developed in [8] may be designated as the parameterized Al-Alaoui operator. However for brevity Al-Alaoui operator is used to refer to either of them. In [10] the  $\alpha$ -approximation is proposed.

The paper is divided into 5 sections including the introduction and conclusion. The second section presents the  $\alpha$ -approximation, the third section introduces Al-Alaoui operator, and the fourth section shows that the  $\alpha$ -approximation and the Al-Alaoui operator are the same operator.

## 2 The $\alpha$ -Approximation

The  $\alpha$ -approximation for discretization of analog systems was recently introduced in [10]. The approximation starts from the well known mapping of the  $s$ -to- $z$  domains shown in equation (1)

$$z = e^{sT} \quad (1)$$

where  $T$  is the sampling period.

Starting from equation (1), the following equivalent relation can be formulated

$$\begin{aligned} z = e^{sT} &= e^{s((1-\alpha)T + \alpha T)} \\ &= \frac{e^{(1-\alpha)Ts}}{e^{-\alpha Ts}}, \quad \text{for } \alpha \in [0, 1] \end{aligned} \quad (2)$$

After the numerator and denominator on the right hand side of equation (2) have been expanded in series and all member of the second and higher orders neglected, expression (2) becomes

$$z = \frac{\sum_{n=0}^{\infty} \frac{[(1-\alpha)T]^n}{n!}}{\sum_{k=0}^{\infty} (-1)^k \frac{(\alpha Ts)^k}{k!}} \simeq \frac{1 + (1-\alpha)Ts}{1 - \alpha Ts} \quad (3)$$

Solving equation (3) for the complex variable  $s$  yields

$$s = f(z, \alpha) = \frac{1}{T} \frac{z-1}{1 + \alpha(z-1)} \quad (4)$$

Equation (4) defines the  $s$ -to- $z$   $\alpha$ -approximation.

### 3 Al-Alaoui Operator

Al-Alaoui operator is obtained by interpolating the trapezoidal and the rectangular integration rules to obtain the following class of integrators [8]

$$H(z) = aH_{Rect}(z) + (1-a)H_{Trap}(z) \quad (5)$$

Using the backward rectangular rule for the rectangular integration rule yields

$$H(z) = a \frac{Tz}{z-1} + (1-\alpha) \frac{Tz+1}{2z-1}, \quad \text{for } 0 \leq a \leq 1 \quad (6)$$

where  $T$  is the sampling period.

Equating  $H(z)$ , the transfer function of the resulting IIR digital integrator as expressed in (6) to the transfer function  $1/s$  of an ideal analog integrator, yields the parameterized  $s$ -to- $z$  transformation shown in equation (7)

$$s = \frac{2(z-1)}{T[(1-a) + (1+a)z]} \quad (7)$$

In [7] a fixed value of  $a$  was used,  $a = 3/4$ , and the  $s$ -to- $z$  transformation of equation (8) is obtained

$$s = \frac{8(z-1)}{7T(z + \frac{1}{7})} \quad (8)$$

It is to be noted that the forward rectangular rule,  $H_{ForwardRect}(z) = T/(z-1)$ , is used in [7] which results in a non-minimum phase transfer function  $H(z)$  and a stabilizing approach is used [7], [12]. The same results are obtained, without the need for stabilization, by using the backward rectangular rule. The above procedure is equivalent to interpolating directly the bilinear operator (Tustin), and the backward difference operator.

### 4 Al-Alaoui Operator and the $\alpha$ -Approximation

In this section it will be shown that Al-Alaoui Operator, equation (7), and the  $\alpha$ -approximation, equation (4) are one and the same. Equation (7) may be reformulated as follows by letting  $\alpha = (1+a)/2$ , and thus  $1-\alpha = (1-a)/2$

$$\begin{aligned} s &= \frac{2(z-1)}{T[(1-a) + (1+a)z]} = \frac{z-1}{T\left[\frac{1-a}{2} + \frac{(1+a)z}{2}\right]} \\ &= \frac{z-1}{T[(1-\alpha) + \alpha z]} = \frac{1}{T} \frac{z-1}{1 + \alpha(z-1)} \end{aligned} \quad (9)$$

Thus Al-Alaoui operator and the  $\alpha$ -approximation are one and the same.

## 5 Conclusion

It was shown that Al-Alaoui operator and the  $\alpha$ -approximation are one and the same. Al-Alaoui operator is a credible alternative to other discretization methods with wide range of applications in integer and fractional order discretization.

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