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Applicatin of the α-Approximation for Discretization of Analogue Systems

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Abstract: The method for discretization of analogue systems using the α-approximation is presented. A generalization of some of the existing transformation methods is also done. A comparative analysis, through the corresponding examples involving several known discretization methods, is carried out. It is demonstrated that the application of this α-approximation allows the reduction of discretization error compared to other approximation methods. The frequency characteristics of the discrete system obtained by these transformations are approximately equal to these of the original analogue system in the basic frequency range.

Keywords: Discretization, α-approximation, Digital filters

1 Introduction

The design of a classical analogue system in combination with a digital system often requires formulation of a discrete model or a discrete equivalent. This can be done either by employing the method of invariable response to a pulse or step excitation or by a method involving hold circuits or by a series of other approximate methods. The procedure of transformation of a continuous transfer function to a digital equivalent (pulse transfer function) should preserve the essential properties of the analogue system. First of all, the transformation has to be rational, i.e., starting from an analogue transfer function, which is a real rational function of the complex frequency s, one should obtain a rational function of the complex variable z. Secondly, for the purpose of preserving stability of the system, the poles of the

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analogue transfer function lying in the left half of the s-plane have to be mapped inside the unit circle centered at the origin of the z-plane. Thirdly, it is desirable that the transformation does not increase the order of the transfer function, that the steady state gains are equal, etc. In the process of discretization of a continuous system, one can use the well-known mapping of the s-domain into z-domain by substitution

$$z = e^{sT}, \tag{1}$$

where T is the sampling period. Transformation (1) maps the left half of the s-plane into the interior of the unit circle in the z-plane. This means that the stability of the discrete system has been preserved if all poles of the discrete system are inside the unit circle.

One of the basic goals of discretization is fulfilling the need for practical realizations of, e.g., adequate control laws or some other digital systems focused at the goal that the digital equivalent is as close as possible to the corresponding continuous system. For example, the synthesis of digital filters of IIR type is most often carried out by an appropriate transformation of the transfer function of the corresponding analogue filter. This approach to the synthesis procedure is justified by several reasons. First of all, the procedure of synthesis of analogue filters (particularly low frequency filters) has been studied for more than sixty years so that there are developed procedures for many important practical situations.

The paper is divided into four sections including the introduction and conclusions. The second section presents a brief review of the basic discretization methods of the classic analogue systems. The third section presents the novel method for discretization of continuous systems together with a comparative analysis through the corresponding examples of several other known methods for discretization of such systems.

2 A brief review of the basic discretization methods

The basic definition of the pulse transfer function is

$$G(z) = \mathrm{Z}[g(t)] = \sum_{k=0}^{\infty} g(kT)z^{-k} = \mathrm{Z}\left\{\mathrm{L}^{-1}[G(s)]\right\}, \tag{2}$$

where Z and $\mathrm{L}^{-1}$are the operators representing the Z-transform and inverse Laplace transform, respectively. The pulse transfer function $H(z)$ of a system containing zero-order hold can be written as

$$H(z) = (1-z^{-1})\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{z}{z-e^{sT}}\frac{G(s)}{s}ds, e^{\sigma T} < |z| \tag{3}$$

where T is the sampling period and σ is a real number such that all poles of function $G(s)/s$ have real parts smaller than σ and $\lim_{s\to\infty} G(s) = 0$. Applying the calculus of residues to (3), after rearrangement, one could write

$$H(z) = (1-z^{-1})\sum_k \text{Res}\left\{\frac{z}{z-e^{sT}}\frac{G(s)}{s}\right\}_{s=s_k} \tag{4}$$

Transfer function defined by (3) or (4) can also be obtained on the basis of the expression

$$H(z) = (1-z^{-1})\text{Z}\left\{\text{L}^{-1}\left[\frac{G(s)}{s}\right]\right\} \tag{5}$$

Notice that from (2) one can formally obtain $G(z)$ by substitution $s = \ln(z)/T$, i.e.,

$$G(z) = G^*(s)\Big|_{s=(\ln z)/T} \tag{6}$$

Where $G^*(s)$ is the Laplace transform of the pulse sequence $g^*(t) = \sum_{k=0}^{\infty} g(kT)\,\delta(t-kT)$. The above described transformations (2), (4), and (5) are direct and they allow transformation of transfer function $G(s)$ of a continuous system to a zero-hold equivalent pulse transfer function [1], [2]. It should be noticed that besides these direct discretization methods there are also indirect discetization methods which are approximate and are based to a transformation of the type

$$s = f(z), \tag{7}$$

where $f(z)$ is the corresponding real rational function of complex variable z obtained on the basis of relation (1).

For the purpose of discretization of an analogue system, the use is often made of the so called method of *matching poles and zeros* (MPZ) [3].

It is shown [4] that when $G(s)$=1/s^n , then

$$H_J(z) = \frac{T^n}{n!}\frac{B_n(z)}{(z-1)^n}, \tag{8}$$

where $B_n(z)$ are given for several values of $n \in N$.

$$B_1(z) = 1$$

$$B_2(z) = z+1$$

$$B_3(z) = z^2+4z+1$$

$$B_4(z) = z^3+11z^2+11z+1$$

$$B_5(z) = z^4 + 26z^3 + 66z^2 + 26z + 1$$

$$B_6(z) = z^5 + 57z^4 + 302z^3 + 302z^2 + 57z + 1$$

....

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \cdots + b_n^n$$

Here $b_k^n = k b_k^{n-1} + (n-k+1) b_{k-1}^{n-1}$, $k = \overline{2, n-1}$, and $b_1^n = b_n^n = 1$. It can also be established that $B_2(z)$ and $B_3(z)$ have one unstable zero (zero outside of the unit circle with the origin of the z-plane), $B_4(z)$ and $B_5(z)$ two unstable zeros, $B_6(z)$ three unstable zeros, etc. Other details concerning zeros in the discrete transfer function and Jury's polynomials $B_n(z)$ are available in the literature [4],[5],[6].

3 α-approximation

Starting from basic relation (1), the following equivalent relation can be written

$$\begin{aligned} z &= e^{sT} = e^{s((1-\alpha)T + \alpha T)} \\ &= \frac{e^{(1-\alpha)Ts}}{e^{-\alpha Ts}}, \quad \alpha \in [0,1] \end{aligned} \tag{9}$$

After the numerator and denominator on the right hand side of (9) have been expanded in series and all member of the second and higher orders neglected, expression (9) becomes

$$z = \frac{\sum_{n=0}^{\infty} \frac{[(1-\alpha)Ts]^n}{n!}}{\sum_{k=0}^{\infty} (-1)^k \frac{(\alpha Ts)^k}{k!}} \approx \frac{1 + (1-\alpha)Ts}{1 - \alpha Ts} \tag{10}$$

By solving (10) for complex variable s, the α-approximation of first order is obtained

$$s = f(z, \alpha) = \frac{1}{T} \frac{z-1}{1 + \alpha(z-1)} \tag{11}$$

Parameter α can be obtained by a physical explanation (Fig. 1) in the sense of a continuous increase of the part of the surface of the basic rectangle of integration sum, in particular from the surface of the basic rectangle of the lower integration sum to the surface of the basic rectangle of the upper integration sum. This enables the possibility of vertical modification of Z-transform such as, e.g., modified Z-transform along time axis. Generally speaking, direct application of the first order α-approximation for the purpose of discretization of continuous systems should

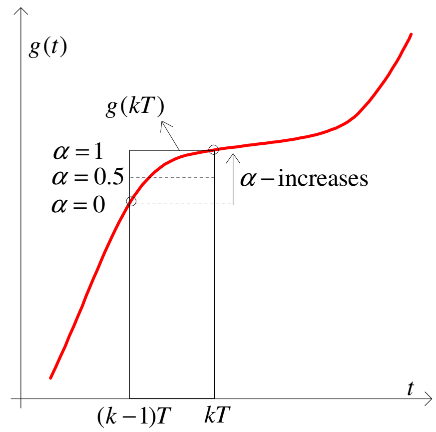

Fig. 1. Physical interpretation of the α-approximation of first order, for variable parameter α

ensure mapping stability (a stable analogue system is mapped into a stable discrete system). In order to analyze this problem, let us use Fig. 2 which shows mapping of the left half of the s-plane (a) into a circle in the z-plane (b) by applying the suggested approximation (11). As is known from the theory of stability of digital systems, mapping is stable if the left half of the s-plane is mapped inside the unit circle of the z-plane. Fig. 2 shows clearly that for values of parameter $\alpha \geq 0.5$ the first order α-approximation results to a stable mapping. If the value of parameter α is less than 0.5, the left half of the s-plane is mapped into a unit circle in the z-plane exceeding the boundaries of the unit circle. The center and radius of the corresponding circle are $C(1-1/(2\alpha), j0)$ and $R = 1/(2\alpha)$, respectively.

It should be noted that approximation $s = \frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}$, which has been introduced and analyzed [7] with respect to other approximations, corresponds to the α-approximation for $\alpha = 7/8$. This shows that the α-approximation, depending on the value of α, unifies a number of the approximation, some of them are presented in Table 1. In other words, the α-approximation can offer an additional degree of freedom in the course of an efficient discretization of a particular analogue system.

The efficiency of the α-approximation may be looked at on the basis of the following equation:

$$\frac{1}{s^n} = \prod_{k=1}^{n} \frac{1+\alpha_k(z-1)}{z-1} T \tag{12}$$

This means that a separate parameter $\alpha_k \in [0,1]$, $k = 1, n$ is attached to each integrator $1/s$ offering a considerable degree of freedom in achieving efficient discretiza-

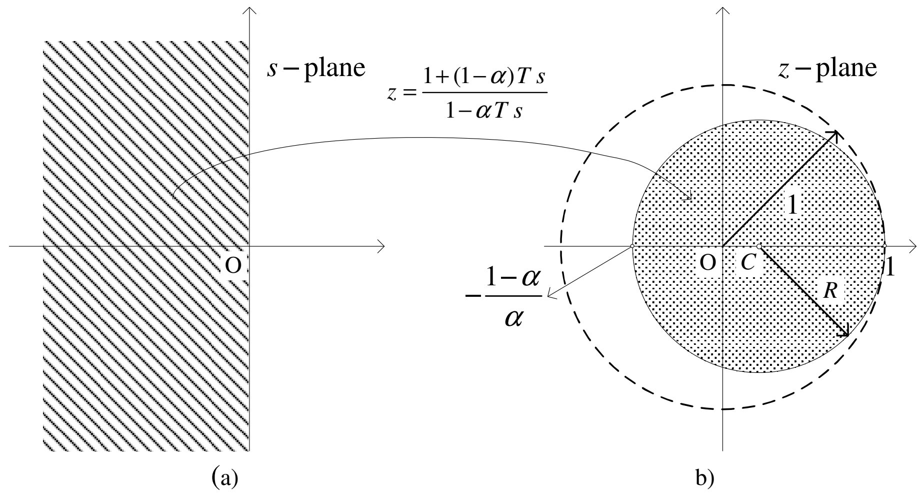

Fig. 2. Mapping of the left half of the s-plane applying the α-approximation first order to a circle of radius R centered at C in the z-plane

Table 1. Generalization of some known transformations for discretization of analogue systems [8],[9],[10],[11],[12]

r	α	$s-z$ approximation	Name of approximation
-	0	$(z-1)/T$	Euler approximation of first order (FD)
1	$\frac{1}{2}$	$\frac{2}{T}\frac{z-1}{z+1}$	Tustin approximation (BL)
0	1	$\frac{1}{T}\frac{z-1}{z}$	Euler approximation of second order (BD)
-	$\alpha \in [0,1]$	$\frac{1}{T}\frac{z-1}{1+\alpha(z-1)}$	$\alpha-$approximation of first order
$r \in [0,1]$	$\alpha \in [0.5,1]$ $\alpha = 1/(1+r)$	$\frac{1+r}{T}\frac{z-1}{z+r}$	Parametric BD-BL approximation
-	-	$s = u\,\frac{1-z}{-v+wz}$	General form of first-order $s-z$ approximation

tion of particular continuous systems.

Example 1. Let be given an analogue transfer function describing suppressed oscillations

$$G(s) = \frac{1}{s^2+0.4s+0.68} \tag{13}$$

Determine discrete transfer function $G(z)$ by using previously introduced standard approximations, given in Table 1, and make a comparison with the α-approximation given in the same table.

a$_1$) The use is made directly by Euler's approximation of first order from Table 1 with $T = 1$ to obtain

$$G_{E1}(z) = \frac{25}{25z^2 - 40z + 32} \tag{14}$$

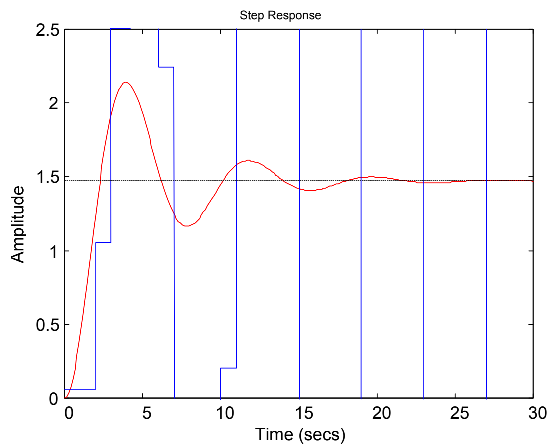

Fig. 3. Step response of the system in the continuous and discrete domains for $G(s)$ and $G_{E1}(z)$

a$_2$) By using directly the Tustin approximation from Table 1 with $T = 1$, one obtains

$$G_T(z) = \frac{25(z+1)^2}{137z^2 - 166z + 97} \tag{15}$$

a$_3$) By using directly the Euler approximation of second order from Table 1 with T=1, one obtains

$$G_{E2}(z) = \frac{25z^2}{52z^2 - 60z + 25} \tag{16}$$

The α-approximation of discretization will be illustrated by the above example and compared to the results obtained by (a$_1$) to (a$_4$). By applying direct α-approximation of the first order (11) to relation (13), one obtains discrete transfer function

$$G_\alpha(z) = \frac{a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0}, \tag{17}$$

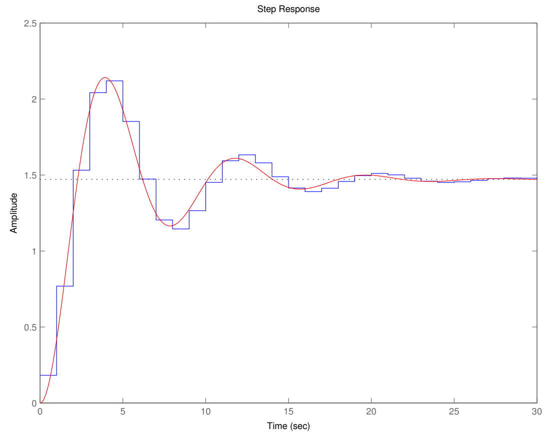

Fig. 4. Step response of the system in the continuous and discrete domains for $G(s)$ and $G_T(z)$

where

$$\begin{array}{ll} a_0 = 25T^2(1-\alpha)^2 & b_0 = 17\alpha^2T^2 + 2\alpha T(5-17T) + 17T^2 - 10T + 25 \\ a_1 = 50\alpha(1-\alpha)T^2 & b_1 = -34\alpha^2T^2 - 2\alpha T(10-17T) + 10(T-5) \\ a_2 = 25\alpha^2T^2 & b_2 = 17\alpha^2T^2 + 10\alpha T + 25 \end{array}$$

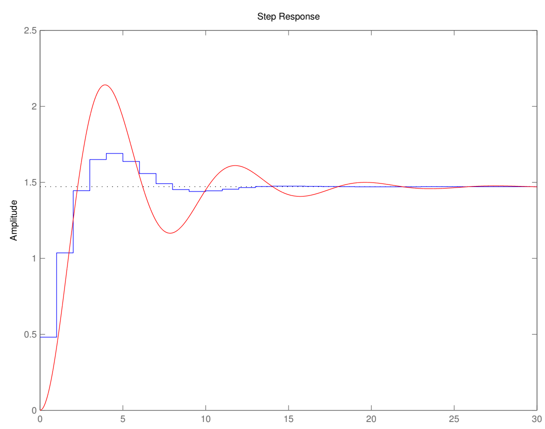

Fig. 5. Step response of the system in the continuous and discrete domains for $G(s)$ and $G_{E2}(z)$

Determine such α and T that poles z_k of discrete transfer function $G_\alpha(z)$ coincide with poles p_k of transfer function $G(z)$ on the basis of relation z_k=e^{Tp_k}. This means that discrete transfer function $G_\alpha(z)$ should have poles in the z-plane $z_{1,2} = e^{-0.2\pm j0.8} = 0.5704 \pm j0.5873$. By numerical solution of equations $\frac{b_0}{b_2} = z_1 z_2 = e^{-0.4}$, $\frac{b_1}{b_2} = -(z_1+z_2) = -2e^{-0.2}\cos(0.8)$, one obtains α=0.53401 and T=1.06063.

a$_4$) By introducing the obtained parameters α and T in the right hand side of (17), one obtains discrete transfer function

$$G_\alpha(z) = \frac{2.812\,(2.67\,z+2.329)^2}{90.29\,z^2 - 103z + 60.52} \tag{18}$$

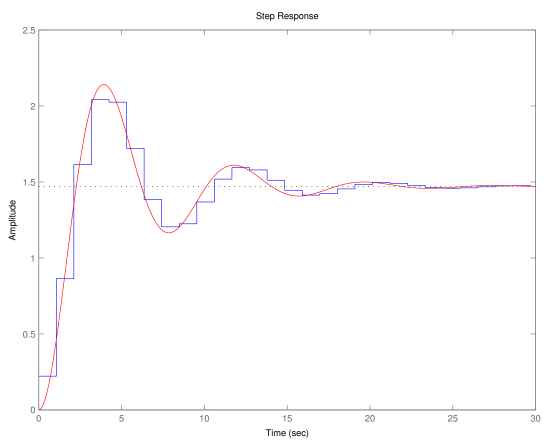

Fig. 6. Step response of the system in the continuous and discrete domains for $G(s)$ and $G_\alpha(z)$

It can be seen that, with the α-approximation of first order, it is possible to make corrections of the amplitude and phase errors depending on the values of parameters α and T of the second order system which is most often met in practice. If a priory discretization period T is fixed, there remains parameter α available for optimization of the discrete transfer function in accordance with the practical need important for achieving minimum errors in the phase and amplitude frequency characteristics.

E.g., for the analogue system (13), by relation (12) directly and by specifying discretization time T, the discretized analogue system becomes

$$G_{\alpha 1,\alpha 2}(z) = \frac{1}{\left(\frac{1}{T}\frac{z-1}{1+\alpha_1(z-1)}\right)\left(\frac{1}{T}\frac{z-1}{1+\alpha_2(z-1)}\right)+0.4\left(\frac{1}{T}\frac{z-1}{1+\alpha_1(z-1)}\right)+0.68} \tag{19}$$

a_5) If discretization time is adopted to be T=1, in accordance with the previous examples, and α_1 and α_2 are determined so that poles z_k of discrete transfer function $G_{\alpha 1,\alpha 2}(z)$ coincide with poles p_k of transfer function $G(s)$ on the basis of relation $z_k = e^{T p_k}$,one obtains α_1=0.71320 and α_2=0.32120 and discrete function (19) becomes

$$G_{\alpha_1,\alpha_2}(z) = \frac{0.22908\,(z+2.1133)\,(z+0.40213)}{1.2843\,z^2 - 1.4651\,z + 0.86086} \tag{20}$$

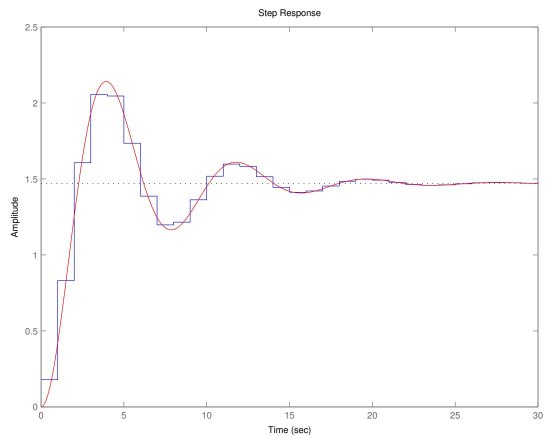

Fig. 7. Step response of the system in the continuous and discrete domains for $G(s)$ and $G_{\alpha 1,\alpha 2}\,(z)$

On the basis of the presented step responses in the continuous and discrete domains, it is straightforward to notice that errors of the approximation relations of Table 2 and of the α-approximation of the first order in this example are smaller compared to those of the classical approximations of Table 1 for $\alpha \in \{0, 0.5, 1\}$.

Let us verify numerically these results and present them in Table 2 by using the following integral criterion of the squared error for each cell over the interval of observation in all previous examples, a_1) to a_5), with reference to the exact value.

$$J = \sum_{k=0}^{N-1} \left(\int_{kT}^{(k+1)T} (g(t) - g_{ap}(kT))dt \right)^2, \tag{21}$$

where:

$g(t)$ - exact value of the step response of continuous system in $G(s)$, (13)

$g_{ap}(kT)$ - the corresponding step response of discrete system $G_{ap}(z)$ obtained by approximations a_1) to a_5),

Table 2. The values of the integral criterion of the squared error for each cell in the interval of observation $N = 30$ of the previous examples a_k, $k = 1, \ldots, 5$

Example1.	(a_1)	(a_2)	(a_3)	(a_4)	(a_5)
J	-	0.0719	0.7598	0.0714	0.0022

N- number of cells in the interval of observation.

By using frequency characteristic given in Fig. 8, one can notice the advantageous merits of the α-approximation for discretization of analogue systems which gave characteristic a_5 in example a_1).

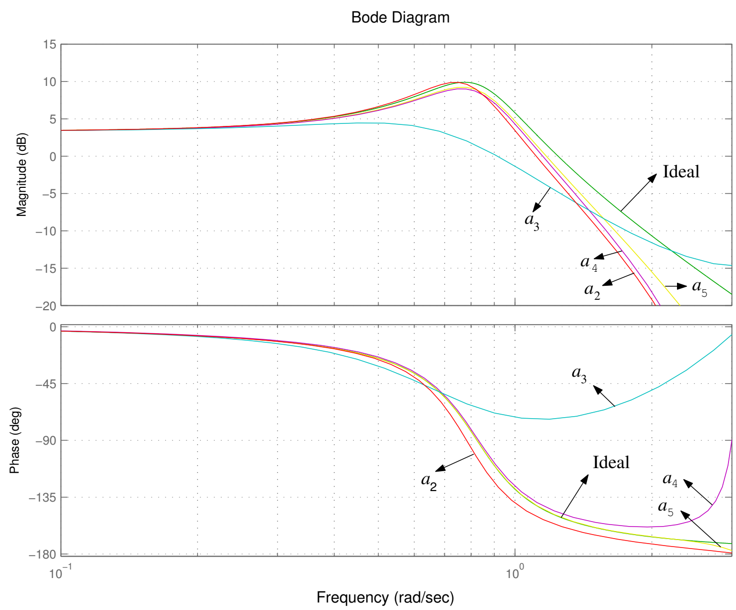

Fig. 8. The amplitude and phase characteristics of the analogue $G(s)$ (-ideal) and the corresponding discrete approximations $G_T(z)$ (-a_2), $G_{E2}(z)$ (-a_3), $G_\alpha(z)$ (-a_4), and $G_{\alpha1,\alpha2}(z)$ (-a_5) for Example 1

Example 2. Let be given a transfer function describing stationary oscillations

$$G(s) = \frac{0.64}{s^2 + 0.64} \tag{22}$$

Determine pulse transfer function $G(z)$, comprising the zero-order hold circuit, by using previously introduced standard approximations given in Table 1 and make a comparison with the α-approximation given in the same table.

b$_1$) Let us form function

$$G_{HO}(s^{-1}) = \frac{0.64\,s^{-3}}{1+0.64s^{-2}}(1-e^{-sT}). \tag{23}$$

By the corresponding substitution s in (23) on the basis of relation (12) and taking that T=1, one obtains zero-hold equivalence pulse transfer function

$$G_{HO}^{\alpha_1,\alpha_2}(z) = \frac{0.64\,(1-z^{-1})}{\dfrac{z-1}{1+\alpha_0(z-1)}\left(\dfrac{z-1}{1+\alpha_1(z-1)}\dfrac{z-1}{1+\alpha_2(z-1)}+0.64\right)} \tag{24}$$

Let us determine such α_0, α_1 and α_2 so that poles z_k of discrete transfer function $G_{\alpha 1,\alpha 2}(z)$ coincide with poles p_k of transfer function $G(s)$ on the basis of relation $z_k = e^{Tp_k}$, one obtains α_0=0.5, α_1=0.0951 and α_2=0.9049 and discrete function (24) becomes

$$G_{HO}^{\alpha_1,\alpha_2}(z) = \frac{0.026104\,(z+0.10511)(z+1)(z+9.5134)}{z\,(z^2-1.3934\,z+1)} \tag{25}$$

Fig. 9 shows step response of $G_{HO}^{\alpha_1,\alpha_2}(z)$ together with step response of continuous function $G(s)$.

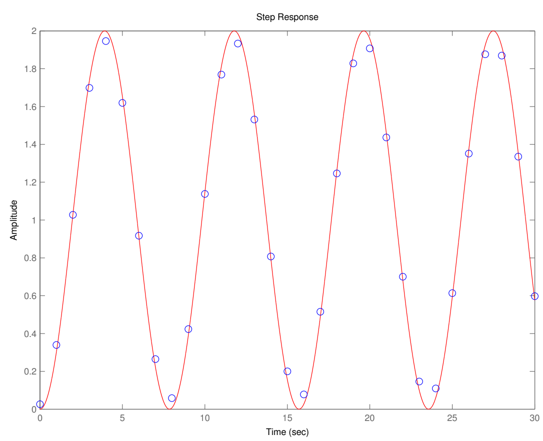

Fig. 9. Step response of the system in continuous and discrete domains for $G(s)$ and $G_{HO}^{\alpha_1,\alpha_2}(z)$

b$_2$) By using directly α-approximation with T=1, one obtains pulse transfer function

$$G_{HO}^{\alpha} = \frac{16}{z}\frac{(1+\alpha(z-1))^3}{(16\alpha^2+25)z^2-2(16\alpha^2-16\alpha+25)z+16\alpha^2-32\alpha+41}, \tag{26}$$

which is stable for parameter $\alpha \geq 0.5$. This implies all α-transformations covered by the set of values $0.5 \leq \alpha \leq 1$ where also belong the Tustin and Euler transformations of second order. If it is assumed that the corresponding roots of the characteristic equation are at the boundary of stability in both discrete and continuous systems, it follows that $\alpha = 0.5$ (Tustin's approximation), i.e., the discrete transfer function is

$$G_{HO}^{T}(z) = \frac{2\,(z+1)^3}{(29z^2 - 42z + 29)z} \tag{27}$$

Fig. 10 shows step response of $G_{HO}^{T}(z)$ together with the step response of continuous function $G(s)$.

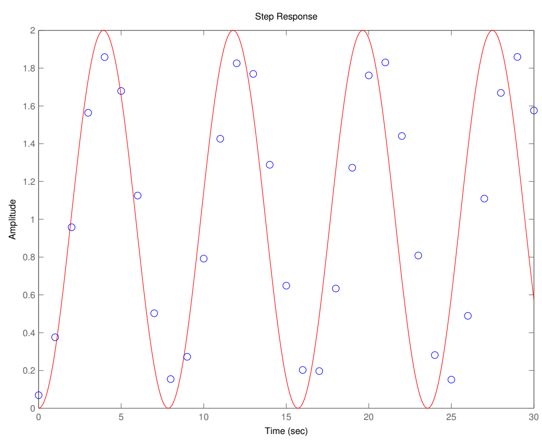

Fig. 10. Step response of the system in continuous and discrete domains $G(s)$ and $G_{HO}^{T}(z)$

On the basis of Fig. 9, one may conclude that in Example 2 the step response of the discrete equivalent (25) obtained by using α-approximation coincides fairly with the step response of the continuous system (23). The maximum discrepancy $\varepsilon = \max|g(kT) - g_{ap}(kT)| = 0.052$, obtained in the interval of observation $t \in [0,30]$ corresponds to the relative error of less than 3 % with respect to the exact value. The maximum error obtained in this example, applying the α-approximation for $\alpha = 0.5$ (Tustin's approximation for the same interval of observation), is $\varepsilon =$ 1.0001. This error arises as a consequence of the phase delay brought about by the Tustin approximation. It can be best shown by using frequency diagrams shown in Fig. 11.

Example 3. Determine discrete model of the continuous system represented by the Butterworth filter of sixth order having transfer function $G_{BF}(s)$ for the purpose of synthesizing IIR filter functions having sampling period T =0.5, i.e., $\Omega = \omega T =$

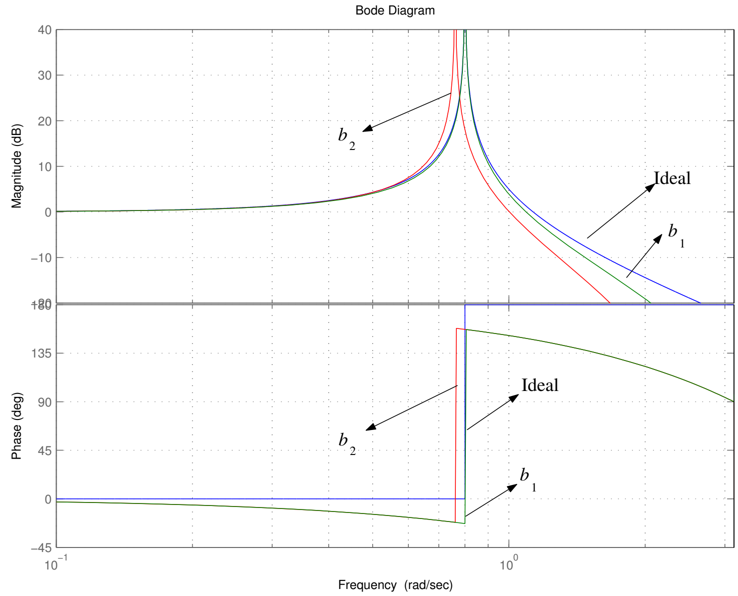

Fig. 11. The amplitude and phase characteristics of the analogue $G(s)$ (-ideal) and digital $G_{HO}^{\alpha_1,\alpha_2}(z)(-b_1), G_{HO}^{T}(z)(-b_2)$ for Example 2.

$\omega/2$, applying approximation polynomials [13]. Then, compare the amplitude and phase characteristics of the analogue and digital filters and also their step responses.

$$G_{BF}(s) = \frac{1}{(s^2+2\cos\frac{5\pi}{12}s+1)(s^2+2\cos\frac{\pi}{12}s+1)(s^2+\sqrt{2}s+1)} \tag{28}$$

Applying α-approximation on the basis of relation (12) on the previously obtained relation of the analogue Butterworth filter (28), one obtains equivalent transfer function $H_{BF}(z)$of the digital Butterworth filter

$$H_{BF}(z) = K\frac{(z+1.6406)(z+0.99664)(z+0.86104)(z+0.84193)(z+0.79103)(z+0.55494)}{(z^2-1.2236z+0.38063)(z^2-1.3175z+0.49307)(z^2-1.5562z+0.77196)}, \tag{29}$$

where $K = 1.1816 \times 10^{-4}$.

4 Conclusion

For the purpose of discretization of continuous systems depending on their purpose, a series of approximation methods has been developed. A generalization of some of the existing approximations for discretization (α-approximation) of analogue systems has been carried out. This paper has presented α-approximation

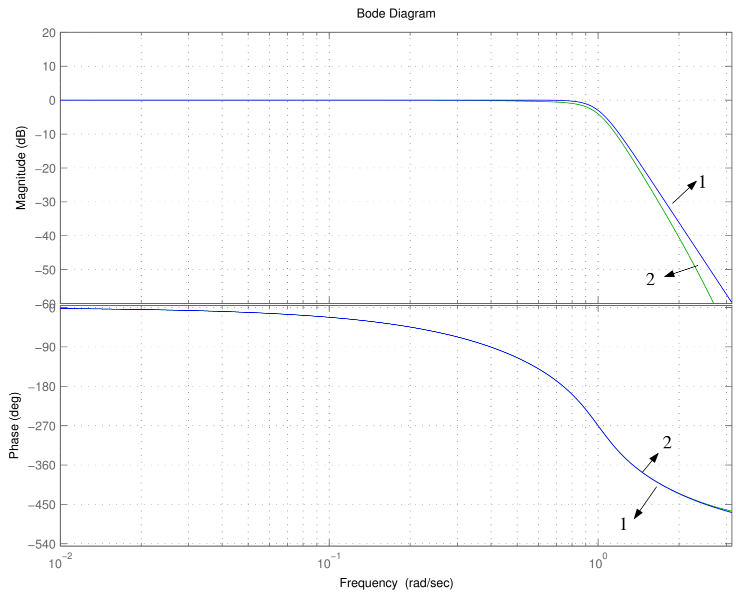

Fig. 12. The amplitude and phase characteristics of the analogue $G_{BF}(s)$ (-1) and digital $H_{BF}(z)$ (-2) Butterworth filters

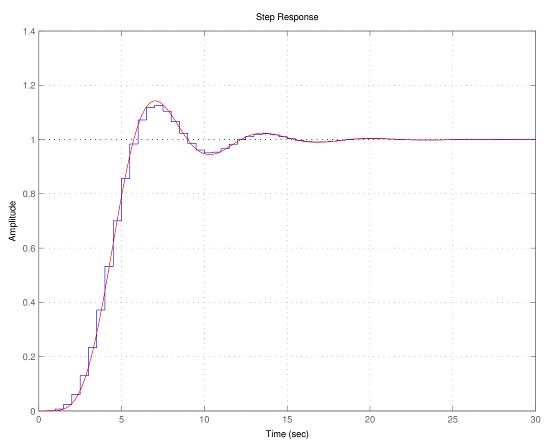

Fig. 13. The step responses of the analogue $G_{BF}(s)$ and digital $H_{BF}(z)$ Butterworth filters

offering more efficient discretization of analogue systems. The application of this α-aproximation has also some practical implications, e.g., in the synthesis of IIR filters for digital signal processing it allows modular solutions.

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